

# Adaptive choosability of planar graphs with sparse short cycles

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## Abstract

Given a (possibly improper) edge-colouring  $F$  of a graph  $G$ , a vertex colouring of  $G$  is *adapted to  $F$*  if no colour appears at the same time on an edge and on its two endpoints. A graph  $G$  is called *adaptably  $k$ -choosable* (for some positive integer  $k$ ) if for any list assignment  $L$  to the vertices of  $G$ , with  $|L(v)| \geq k$  for all  $v$ , and any edge-colouring  $F$  of  $G$ ,  $G$  admits a colouring  $c$  adapted to  $F$  where  $c(v) \in L(v)$  for all  $v$ . This paper proves that a planar graph  $G$  is adaptably 3-choosable if one of the following holds: (1) Any two triangles in  $G$  do not intersect and no triangle is adjacent to a 4-cycle or a 5-cycle. Moreover, any 6-cycle is adjacent to at most two triangles. (2) Any two triangles in  $G$  have distance at least 2 and no triangle is adjacent to a 4-cycle. (3) Any two triangles in  $G$  have distance at least 2 and no triangle is adjacent to a 5-cycle.

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## 1 Introduction

Let  $G$  be a multigraph (i.e., multiple edges are allowed), and let  $F : E(G) \rightarrow \mathbb{N}$  be a (possibly improper) colouring of the edges of  $G$ . A  $k$ -colouring  $c : V(G) \rightarrow$

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$\{1, \dots, k\}$  of the vertices of  $G$  is *adapted* to  $F$  if for every  $uv \in E(G)$ ,  $c(u) \neq c(v)$  or  $c(v) \neq F(uv)$ . In other words, the same colour never appears on an edge and both its endpoints. If there is an integer  $k$  such that for any edge colouring  $F$  of  $G$ , there exists a vertex  $k$ -colouring of  $G$  adapted to  $F$ , we say that  $G$  is *adaptably  $k$ -colourable*. The smallest  $k$  such that  $G$  is adaptably  $k$ -colourable is called the *adaptable chromatic number* of  $G$ , denoted by  $\chi_{ad}(G)$ .

The concept of adapted colouring of a graph was introduced by Hell and Zhu in [9], and has connections with matrix partition of graphs, graph homomorphisms, and full constraint satisfaction problems [3, 4, 5, 8].

Let  $L : V(G) \rightarrow 2^{\mathbb{N}}$  be a list assignment to the vertices of a graph  $G$ , and  $F$  be a (possibly improper) edge colouring of  $G$ . A colouring  $c$  of  $G$  adapted to  $F$  is an  *$L$ -colouring adapted to  $F$*  if for any vertex  $v \in V(G)$ , we have  $c(v) \in L(v)$ . If for any edge colouring  $F$  of  $G$  and any list assignment  $L$  with  $|L(v)| \geq k$  for all  $v \in V(G)$  there exists an  $L$ -colouring of  $G$  adapted to  $F$ , we say that  $G$  is *adaptably  $k$ -choosable*. The smallest  $k$  such that  $G$  is adaptably  $k$ -choosable is called the *adaptable choice number* of  $G$ , denoted by  $\text{ch}_{ad}(G)$ . The concept of adapted list colouring of graphs and hypergraphs was introduced by Kostochka and Zhu in [10].

Adapted list colouring can be used as a model for scheduling problems. Compared to the original list colouring model, the adapted list colouring allow different constraints for different colours. For example, suppose there is a set of basketball games that need to be scheduled into a set of time slots. The time slots are the colours. The constraints are (1): each game has a list of permissible time slots, and (2): some pairs of games cannot be assigned to the same time slot. This problem is modeled as a list colouring problem. It may happen that two games  $a, b$  cannot be both assigned to time slot  $i$ , however, they can be both assigned to time slot  $j$ . The adapted list colouring of graphs provides a model for this problem.

In ordinary colouring of a graph, parallel edges can be replaced by a single edge. However, for adapted colouring, parallel edges can have different colours and hence cannot be replaced by a single edge. Indeed, if  $G$  is a simple graph, and  $G'$  is obtained from  $G$  by replacing each edge of  $G$  with  $k$  parallel edges with colours  $1, 2, \dots, k$ , then  $G$  is  $k$ -colourable if and only if  $G'$  is adaptably  $k$ -colourable.

In this paper, we shall be interested in adaptable colouring of simple graphs. In the remainder of the paper, all graphs are assumed to be simple.

Since a proper vertex  $k$ -colouring of a graph  $G$  is adapted to any edge colouring of  $G$ , we have  $\chi_{ad}(G) \leq \chi(G)$  and  $\text{ch}_{ad}(G) \leq \text{ch}(G)$  for any graph  $G$ , where  $\chi(G)$  is the usual chromatic number of  $G$ , and  $\text{ch}(G)$  is the usual choice number of  $G$ . In particular, if  $G$  is a planar graph, then  $\chi_{ad}(G) \leq 4$ . This bound turns out to be sharp. It was shown in [9] that there exist simple planar graphs that are not adaptably 3-colourable.

