

# 4-colorable 6-regular toroidal graphs

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## Abstract

This paper proves two conjectures of Collins, Fisher and Hutchinson about the chromatic number of some circulant graphs. As a consequence, we characterize 4-colorable 6-regular toroidal graphs .

## 1 Introduction

A graph  $G$  is  $k$ -regular if every vertex of  $G$  is incident with exactly  $k$  edges. A graph is *Eulerian* if all its vertices have even degree. A *surface* is a compact Hausdorff topological space which is locally homeomorphic to  $\mathcal{R}^2$ . Any surface falls in one of the following two infinite classes: take a sphere and attach  $n$

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handles, or take a sphere and attach  $m$  crosscaps. We denote these surfaces by  $S_n$  and  $\tilde{S}_m$  respectively. Surfaces  $S_n$  are *orientable*, that is, it is possible to assign a local sense of clockwise and anticlockwise so that along any path between any two points in the surface the local sense is consistent. Surfaces  $\tilde{S}_m$  are *nonorientable*. A graph *embeds* on a surface if it can be drawn there without edge crossings. The complement of an embedded graph on a surface is a set of open regions. The maximal connected open regions are called the *faces* of the embedding. An embedding is a *triangulation* if each face is bounded by exactly three edges. In Figure 1, the left-hand diagram is a triangulation of the Klein bottle and the right-hand diagram represents a triangulation of the projective plane.

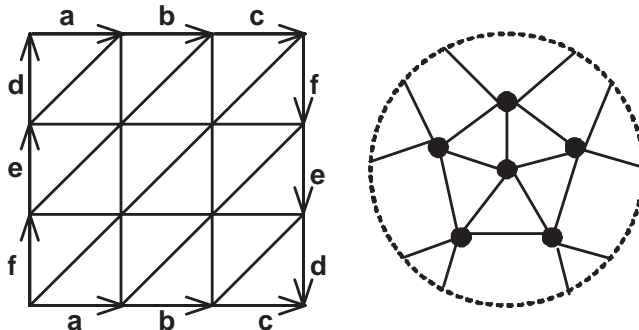


Figure 1.

We say that a graph  $G$  is  $k$ -colorable if  $V(G)$  can be colored by at most  $k$  distinct colors so that adjacent vertices are colored by different colors. The *chromatic number* of  $G$ , denoted by  $\chi(G)$ , is the smallest  $k$  such that  $G$  is  $k$ -colorable. A cycle  $C$  of a graph  $G$  embedded in a surface  $S$  is called *contractible* if, when viewed as a closed curve on  $S$ , it is homotopic to a point, otherwise it is called *noncontractible*. A classic result of Heawood and Kempe states that every Eulerian triangulation of the plane is 3-colorable. The condition that the graph be embedded in plane is essential. For example  $K_7$  can be embedded as an Eulerian triangulation of the torus, but has chromatic number 7. However, it is recently proved by Hutchinson, Richter and Seymour [11] that every Eulerian triangulation  $G$  of an orientable surface is 4-colorable if every noncontractible cycle of  $G$  is sufficiently large.

As an effort to understand graphs embedded on orientable surfaces of higher genus, people has studied non-4-colorable graphs embedded in the torus [3, 4, 13]. It turns out that 6-regular triangulations of the torus play special roles. The following result is proved by Thomassen [13].

**Theorem 1 [13]** All 6-regular graphs  $G$  embedded on torus are 5-colorable, except  $G = K_7$  or  $G = T_{11}$ .

A natural question is to characterize 4-colorable 6-regular graphs embedded on torus. A characterization of 6-regular graphs embedded on torus is given by Altshuler [2]. Imagine a rectangular grid of  $m$  rows and  $n$  columns. We label the vertex on the  $i$ th row and  $j$ th column by  $(i, j)$ . The 6-regular right-diagonal grid  $G[m \times n]$  is a graph with vertex set  $V = \{(i, j) : 1 \leq i \leq m, 1 \leq j \leq n\}$ , where the neighbors of  $(i, j)$  are  $(i, j-1), (i, j+1), (i-1, j), (i+1, j), (i+1, j-1), (i-1, j+1)$ . Here the arithmetic in the first coordinate is modulo  $m$  and in the second coordinate modulo  $n$ . It is obvious that  $G[m \times n]$  can be embedded on the torus. See Figure 2 for a 6-regular right-diagonal grid. Suppose  $1 \leq k \leq m$  is an integer. The 6-regular right-diagonal shifted grid  $G[m \times n, k]$  is almost the same as  $G[m \times n]$ , except that there is a rotation before the vertices in the  $n$ th column are joined to the first column. Specifically,  $(i, n)$  is now adjacent to  $(i+k-2, 1)$  and  $(i+k-1, 1)$  as well as to  $(i+1, n), (i+1, n-1), (i, n-1), (i-1, n)$  for  $i = 1, 2, \dots, m$ . See Figure 3 for a 6-regular right-diagonal shifted grid. For any  $m, n$ ,  $G[m \times n]$  is the same as  $G[m \times n, 1]$ .

