# Coloring the Square of a $K_4$-minor Free Graph


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Revised, December 24, 2002



**Abstract**

Let $G$ be a $K_4$-minor free graph with maximum degree $\Delta$. We prove that the chromatic number of the square of $G$ is at most (i) $\Delta + 3$ if $2 \leq \Delta \leq 3$; or (ii) $\lfloor 3\Delta/2 \rfloor + 1$ if $\Delta \geq 4$. Examples are given to show the bounds can be attained.



*Key words*: Series-parallel graph, $K_4$-minor free graph, square, chromatic number


## 1 Introduction

All graphs considered in this paper are finite, loopless, and without multiple edges. For a graph $G$, let $V(G)$, $E(G)$, $|G|$, $\Delta(G)$, and $\delta(G)$ denote, respectively, its vertex set, edge set, number of vertices, maximum degree, and minimum degree. For $x \in V(G)$, let $N_G(x)$ denote the set of neighbors of $x$ in $G$ and let $d_G(x)$ denote the degree, i.e., the number of neighbors, of $x$ in $G$. A vertex of degree $k$ is called a $k$-vertex. For two vertices $u, v \in V(G)$, let $\mathrm{dist}_G(u, v)$ denote the distance between $u$ and $v$, i.e., the length of a shortest path connecting them.

A $k$-coloring of a graph $G$ is a mapping $\phi$ from $V(G)$ to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(x) \neq \phi(y)$ for every edge $xy$ of $G$. We call $G$ $k$-colorable if it has a $k$-coloring. The *chromatic number* $\chi(G)$ of $G$ is the smallest integer $k$ such that $G$ is $k$-colorable. The *square* $G^2$ of a graph $G$ is the graph defined on the vertex set $V(G)$ such that two vertices $u$ and $v$ are adjacent in $G^2$ if and only if $1 \leq \mathrm{dist}_G(u, v) \leq 2$. A mapping $\phi$ is a $k$-coloring of $G^2$ if and only if $\phi(u) \neq \phi(v)$ whenever $1 \leq \mathrm{dist}_G(u, v) \leq 2$.

Since $G^2$ contains a clique of order at least $\Delta(G)+1$, we have $\chi(G^2) \geq \Delta(G)+1$ for any graph $G$. If $G$ is a tree, then $\chi(G^2) = \Delta(G)+1$. On the other hand, $\Delta(G^2) \leq \Delta^2(G)$. Hence $\chi(G^2) \leq \Delta^2(G)+1$ for any graph $G$. The 5-cycle $C_5$ and the Petersen graph satisfy $\chi(G^2) = \Delta^2(G)+1$.

Wegner [9] first investigated the chromatic number of the square of a planar graph. Wegner proved that $\chi(G^2) \leq 8$ for every planar graph $G$ with $\Delta(G) = 3$ and conjectured that the upper bound could be reduced to 7, which has been confirmed by Thomassen [7]. For planar graphs of maximum degree $\Delta \geq 4$, Wegner [9] proposed the following conjecture. The bounds are sharp if the conjecture is true.

**Conjecture 1** *Let $G$ be a planar graph. Then*

$$\chi(G^2) \leq \begin{cases} \Delta(G)+5 & \text{if } 4 \leq \Delta(G) \leq 7; \\ \lfloor 3\Delta(G)/2 \rfloor + 1 & \text{if } \Delta(G) \geq 8. \end{cases}$$

This conjecture remains open. The best upper bound so far is $5\Delta(G)/3 + 78$ established by Molloy and Salavatipour [5]. This improves other recently obtained upper bounds: $\lfloor 9\Delta/5 \rfloor + 2$ for $\Delta(G) \geq 749$ ([1]), $\lceil 9\Delta/5 \rceil + 1$ for $\Delta(G) \geq 47$ ([2]), and $2\Delta(G) + 25$ ([6]). For planar graphs of large girth, better upper bounds for $\chi(G^2)$ are known. Wang and Lih [8] proved that if $G$ is a planar graph with girth $g(G)$, then $\chi(G^2) \leq \Delta(G) + 5$ when $g(G) \geq 7$, $\chi(G^2) \leq \Delta(G) + 10$ when $g(G) \geq 6$, and $\chi(G^2) \leq \Delta(G) + 16$ when $g(G) \geq 5$.

A graph $G$ has a graph $H$ as a *minor* if $H$ can be obtained from a subgraph of $G$ by contracting edges, and $G$ is called *$H$-minor free* if $G$ does not have $H$ as a minor. A graph $G$ is called a *series-parallel* graph if $G$ can be obtained from $K_2$ by applying a sequence of operations, where each operation is either to duplicate an edge (i.e., replace an edge with two parallel edges) or to subdivide an edge (i.e., replace an edge with a path of length 2). It is well-known [3] that a graph $G$ is an outerplanar graph if and only if $G$ is $K_4$-minor free and $K_{2,3}$-minor free. A graph $G$ is $K_4$-minor free if and only if each block of $G$ is a series-parallel graph. Thus the class of $K_4$-minor free graphs is a class of planar graphs that contains both outerplanar graphs and series-parallel graphs. We will establish the following in this paper.

**Theorem 1** *Let $G$ be a $K_4$-minor free graph. Then*

$$\chi(G^2) \leq \begin{cases} \Delta(G) + 3 & \text{if } 2 \leq \Delta(G) \leq 3; \\ \lfloor 3\Delta(G)/2 \rfloor + 1 & \text{if } \Delta(G) \geq 4. \end{cases}$$

Before proving Theorem 1, we show that the result is best possible. For $\Delta = 2$, we have $\Delta(C_5) = 2$ and $\chi(C_5^2) = 5$. For $\Delta = 3$, let $G$ be the graph consisting of three internally disjoint paths joining two vertices $x$ and $y$, where two of the paths are of length 2, and the third one is of length 3. Then $\Delta(G) = 3$ and $\chi(G^2) = \chi(K_6) = 6$. For $\Delta = 2k \geq 4$, let $G_{2k}$ be the graph consisting of $k$ internally disjoint paths joining $x$ and $y$, $k$ internally disjoint paths joining $x$ and $z$, and $k$ internally disjoint paths joining $y$ and $z$. All these paths are of length 2, except one path joining $x$ and $y$ is of length 1, and one path joining $x$ and $z$ is of length 1. Then $\Delta(G_{2k}) = 2k$ and $\chi(G_{2k}^2) = \chi(K_{3k+1}) = 3k+1$. For $\Delta = 2k+1 \geq 5$, let $G_{2k+1}$ be obtained from $G_{2k}$ by adding a new path of length 2 joining $y$ and $z$. Then $\Delta(G_{2k+1}) = 2k+1$ and $\chi(G_{2k+1}^2) = \chi(K_{3k+2}) = 3k+2$.

## 2 Proof of Theorem 1

Let $u \sim v$ denote that $u$ and $v$ are adjacent in $G$, that is, $uv \in E(G)$. Define $S_G(u) = \{x \mid d_G(x) \geq 3$ such that $u \sim x$ or there exists a 2-vertex $z$ satisfying $u \sim z$ and $z \sim x\}$. Let $D_G(u) = |S_G(u)|$. It is well-known [4] that every $K_4$-minor free graph contains a vertex of degree at most two.

**Lemma 2** *Let $G$ be a $K_4$-minor free graph. Then one of the following holds.*

(i) $\delta(G) \leq 1$;

(ii) *There exist two adjacent 2-vertices;*

(iii) *There exists a vertex $u$ with $d_G(u) \geq 3$ such that $D_G(u) \leq 2$.*

**Proof.** Suppose that the lemma is false. Let $G$ be a counterexample, i.e., $G$ is a $K_4$-minor free graph satisfying the following properties:

(a) $\delta(G) = 2$;

(b) Every 2-vertex is adjacent to two vertices of degree $\geq 3$;

(c) Every vertex $u$ with $d_G(u) \geq 3$ has $D_G(u) \geq 3$.

Let $H$ be the graph obtained from $G$ by removing all 2-vertices and joining any two nonadjacent vertices that had a 2-vertex as a common neighbor. Then $H$ has minimum degree at least 3 because $N_H(v) = S_G(v)$ for each vertex $v$ with $d_G(v) \geq 3$. However, $H$ is a $K_4$-minor free graph by its very construction. A contradiction is henceforth obtained. □

**Proof of Theorem 1**. The case for $\Delta(G) = 2$ follows directly from the well-known Brooks' Theorem since $\Delta(G) = 2$ implies $\Delta(G^2) \leq 4$. In the following we assume that $\Delta(G) \geq 3$.