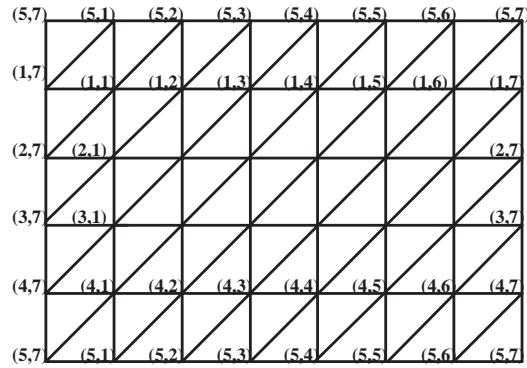


Figure 2. 6-regular right-diagonal grid  $G[5 \times 7]$ .

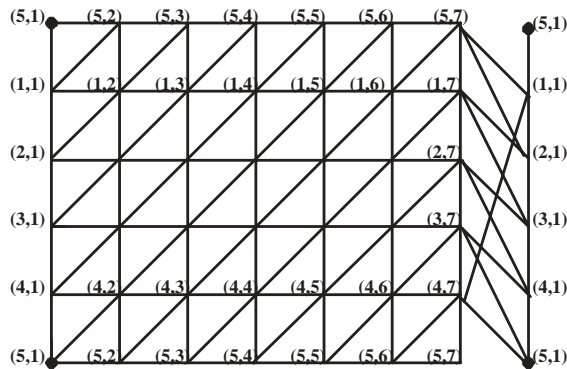


Figure 3. 6-regular right-diagonal shifted grid  $G[5 \times 7, 3]$ .

It is obvious that for any positive integers  $m, n$  and  $1 \leq k \leq m$ ,  $G[m \times n, k]$  is a 6-regular toroidal graph. The following result of Altshuler [2] shows that the converse is true.

**Theorem 2** [2] *Every 6-regular toroidal graph is isomorphic to a 6-regular right-diagonal shifted grid  $G[m \times n, k]$  for some integers  $m, n, k$ .*

Collins, Fisher and Hutchinson [3], Collins and Hutchinson [4] investigated the chromatic numbers of such shifted grids, and they determined the chromatic number of all the  $(m \times n; i)$  shifted grids with  $m \geq n \geq 2$ .

**Theorem 3** [4] *Let  $G$  be a 6-regular right-diagonal  $(m \times n; i)$  shifted grid on the torus, with  $3 \leq m, n$ . Then  $G$  is 4-colorable except  $G \in \{G[3 \times 3, 2], G[3 \times 3, 3], G[5 \times 3, 2], G[5 \times 3, 3], G[5 \times 5, 3], G[5 \times 5, 4]\}$ .*

**Theorem 4** [4] *Let  $G = G[m \times n, k]$  be a 6-regular right-diagonal shifted grid on the torus. If  $m$  is even then  $G$  is 4-colorable. If  $m$  is odd and  $k = 1$  then  $G$  is not 4-colorable. If  $m$  is odd and  $k \geq 2$  then  $G$  is isomorphic to  $G[r \times s, t]$  for some  $r, s, t$  such that  $r, s \neq 2$ ,  $rs = 2m$  and  $1 \leq t \leq r$ , and so its colorability is settled either in Theorem 3 or reduces to the colorability of a shifted grid of the form  $G[m \times 1, i]$ .*

Therefore, to characterize 4-colorable 6-regular toroidal graphs, it remains to consider the shifted grids of form  $G[m \times 1, i]$ .

Let us denote by  $x \pmod{n}$  the unique integer  $t \in \{0, 1, \dots, n-1\}$  such that  $t \equiv x \pmod{n}$  and write  $|x|_n$  for the value of  $\min\{t, n-t\}$ . For any subset  $S$  of  $\{1, 2, \dots, \lfloor n/2 \rfloor\}$ , let  $G_n[S]$  denote the *circulant graph* on the  $n$  vertices  $\{0, 1, \dots, n-1\}$  with  $i$  and  $j$  adjacent whenever  $|i-j|_n \in S$ . The set  $S$  is called the *generating set* of  $G_n[S]$ . Collins, Fisher and Hutchinson [3], Collins and Hutchinson [4] observed that every 6-regular shifted grid  $G[m \times 1, i]$  is a circulant graph with generating set of the form  $S = \{1, r, r+1\}$ . See Figure 4 for a 6-regular right-diagonal  $G[12 \times 1, 5]$  shifted grid which is just the circulant graph  $G_{12}[1, 3, 4]$ .

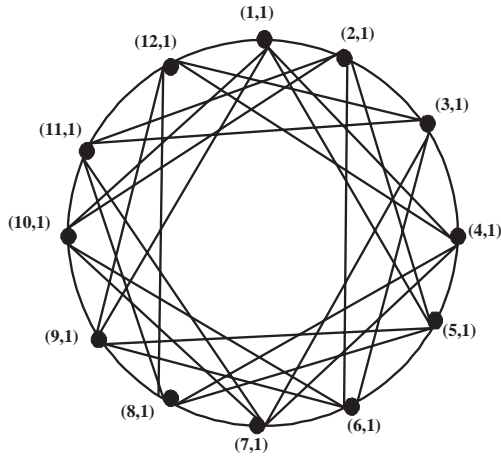


Figure 4. 6-regular right-diagonal shifted grid  
 $G[12 \times 1, 5] = G_{12}[1, 3, 4]$ .

Therefore, to characterize 4-colorable 6-regular toroidal graphs, it remains to determine which circulant graphs  $G_n[S]$  with  $S = \{1, r, r + 1\}$  are 4-colorable.

Based on some theoretical work as well as a computer implementation of a Boolean algebra-based dynamic programming algorithm, Collins, Fisher and Hutchinson [3], Collins and Hutchinson [4] proposed the following conjecture:

**Conjecture 1 [3, 6]** *If  $n \geq 2(r + 1) \geq 6$ , then  $G_n[1, r, r + 1]$  is 4-colorable unless*

1.  $n = 2r + 2, 2r + 3, 3r + 1$  or  $3r + 2$  and is not divisible by 4, or
2.  $r = 2$  and  $n$  is not divisible by 4, or
3.  $(r, n) \in \{(3, 13), (3, 17), (3, 18), (3, 25), (4, 17), (6, 17), (6, 25), (6, 33), (7, 19), (7, 25), (7, 26), (9, 25), (10, 25), (10, 26), (10, 37), (14, 33)\}$ .

Their program [3] verified Conjecture 1 for  $r \leq 11$ , and the theoretical work in [3] proved Conjecture 1 for  $n \geq 10(r + 1)$ .

In this paper, we shall prove Conjecture 1. Combining the results of this paper with results in [4] and [2], we completely characterize 4-colorable 6-regular toroidal graphs.

An embedding of a graph in a surface  $S$  is said to be *evenly embedded* if every region is bounded by an even number of edges. An embedding of a graph with each facial cycle has length four is called a *quadrangulation*.

The *representativity* of a graph  $G$  embedded in a surface  $S$  is the minimum number of points in  $C$  intersect  $G$  taken over all noncontractible  $C$  in  $S$ .

It is obvious that every graph  $G$  evenly embedded in the plane is 2-colorable. Again the condition that  $G$  be embedded in plane is essential. However it is proved by Hutchinson [10] that any graph evenly embedded in an orientable surface is 3-colorable if its representativity is sufficiently high. The result is further generalized in [5] that any graph evenly embedded in an orientable surface has circular chromatic number “close” to 2 if its representativity is sufficiently high (see Section 3 for the definition of circular chromatic number of a graph). Recently, Archdeacon, Hutchinson, Nakamoto, Negami and Ota [1] proved that graphs evenly embedded in torus with representativity at least 9 is 3-colorable, and conjectured that the number 9 can be reduced to 6. For a quadrangulation  $G$  on torus, it is proved in [1] that  $G$  is 3-colorable if and only if there exists “an independent diagonal curve for  $G$  which is essential on the torus” (see [1] for the definitions). However, it seems difficult to determine the existence of such a curve. A condition easier to check is still sought after. Similar to 6-regulation triangulations on torus, 4-regular quadrangulations of the torus play special roles in the study of graphs evenly embedded in torus. Among such graphs, again a class of circulant graphs, those circulant graphs  $G_n[1, r]$  with generating set  $\{1, r\}$  turns out to be difficult to handle. Based on some computational and some theoretical results, Collins and Hutchinson [4] proposed the following conjecture on the chromatic number of circulant graphs with generating set of the form  $S = \{1, r\}$ .

**Conjecture 2** [3, 6] *If  $n \geq 2r \geq 4$ , then  $G_n[1, r]$  is 3-colorable unless*

1.  $r = 2$  and  $n$  is not divisible by 3, or
2.  $n = 2r + 1$  and is not divisible by 3, or
3.  $r = 5$  and  $n = 13$ .

This conjecture has also been verified for  $r \leq 15$  (by a computer implementation of a Boolean algebra-based dynamic programming algorithm [3]) and for  $n \geq 4r$  (by a theoretical proof [4]). We confirm Conjecture 2 in this paper.

To confirm conjectures 1 and 2, we need to find proper colorings of the corresponding circulant graphs. Such colorings are produced by the multiplier method. The multiplier method was introduced in [9], and used in [8, 12], for studying the circular chromatic number of circulant graphs and

distance graphs. By using this method, we completely settle Conjecture 2, and settle Conjecture 1 up to a few exceptions, which settled with some ad hoc methods.

## 2 Preliminaries

Suppose that  $G$  is a graph and  $k \geq 2d \geq 2$ . A  $(k, d)$ -coloring of  $G$  is a coloring  $c$  of the vertices of  $G$  with colors  $\{0, 1, \dots, k-1\}$  such that for any edge  $xy$  of  $G$ , we have  $|c(x) - c(y)|_k \geq d$ , namely  $d \leq |c(x) - c(y)| \leq k - d$ . The *circular chromatic number*  $\chi_c(G)$  of  $G$  is defined as

$$\chi_c(G) = \inf\{k/d : G \text{ has a } (k, d)\text{-coloring}\}.$$

As an example, we have  $\chi_c(C_5) = 5/2$  for 5-cycle  $C_5$ . The circular chromatic number  $\chi_c(G)$  (also called the “*star chromatic number*” and denoted by  $\chi^*(G)$ ) is a refinement of the chromatic number  $\chi(G)$  (see [14] for a survey on the subject). It is known [14] that for any graph  $G$ , we have

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

**Lemma 1** *Let  $G_n[S]$  be a circulant graph and  $d$  a positive integer. If there exists an integer  $j$  such that  $|js|_n \geq d$  for all  $s \in S$ , then  $G_n[S]$  is  $(n, d)$ -colorable, and hence  $\chi_c(G_n[S]) \leq n/d$ .*