Our proof proceeds by induction on the number of vertices of $G$. To make notation simpler, define $K(\Delta) = 6$ if $\Delta = 3$ and $K(\Delta) = \lfloor 3\Delta/2 \rfloor + 1$ if $\Delta \geq 4$.

The conclusion follows immediately if $|G| \leq 6$. Now assume that $|G| \geq 7$ and Theorem 1 holds for $K_4$-minor free graphs $H$ with $|H| < |G|$.

If $G$ has a vertex $x$ of degree 1, then let $H = G - x$. If $G$ has two adjacent 2-vertices $x$ and $y$, then let $H$ be the graph obtained from $G$ by deleting $x$ and adding an edge joining the two neighbors of $x$. In each of the above cases, $H$ is a $K_4$-minor free graph with $\Delta(H) \leq \Delta(G)$. Moreover, $G^2 - x$ is a subgraph of $H^2$. By the induction hypothesis, $H^2$ is $K(\Delta)$-colorable. So $G^2 - x$ is $K(\Delta)$-colorable. Since $x$ has degree $< K(\Delta)$ in $G^2$, any $K(\Delta)$-coloring of $G^2 - x$ can be extended to a $K(\Delta)$-coloring of $G^2$.

Thus we may assume that $\delta(G) = 2$ and any two 2-vertices of $G$ are not adjacent. By Lemma 2, $G$ has a vertex $u$ such that $d_G(u) \geq 3$ and $D_G(u) \leq 2$. For $t \in S_G(u)$, let $M(u,t)$ denote the set of all 2-vertices in $G$ that are adjacent to both $u$ and $t$ and let $m(t) = |M(u,t)|$.

Obviously $D_G(u) \geq 1$. Assume that $D_G(u) = 1$ and $S_G(u) = \{z\}$. Then all the neighbors of $u$ are either $z$ or some neighbors of $z$. Since $d_G(u) \geq 3$, we see that $m(z) \geq 2$. Let $w \in M(u,z)$ and $H = G - w$. Then $H$ is a $K_4$-minor free graph with $\Delta(H) \leq \Delta(G)$. Moreover, $G^2 - w = H^2$. By the induction hypothesis, $H^2$ has a $K(\Delta)$-coloring $\phi$. Since $d_{G^2}(w) \leq \Delta(G) + 1 < K(\Delta)$, $\phi$ can be extended to a $K(\Delta)$-coloring of $G^2$.

Assume that $D_G(u) = 2$. Let $S_G(u) = \{x, y\}$. Thus all the neighbors of $u$ are either $x$, $y$, or some neighbors of $x$ or $y$. Without loss of generality, we suppose $m(x) \geq m(y)$. Since $d_G(u) \geq 3$, we have $m(x) \geq 1$. Let $w \in M(u,x)$. The proof is divided into the following three cases.

**Case 1.** $x \sim u$.

Let $H = G - w$. Then $H$ is a $K_4$-minor free graph with $\Delta(H) \leq \Delta(G)$ and $G^2 - w = H^2$. By the induction hypothesis, $H^2$ has a $K(\Delta)$-coloring $\phi$. In order to extend $\phi$ to a $K(\Delta)$-coloring of $G^2$, it suffices to show that $w$ has degree at most $K(\Delta) - 1$ in $G^2$. Since $x$ is adjacent to $u$, $d_{G^2}(w) \leq d_G(u) + d_G(x) - m(x) - 1$. ¿From $m(x) + m(y) \geq d_G(u) - 2$ and $m(x) \geq m(y)$, we conclude that $m(x) \geq \lceil (d_G(u) - 2)/2 \rceil = \lceil d_G(u)/2 \rceil - 1$. Hence

$$\begin{aligned} d_{G^2}(w) &\leq d_G(u) + d_G(x) - m(x) - 1 \\ &\leq d_G(u) + d_G(x) - \lceil d_G(u)/2 \rceil \\ &\leq \lfloor 3\Delta(G)/2 \rfloor \\ &= K(\Delta) - 1. \end{aligned}$$