**Proof.** Let  $f(i) = ij \pmod{n}$ . Then  $f$  is an  $(n, d)$ -coloring of  $G_n[S]$ . ■

**Corollary 1** *Let  $G_n[S]$  be a circulant graph and  $t$  a positive integer. If there exists an integer  $j$  such that  $|js|_n \geq n/t$  for all  $s \in S$ , then  $G_n[S]$  is  $t$ -colorable.*

**Proof.** By Lemma 1,  $\chi_c(G_n[S]) \leq \frac{n}{n/t} = t$ . As  $t$  is an integer, we have  $\chi(G_n[S]) \leq t$ . ■

So to prove a circulant graph  $G_n[1, r, r+1]$  is 4-colorable, it suffices to find an integer  $j$  such that  $|j|_n, |jr|_n, |j(r+1)|_n \geq n/4$ . To shorten expressions, we introduce the notation

$$\xi_{n,r}(j) = \min\{|j|_n, |jr|_n, |j(r+1)|_n\}.$$

We write  $\xi(j)$  for  $\xi_{n,r}(j)$  if there is no confusion.

**Lemma 2** *Suppose that  $n = 4p + a$  and  $r = 4m + b$  with  $0 \leq a, b \leq 3$ . If there is an integer  $t \leq p + a - 2\lceil a/4 \rceil$  and a nonnegative integer  $\ell$  such that*

$$\begin{aligned} & \left( \frac{16\ell + 4(1-b)}{4t-a} \right) p + \frac{4\ell a + 4\lceil a/4 \rceil - ab}{4t-a} \leq r \\ & \leq \left( \frac{16\ell + 4(2-b)}{4t-a} \right) p + \frac{4\ell a + 4a - 4\lceil a/4 \rceil - 4t - ab}{4t-a}, \end{aligned}$$

*then the circulant graph  $G_n[1, r, r+1]$  is 4-colorable.*

**Proof.** By Lemma 1, it suffices to exhibit an integer  $j$  with  $\xi(j) \geq n/4$ . We claim that  $j = p + t$  is satisfactory. Since  $1 \leq t \leq p + a - 2\lceil a/4 \rceil$ ,  $|j|_n \geq n/4$ .

It follows from the assumption that

$$\begin{aligned} (16\ell + 4(1-b))p + 4\ell a + 4\lceil a/4 \rceil - ab & \leq (4t-a)r \\ & \leq (16\ell + 4(2-b))p + 4\ell a + 4a - 4\lceil a/4 \rceil - 4t - ab, \end{aligned}$$

Since  $m = (r-b)/4$  and  $n = 4p + a$ , easy calculation shows that

$$\ell n + (1-b)p + \lceil a/4 \rceil \leq tr - am \leq \ell n + (2-b)p + a - \lceil a/4 \rceil - t.$$

Observe that

$$\begin{aligned} jr & = (p+t)r = tr + p(4m+b) \\ & = tr - am + bp + m(4p+a) \\ & = tr - am + pb + mn, \end{aligned}$$

so  $jr \equiv tr - am + pb \pmod{n}$  and  $j(r+1) \equiv tr - am + pb + p + t \pmod{n}$ .

The above inequalities imply that

$$\ell n + p + \lceil a/4 \rceil \leq tr - am + pb \leq \ell n + 2p + a - \lceil a/4 \rceil - t.$$

Hence  $|jr|_n \geq n/4$ . Similarly,

$$\ell n + 2p + t + \lceil a/4 \rceil \leq tr - am + pb \leq \ell n + 3p + a - \lceil a/4 \rceil,$$

implying that  $|j(r+1)|_n \geq n/4$ . Thus  $\xi(j) \geq n/4$ , and the proof is complete.  $\blacksquare$

In the remainder of this paper, for integers  $\ell, a, b, p$ , let

$$I_{\ell, t, a, b, p}^L = \left( \frac{16\ell + 4(1-b)}{4t-a} \right) p + \frac{4\ell a + 4\lceil a/4 \rceil - ab}{4t-a}$$

and

$$I_{\ell,t,a,b,p}^R = \left( \frac{16\ell + 4(2-b)}{4t-a} \right) p + \frac{4\ell a + 4a - 4\lceil a/4 \rceil - 4t - ab}{4t-a}.$$

We denote the closed interval  $[I_{\ell,t,a,b,p}^L, I_{\ell,t,a,b,p}^R]$  by  $I_{\ell,t,a,b,p}$ . Lemma 2 can be re-stated as follows:

*Suppose  $n = 4p + a$  and  $r = 4m + b$  with  $0 \leq a, b \leq 3$ . If  $r$  lies in an interval  $I_{\ell,t,a,b,p}$  or some positive integer  $t \leq p + a - 2\lceil a/4 \rceil$  and a nonnegative integer  $\ell$ , then  $G_n[1, r, r + 1]$  is 4-colorable.*

In the sequel, we will omit the subscripts  $a, b$  and  $p$ , and denote the interval  $I_{\ell,t,a,b,p}$  by  $I_{\ell,t}$ , when they are clear from the context. A major part of this paper is devoted to for all  $r$ , there exist integers  $t$  and  $\ell$  such that  $r \in I_{\ell,t}$ .

For  $a < b \in R$ ,  $i = 0, 1, 2, 3$ , we shall denote by  $[a, b]_i$  (respectively  $(a, b)_i$ ) those integers  $t \in [a, b]$  (respectively  $t \in (a, b)$ ) for which  $t \equiv i \pmod{4}$ .

For convenience, we shall write “the intervals  $I_j$  ( $j = 1, 2, \dots, t$ ) overlap each other” to mean that

$$I_1^L \leq I_{j-1}^L \leq I_j^L \leq I_{j-1}^R \leq I_j^R \leq I_t^R \quad \text{for } j = 2, 3, \dots, t-1, t.$$

### 3 The main results

The main results of this paper are the confirmation of Conjectures 1 and 2. For conjecture 1, the main part is to prove it for  $n \geq 303$ . Then with the help of computer, we confirm it for all integer  $n$ .

**Theorem 5** *Conjecture ?? is true.*

The proof of Theorem 5 is left to Section 5. Here we show that the circulant graphs excluded by Conjecture 1 are not 4-colorable. The authors of [4] must have proved this. However, no proof is found in [4]. For the completeness, we include here a short proof.

If  $r = 2$ , then in  $G_n[1, r, r + 1]$ , the set  $\{i, i + 1, i + 2, i + 3\}$  induces a copy of  $K_4$ . Assume  $f$  is a 4-coloring of  $G_n[1, r, r + 1]$ . Then  $f(i) = f(i + 4)$  for all  $i$ . This is possible only if  $n$  is a multiple of  $n$ . If  $n = 2r + 3$ , then  $f(i) = 2i \pmod{n}$  is an isomorphism from  $G_n[1, r, r + 1]$  to  $G_n[1, 2, 3]$ . If  $n = 3r + 1$  or  $3r + 2$ , then the mapping  $f(i) = 3i \pmod{n}$  is an isomorphism

from  $G_n[1, r, r + 1]$  to  $G_n[1, 2, 3]$ . So in all these cases,  $G_n[1, r, r + 1]$  is 4-colorable only if  $n$  is a multiple of 4. If  $n = 2r + 2$ , then for each  $i$ , the set  $I_i = \{i, i + 1, i + r + 1, i + r + 2\}$  induces a copy of  $K_4$  (where the summation in the index is modulo  $n$ ). If  $f$  is a 4-coloring of  $G_n[1, r, r + 1]$ , then by considering the coloring of  $I_i$  and  $I_{i+1}$ , it is easy to see that  $\{f(i), f(i + r + 1)\} = \{f(i + 2), f(i + r + 3)\}$ . This implies that for any integer  $j$ ,  $\{f(i), f(i + r + 1)\} = \{f(i + 2j), f(i + r + 1 + 2j)\}$ . If  $n = 2r + 2$  is not divisible by 4, then  $r$  is even, say  $r = 2k$ . By taking  $j = k + 1$ , we conclude that  $\{f(i), f(i + r + 1)\} = \{f(i + 1), f(i + k + 2)\}$ , which is an obvious contradiction.

For the 16 small graphs excluded in the conjecture, it is easy to check them one by one.

**Theorem 6** *Conjecture 2 is true.*

The proof of the Theorem 6 is left to Section 4. Again we present a short proof showing that the excluded circulant graphs are not 3-colorable. If  $r = 2$ , then  $\{i, i + 1, i + 2\}$  induces a triangle in  $G_n[1, r]$ . It follows that  $G_n[1, r]$  is 3-colorable if and only if  $n$  is a multiple of 3. If  $n = 2r + 1$ , then  $f(i) = 2i \pmod{n}$  is an isomorphism from  $G_n[1, r]$  to  $G_n[1, 2]$ , and hence  $G_n[1, r]$  is 3-colorable if and only if  $n$  is a multiple of 3. The case  $r = 5$  and  $n = 13$  is easily seen to be not 3-colorable.

## 4 Proof of Theorem 6

The proof of both Theorems 5 and 6 use the multiplier method, except for one special case of Theorem 5, which is proved differently. The proof of Theorem 5 involves some tedious calculation. We prove Theorem 6 first.

By Lemma 1, to prove  $G_n[1, r]$  is 3-colorable, it suffices to prove that there is an integer  $j$  such that  $n/3 \leq j \leq 2n/3$  and  $n/3 \leq rj \pmod{n} \leq 2n/3$ .

As all our calculations are modulo  $n$ , it is helpful to think of the set  $\{0, 1, \dots, n - 1\}$  as points lie on a circle  $C^n$  of length  $n$ , with  $i$  and  $i + 1$  unit length apart, and  $n - 1$  and 0 also unit length apart.

Let  $p = \lceil n/3 \rceil$ . Then the sequence

$$(pr \pmod{n}), (p + 1)r \pmod{n}, \dots, (n - p)r \pmod{n})$$

of integers is a sequence of points on  $C^n$ , traversing along the circle  $C^n$  in the increasing direction (say clockwise direction, which we shall call the

forward direction), with each step of length  $r$ , where  $2 \leq r \leq n/2$ . Note that the interval  $[n/3, 2n/3]$  contains  $n - 2p + 1$  consecutive integers, namely,  $\{p, p + 1, \dots, n - p\}$ . We need to show that one of the points in the sequence fall into this interval.