**Case 2.** $x \not\sim u$ and $y \not\sim u$.

Similarly as above, let $H = G - w$. Then $H$ is a $K_4$-minor free graph with $\Delta(H) \leq \Delta(G)$. Note that $x$ and $u$ have distance at most 2 in $H$ since $d_G(u) \geq 3$ and $m(x) \geq m(y)$. Hence $G^2 - w = H^2$. It is easy to see that $m(x) + m(y) = d_G(u)$ and $d_{G^2}(w) = d_G(u) + d_G(x) - m(x) + 1$. From $m(x) \geq m(y)$ we conclude that $m(x) \geq \lceil d_G(u)/2 \rceil$. Hence

$$\begin{aligned} d_{G^2}(w) &\leq d_G(u) + d_G(x) - m(x) + 1 \\ &\leq d_G(u) + d_G(x) - \lceil d_G(u)/2 \rceil + 1 \\ &\leq \lfloor 3\Delta(G)/2 \rfloor + 1 \\ &= K(\Delta). \end{aligned}$$

By the induction hypothesis, $H^2$ has a $K(\Delta)$-coloring $\phi$, which acts as a partial coloring of $G^2$ such that all the vertices except $w$ have been properly colored. Since $d_{G^2}(u) \leq \Delta(G) + 2 \leq K(\Delta) - 1$, $u$ has at most $K(\Delta) - 2$ colored neighbors in $G^2$ in this partial coloring. Hence there are at least two choices for properly coloring $u$ before coloring $w$. One of these two colorings of $u$ will imply that the neighbors of $w$ have at most $K(\Delta)-1$ colors. So the coloring $\phi$ can be extended to a $K(\Delta)$-coloring of $G^2$.

**Case 3.** $x \not\sim u$ and $y \sim u$.

If $m(x) = m(y)$, we may interchange $x$ and $y$, and it falls under Case 1. So assume that $m(x) > m(y)$.

First suppose $d_G(u)$ is odd. Then $m(x) + m(y) = d_G(u) - 1$ is even.

If $m(x) \geq m(y) + 4$, then $m(x) \geq (d_G(u) + 3)/2$. This implies that

$$\begin{aligned} d_{G^2}(w) &\leq d_G(u) + d_G(x) - m(x) + 1 \\ &\leq \lfloor 3\Delta(G)/2 \rfloor \\ &= K(\Delta) - 1. \end{aligned}$$

Let $H = G - w$. As before, $H^2 = G^2 - w$ and every $K(\Delta)$-coloring of $H^2$ can be extended to a $K(\Delta)$-coloring of $G^2$.

If $m(x) = m(y) + 2$, we have $m(x) = (d_G(u) + 1)/2$. If $d_G(u) < \Delta(G)$, then $d_{G^2}(w) \leq \lfloor 3\Delta(G)/2 \rfloor = K(\Delta) - 1$, and the same argument applies to this case. Thus we assume that $d_G(u) = \Delta(G)$.

If $x \sim y$, then $d_{G^2}(w) \leq d_G(u) + d_G(x) - 1 - (m(x) - 1) \leq \lfloor 3\Delta(G)/2 \rfloor$. Any $K(\Delta)$-coloring of $G^2 - w$ can be extended to a $K(\Delta)$-coloring of $G^2$. Assume now that $x \not\sim y$. In this case, the calculation shows that $d_{G^2}(w) \leq K(\Delta)$. We now show how to choose a $K(\Delta)$-coloring of $G^2 - w$ so that two neighbors of $w$ in $G^2$ have the same color. Such a $K(\Delta)$-coloring then can be extended to $w$.

Since $d_G(u) = \Delta(G) \geq 5$, we may choose some $w' \in M(u, y) \neq \emptyset$. Let $H = G - w - w' + xy$. It is easy to see that $H$ is also a $K_4$-minor free graph with $\Delta(H) \leq \Delta(G)$. Moreover, $G^2 - w - w'$ is a subgraph of $H^2$. By the induction hypothesis, $H^2$ has a $K(\Delta)$-coloring $\phi$. Note that $x$ has a color different from those of $y$ and all neighbors of $y$ in $H$. Going back to the graph $G$, we color $w'$ with the color $\phi(x)$. This coloring can be extended to $w$ because $x$ and $w'$ are neighbors of $w$ in $G^2$ with the same color.