First consider the case that  $r \leq n - 2p + 1$ . Then each step is of length  $\leq n - 2p + 1$ , and the total distance traversed by the sequence is  $(n - 2p)r$ . If  $r \geq 3$ , then  $(n - 2p)r \geq 2n/3$ , and hence one of the points lie in the interval  $[n/3, 2n/3]$  and we are done. If  $r = 2$  and  $n$  is a multiple of 3, then  $(n - 2p)r = 2n/3$  and hence also one of the points lie in the interval  $[n/3, 2n/3]$  and we are done. The case  $r = 2$  and  $n$  is not a multiple of 3 is the case which we know to be 4-chromatic.

Assume now that  $r \geq n - 2p + 2 \geq (n + 2)/3$ . Then  $(2n + 4)/3 \leq 2r \leq n$ . For simplicity, we assume that  $n \geq 35$ . The case  $n \leq 34$  can be checked one by one and is included in the part proved in [3].

Consider two sequences of points on the circle:

$$A = (pr \pmod{n}, (p + 2)r \pmod{n}, (p + 4)r \pmod{n}, \dots, (p + 2t)r \pmod{n}),$$

and

$$B = ((p + 1)r \pmod{n}, (p + 3)r \pmod{n}, (p + 5)r \pmod{n}, \dots, (p + 2t' + 1)r \pmod{n}),$$

where  $t$  is the largest integer such that  $p + 2t \leq n - p$  and  $t'$  is the largest integer such that  $p + 2t' + 1 \leq n - p$ .

The consecutive points in each of the two sequences can be viewed as traversing the circle backward, with step length  $n - 2r \leq p - 1 \leq n - 2p + 1$  (if viewed traversing the circle forward, then the step length is  $2r$ , which is more than  $2/3$  of the circle). Sequence  $A$  contains  $t + 1$  points, so traversed a distance  $t(n - 2r)$ . Note that  $p + 2t \geq n - p - 1$  and hence  $t \geq (n - 2p - 1)/2$ . It is straightforward to verify that if  $n \geq 2r + 5$ , then sequence  $A$  traverses a distance  $\geq 2n/3$  and hence contains a point of the interval  $[n/3, 2n/3]$ .

If  $n = 2r + 3$  or  $2r + 4$ , then each of the sequence traverses a distance  $\geq n/3$  and one of the sequence starts at point  $x_0 = pr \pmod{n}$ , the other starts at  $x_0 + r \pmod{n}$ . As  $r \geq n/2 - 2$ , so  $x_0 + r$  is ‘‘almost’’ opposite to  $x_0$  in the circle. It follows that one of the two sequences contains a point in the interval  $[n/3, 2n/3]$ . If  $n = 2r$ , then depending on  $r$  is even or odd,  $(r - 1)r \pmod{n}$  or  $r^2 \pmod{n}$  is the needed point. If  $n = 2r + 1$ , then we only need to consider the case  $n$  is a multiple of 3, and in this case  $G_n[1, r]$  is isomorphic to  $G_n[1, 2]$  and is easily seen to be 3-colorable. If  $n = 2r + 2$ , then identifying  $i$  and  $i + r + 1$ , we obtain a cycle, which is then 3-colorable. ■

## 5 Proof of Theorem 5

If  $n = 2r + 2$ , then we only need to consider the case that  $r \equiv 1$ , or  $3 \pmod{4}$ . If  $r = 4m + 1$ , then  $\xi(n/4) \geq n/4$  and hence  $G_n[1, r, r + 1]$  is 4-colorable by Lemma 1. If  $r = 4m + 3$ , then we partition the vertex set of  $G_n[1, r, r + 1]$  into four independent sets (color classes)  $D_i = \{i, i + 2, i + 4, \dots, i + 4m + 2\}$ ,  $i = 0, 1, 4m + 4$  and  $4m + 5$ .

The case  $n = 2r + 3$  is excluded from Theorem 5. From now on, we assume that  $n \geq 2r + 4$ . Assume  $n = 4p + a$  and  $r = 4m + b$  for some  $a, b \in [0, 3]$ . In this scenario,  $r$  is contained in the set  $[1, 2p + \lfloor a/2 \rfloor - 2]_b$ . We shall have completed the proof, with the aid of Lemmas 1 and 2, if we can show that  $[1, 2p + \lfloor a/2 \rfloor - 2]_b$  is contained in a union of intervals  $I_{\ell, t}$  with  $t \leq p + a - 2\lceil a/4 \rceil$ , namely  $r$  lies in some interval  $I_{\ell, t}$ .

We divide the proof into a few cases according to the modulo 4 values of  $n$  and  $r$ . We do one case, i.e., the case  $n \equiv 0 \pmod{4}$  and  $r \equiv 3 \pmod{3}$ , in detail. All the remaining cases can be carried out by the readers without difficulties (but with a lot of patience and care).

Suppose that  $n = 4p$  and  $r = 4m + 3$ . In this case  $I_{\ell, t} = [\frac{4\ell-2}{t}p, \frac{4\ell-1}{t}p - 1]$ .

**Claim 1** *If  $p \geq 16$ , then*

$$[3, p - 1] \subseteq \left( \bigcup_{t=3}^{p-3} I_{1, t} \right),$$

$$I_{1, 2} = [p, \frac{3}{2}p - 1],$$

$$I_{3, 7} \subseteq [\frac{3}{2}p - 1, \frac{3}{2}p]$$

and

$$[\frac{3}{2}p, 2p - 3] \subseteq \left( \bigcup_{\ell=2}^{\lfloor (p-3)/2 \rfloor} I_{\ell, 2\ell} \right).$$

**Proof.** If  $t \geq 9$  then  $t \leq p - 3$  implies that

$$I_{1, t}^R - I_{1, t-1}^L = \frac{(t-3)p}{t(t-1)} - 1 \geq \frac{t-9}{t(t-1)} \geq 0.$$

If  $4 \leq t \leq 8$  then  $t + 8 \leq p$  implies that

$$I_{1, t}^R - I_{1, t-1}^L \geq \frac{6t-24}{t(t-1)} \geq 0.$$

Therefore the intervals  $I_{1,t}$  ( $t = p - 3, p - 4, \dots, 3$ ) overlap each other when  $p \geq 16$ . We note that  $I_{1,p-3}^L = \frac{2p}{p-3} \leq 3$  and  $I_{1,3}^R = p - 1$ , which lead to that  $[3, p - 1]$  is covered by the set  $\bigcup_{t=3}^{p-3} I_{1,t}$ . A routine calculation shows that the intervals  $I_{\ell,2\ell}$  ( $t = 2, 3, \dots, \lfloor \frac{p-3}{2} \rfloor$ ) overlap each other when  $p \geq 16$ . Note that  $I_{2,4}^L = \frac{3}{2}p$  and  $I_{\ell,2\ell} \geq 2p - 3$  when  $\ell = \lfloor \frac{p-3}{2} \rfloor$ , which lead to that  $[\frac{3}{2}p, 2p - 3]$  is covered by the set

$$\bigcup_{\ell=2}^{\lfloor (p-3)/2 \rfloor} I_{\ell,2\ell}.$$

These complete the proof of the claim. ■

It follows from Claim 1 that  $[1, 2p - 2]_3$  is contained in a union of intervals  $I_{\ell,t}$  with  $t \leq p$ . Therefore for  $n \equiv 0 \pmod{4}$ , any  $r \equiv 3 \pmod{4}$ , is contained in an appropriate interval  $I_{\ell,t}$ . By Lemma 2, this implies that  $G_n[1, r, r + 1]$  is 4-colorable.

Note that in Claim 1, it is required that  $p \geq 16$ . This is equivalent to require that  $n \geq 64$ . So within the case  $n = 4p$  and  $r = 4m + 3$ , there is still a small number of circulant graphs  $G_n[1, r, r + 1]$  for which Conjecture 1 is not confirmed. We shall discuss these circulant graphs at the very end of the proof.

The following tables explain how the other cases are proved. Let  $n = 4p + a$  and  $r = 4m + b$ . The third column of the table either indicates which intervals  $I_{\ell,t}$  are used to cover the set  $[1, 2p + \lfloor a/2 \rfloor - 2]_b$ , or indicates for which  $j$ ,  $\xi(j) \geq n/4$ . Then it follows from Lemmas 1 or 2 that  $G_n[1, r, r + 1]$  is 4-colorable. Verification of the details in the tables is similar to the proof of Claim 1, and is omitted. For each of these cases, there is also a requirement that  $p$  is bounded below. This means that for each of these cases, a small number of circulant graphs need to be checked separately. This is done at the very end of the proof.

$a$	$b$	$I_{\ell,t}$
0	0	$[1, 2p - 2]_0 \subseteq \left( \bigcup_{t=1}^{p-2} I_{0,t} \right)$ when $p \geq 6$
0	1	$\xi(p) = p \geq n/4$
0	2	$\xi(p) = p \geq n/4$
0	3	$[1, 2p - 2]_3 \subseteq \left( \bigcup_{t=2}^{p-3} I_{1,t} \right) \cup I_{3,7} \cup \left( \bigcup_{\ell=2}^{\lfloor (p-3)/2 \rfloor} I_{\ell,2\ell} \right)$ when $p \geq 16$

Table 1:  $n = 4p$  and  $r = 4m + b$ .

$a$	$b$	$I_{\ell,t}$
1	0	$[1, 2p - 2]_0 - \{ \frac{4p}{3} \} \subseteq \left( \bigcup_{t=2}^{p-3} I_{0,t} \right) \cup \left( \bigcup_{\ell=1}^{\lfloor (p-5)/3 \rfloor} I_{\ell,3\ell+2} \right) \cup I_{0,1}$ when $p \geq 18$
1	1	$[1, 2p - 2]_1 - \{ \frac{4p-1}{3} \} \subseteq I_{0,1} \cup \left( \bigcup_{\ell=1}^{\lfloor (p-4)/3 \rfloor} I_{\ell,3\ell} \right) \cup \left( \bigcup_{\ell=1}^{\lfloor (p-4)/2 \rfloor} I_{\ell,2\ell+1} \right)$ when $p \geq 19$
1	2	$[1, 2p - 2]_2 - \{2\} \subseteq \left( \bigcup_{t=2}^{p-5} I_{1,t} \right) \cup I_{2,7} \cup I_{3,8} \cup \left( \bigcup_{\ell=2}^{\lfloor (p-6)/2 \rfloor} I_{\ell,2\ell+1} \right)$ when $p \geq 48$
1	3	$[1, 2p - 2]_3 \subseteq \left( \bigcup_{t=3}^{p-4} I_{1,t} \right) \cup I_{2,6} \cup \left( \bigcup_{\ell=1}^{\lfloor (p-3)/2 \rfloor} I_{\ell,2\ell} \right)$ when $p \geq 30$

Table 2:  $n = 4p + 1$  and  $r = 4m + b$ .