Now we consider the case that $d_G(u)$ is even. Then $m(x) + m(y) = d_G(u) - 1$ is odd. If $m(x) \geq m(y) + 3$, we have $m(x) \geq d_G(u)/2 + 1$. If $d_G(u) < \Delta(G)$, then

$$\begin{aligned} d_{G^2}(w) &\leq d_G(u) + d_G(x) - m(x) + 1 \\ &\leq d_G(x) + d_G(u)/2 \\ &\leq \Delta(G) + (\Delta(G) - 1)/2 \\ &\leq \lfloor 3\Delta(G)/2 \rfloor \\ &= K(\Delta) - 1. \end{aligned}$$

If $d_G(u) = \Delta(G)$, the same calculation shows that $d_{G^2}(w) \leq d_G(x) + d_G(u)/2 \leq \lfloor 3\Delta(G)/2 \rfloor$. So any $K(\Delta)$-coloring of $G^2 - w$ can be extended to $G^2$.

Assume that $m(x) = m(y) + 1$. Since $d_G(u) \geq 4$, $m(y) = d_G(u)/2 - 1 \geq 1$. Choose $w' \in M(u, y)$. Then

$$\begin{aligned} d_{G^2}(w') &\leq d_G(u) + d_G(y) - m(y) - 1 \\ &\leq d_G(u) + d_G(y) - d_G(u)/2 \\ &\leq \lfloor 3\Delta(G)/2 \rfloor \\ &= K(\Delta) - 1. \end{aligned}$$

Let $H = G - w'$. Then $H^2 = G^2 - w'$, and any $K(\Delta)$-coloring of $H^2$ can be extended to $G^2$. □

# 3 Remarks

A graph $G$ is called *$k$-degenerate* if every subgraph $H$ of $G$ has $\delta(H) \leq k$. The *degeneracy* of $G$ is the minimum $k$ such that $G$ is $k$-degenerate. The *coloring number* $\mathrm{col}(G)$ of a graph $G$ is one plus the degeneracy of $G$.

In most cases of the proof of Theorem 1, we found a vertex $w$ of $G$ such that $d_{G^2}(w) \leq K(\Delta) - 1$ and $G^2 - w$ was a subgraph of the square of a $K_4$-minor free graph $H$ with $|H| = |G| - 1$. Yet there were two cases where $d_{G^2}(w) = K(\Delta)$, in which case $K(\Delta) = \lfloor 3\Delta(G)/2 \rfloor$, so the proof actually establishes the following result.

**Theorem 3** *Let $G$ be a $K_4$-minor free graph. Then*

$$\mathrm{col}(G^2) \leq \begin{cases} \Delta(G) + 3 & \text{if } 2 \leq \Delta(G) \leq 3; \\ \lfloor 3\Delta(G)/2 \rfloor + 2 & \text{if } \Delta(G) \geq 4. \end{cases}$$

For $\Delta(G) = 2$ and 3, the above upper bound for $\mathrm{col}(G^2)$ is the same as the upper bound for $\chi(G^2)$ in Theorem 1, hence it is sharp. If $\Delta(G)$ is an even number $2k \geq 4$, then let $G'_{2k}$ be the graph consisting of vertices $x_1, x_2, x_3, x_4$ and $k$ internally disjoint paths of length 2 joining $x_i$ and $x_{i+1}$ (addition modulo 4). It is easy to verify that $G'_{2k}$ is a $K_4$-minor free graph, $\Delta(G'_{2k}) = 2k$, and $\delta((G'_{2k})^2) = 3k + 1$. So the coloring number of $(G'_{2k})^2$ is at least $3k + 2$. If $\Delta(G)$ is an odd number $2k + 1 \geq 5$, then let $G'_{2k+1}$ be obtained from $G'_{2k}$ as follows. Replace a length 2 path between $x_1$ and $x_2$ by an edge and a length 2 path between $x_3$ and $x_4$ by an edge. Then add one length 2 path connecting $x_1$ and $x_4$ and one length 2 path connecting $x_2$ and $x_3$. Then $\Delta(G'_{2k+1}) = 2k + 1$ and $\delta((G'_{2k+1})^2) = 3k + 2$. So the coloring number of $(G'_{2k+1})^2$ is at least $3k + 3$. In conclusion, Theorem 3 gives sharp bounds.

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