$a$	$b$	$I_{\ell,t}$
2	0	$[1, 2p - 1]_0 - \{\frac{4p}{3}\} \subseteq \left(\bigcup_{t=2}^{p-2} I_{0,t}\right) \cup \left(\bigcup_{\ell=1}^{\lfloor (p-3)/3 \rfloor} I_{\ell,3\ell+1}\right) \cup \left(\bigcup_{\ell=1}^{\lfloor (p-6)/2 \rfloor} I_{\ell,2\ell+2}\right)$ when $p \geq 14$
2	1	$\xi(p+1) \geq \frac{n}{4}$
2	2	$[1, 2p - 1]_2 - \{2\} \subseteq \left(\bigcup_{t=4}^{p-4} I_{1,t}\right) \cup \left(\bigcup_{\ell=1}^{\lfloor (p-4)/2 \rfloor} I_{\ell,2\ell+1}\right)$ when $p \geq 53$
2	3	$[1, 2p - 1]_3 - \{\frac{4p+1}{3}\} \subseteq \left(\bigcup_{t=3}^{p-3} I_{1,t}\right) \cup \left(\bigcup_{\ell=1}^{\lfloor (p-2)/3 \rfloor} I_{\ell,3\ell}\right) \cup I_{1,2}$ when $p \geq 35$

Table 3:  $n = 4p + 2$  and  $r = 4m + b$ .

$a$	$b$	$I_{\ell,t}$
3	0	$[1, 2p - 1]_0 \subseteq \left(\bigcup_{t=2}^{p-1} I_{0,t}\right) \cup \left(\bigcup_{\ell=0}^{\lfloor (p-3)/2 \rfloor} I_{\ell,2\ell+2}\right)$ when $p \geq 22$
3	1	$\xi(p+1) \geq \frac{n}{4}$
3	2	$[1, 2p - 1]_2 - \{\frac{4p+2}{3}\} \subseteq \left(\bigcup_{t=4}^{p-3} I_{1,t}\right) \cup \left(\bigcup_{\ell=1}^{\lfloor (p-2)/3 \rfloor} I_{\ell,3\ell+1}\right) \cup \left(\bigcup_{\ell=1}^{\lfloor (p-2)/2 \rfloor} I_{\ell,2\ell+1}\right)$ when $p \geq 75$
3	3	$[1, 2p - 1]_3 - \{\frac{4p+1}{3}\} \subseteq \left(\bigcup_{t=3}^{p-2} I_{1,t}\right) \cup \left(\bigcup_{\ell=1}^{\lfloor (p-1)/3 \rfloor} I_{\ell,3\ell-1}\right)$ when $p \geq 65$

Table 4:  $n = 4p + 3$  and  $r = 4m + b$ .

In the tables above, the integer  $p$  is required to be large than some constants. This means that some circulant graphs are not covered by the proof above. By examining the proofs, one can see that the number of graphs not covered by the proof is small. In particular, all circulant graphs  $G_n[1, r, r + 1]$  with  $n \geq 303$  are covered by the proof above. As mention in Section 1, Conjecture has been verified for  $r \leq 11$ . So the circulant graphs for which Conjecture 1 have not been verified are contained in the set  $\{G_n[1, r, r + 1] : r \geq 11, n \leq 303, 2r + 4 \leq n, n \neq 3r + 1, n \neq 3r + 2\}$ . Through a computer search, we found that among this set of circulant graphs  $G_n[1, r, r + 1]$ , there is always an integer  $j$  with  $\xi_{n,r}(j) \geq n/4$ , except  $(r, n) \in \{(12, 30), (14, 33), (14, 41), (18, 45), (18, 49)\}$ . So except for these five graphs, all the other circulant graphs can be 4-colored by the multiplier method. Among the five pairs of  $(r, n)$ , the pair  $(r, n) = (14, 33)$  is an excluded case of Conjecture 1. For each of the other four pairs, a 4-coloring of the graph  $G_n[1, r, r + 1]$  is found by an ad hoc method. This completes the proof of Theorem 5.  $\blacksquare$

## 6 Conclusions

Combining Theorem 5 with results in [4] and [2], we characterize 4-colorable 6-regular toroidal graphs in the following theorem.

**Theorem 7** *All 6-regular toroidal graphs are 4-colorable, with the following exceptions:*

- $G$  is a  $(3 \times 3; 2)$  grid, a  $(3 \times 3; 3)$  grid, a  $(5 \times 3; 2)$  grid, a  $(5 \times 3; 3)$  grid, a  $(5 \times 5; 3)$ , or a  $(5 \times 5; 4)$  grid.
- $G$  is an  $(m \times 2, 1)$  grid with  $m$  odd.
- $G = G_n[1, r, r + 1]$  and  $n = 2r + 2, 2r + 3, 3r + 1$  or  $3r + 2$  and  $n$  is not divisible by 4.
- $G = G_n[1, 2, 3]$  and  $n$  is not divisible by 4.
- $G = G_n[1, r, r + 1]$  and  $(r, n) \in \{(3, 13), (3, 17), (3, 18), (3, 25), (4, 17), (6, 17), (6, 25), (6, 33), (7, 19), (7, 25), (7, 26), (9, 25), (10, 25), (10, 26), (10, 37), (14, 33)\}$ .

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