The adaptable choosability of planar graphs was studied in [2]. It was proved in [2] that

- Planar graphs are adaptably 4-choosable.
- Triangle free planar graphs are adaptably 3-choosable.
- There are  $C_4$ -free planar graphs that are not adaptably 3-colourable.
- For any integer  $k \geq 5$ , there are planar graphs that are  $C_t$ -free for all  $5 \leq t \leq k$  and not adaptably 3-colourable.
- For any integer  $k$ , there are planar graphs  $G$  in which any two triangles have distance at least  $k$  and  $G$  is not adaptably 3-choosable.

A natural question is whether some combinations of requirements on triangles, 4-cycles and 5-cycles imply adaptable 3-choosability or adaptable 3-colourability. In 1976, Steinberg conjectured that planar graphs without cycles of length 4 and 5 are 3-colourable. There are many partial results on this problem (see [11] for a survey), however, the conjecture itself is still open. The corresponding question for adaptable choosability and adaptable colourability was asked in [2]: Are planar graphs without 4-cycles and 5-cycles adaptably 3-colourable (or even adaptably 3-choosable)?

In this note we consider planar graphs in which no short cycles are close to each other. Suppose  $A$  and  $B$  are two subgraphs of  $G$ . We say  $A, B$  are *adjacent* if they have an edge in common. We say  $A, B$  are *intersecting* if  $V(A) \cap V(B) \neq \emptyset$ . The distance between  $A$  and  $B$  is defined as  $d_G(A, B) = \min\{d_G(x, y) : x \in V(A), y \in V(B)\}$ .

**Theorem 1** *Suppose  $G$  is a planar graph. Then  $G$  is adaptably 3-choosable if one of the following holds:*

1. *Any two triangles in  $G$  do not intersect and no triangle is adjacent to a 4-cycle or a 5-cycle. Moreover, any 6-cycle is adjacent to at most two triangles.*
2. *Any two triangles in  $G$  have distance at least 2 and no triangle is adjacent to a 4-cycle.*
3. *Any two triangles in  $G$  have distance at least 2 and no triangle is adjacent to a 5-cycle.*

## 2 Proof of Theorem 1

Suppose  $G$  is plane graph and  $C = (v_1, \dots, v_k)$  is the outer face. Let  $L$  be a list assignment of permissible colours to vertices of  $G$ , such that each vertex  $x$  not on  $C$  is assigned a set  $L(x)$  of 3 permissible colours, and each vertex  $v_i$  on  $C$  has either 2 permissible colours or 3 permissible colours. An edge  $e = xy$  on  $C$  is called a *special edge* if  $|L(x)| = |L(y)| = 2$ . Let  $\mathcal{S}$  be the set of special edges of  $G$  (with respect to the list assignment  $L$ ) and let  $\mathcal{T}$  be the set of triangles of  $G$  and let  $\mathcal{C}_i$  be the set of cycles in  $G$  of length  $i$ . We say  $L$  is a *valid list assignment* if one of the following holds:

- If  $A, B \in \mathcal{T} \cup \mathcal{S}$  and  $A \neq B$ , then  $A, B$  do not intersect; if  $A \in \mathcal{T} \cup \mathcal{S}$  and  $B \in \mathcal{C}_4 \cup \mathcal{C}_5$ , then  $A, B$  are not adjacent; if  $A_1, A_2, A_3 \in \mathcal{T} \cup \mathcal{S}$  ( $A_i \neq A_j$  if  $i \neq j$ ) and  $B \in \mathcal{C}_6$ , then there is an  $i \in \{1, 2, 3\}$  such that  $A_i$  is not adjacent to  $B$ .
- If  $A, B \in \mathcal{T} \cup \mathcal{S}$  and  $A \neq B$ , then  $d_G(A, B) \geq 2$ ; if  $A \in \mathcal{T} \cup \mathcal{S}$  and  $B \in \mathcal{C}_4$ , then  $A, B$  are not adjacent.
- If  $A, B \in \mathcal{T} \cup \mathcal{S}$  and  $A \neq B$ , then  $d_G(A, B) \geq 2$ ; if  $A \in \mathcal{T} \cup \mathcal{S}$  and  $B \in \mathcal{C}_5$ , then  $A, B$  are not adjacent; if  $A \in \mathcal{S}$  and  $B \in \mathcal{C}_4$ , then  $A, B$  are not adjacent.

Observe that if a planar graph  $G$  has a valid list assignment then a list assignment which assigns to each vertex 3 permissible colours is a valid list assignment. So  $G$  has a valid list assignment if and only if  $G$  satisfies the condition of Theorem 1.

**Lemma 2.1** *If  $G$  is plane graph and  $L$  is a valid list assignment for  $G$ , then for any edge colouring  $F$  of  $G$ ,  $G$  has an  $L$ -colouring adapted to  $F$ .*

**Proof.** Assume the lemma is not true and  $G$  with edge colouring  $F$  is a minimum counterexample.

First we prove that  $G$  has no separating triangles. Assume to the contrary that  $T = \{x, y, z\}$  is a separating triangle. Let  $G'$  be obtained from  $G$  by deleting the interior of  $T$ . Since  $G$  is a minimum counterexample, there is an  $L$ -colouring  $c$  of  $G'$  adapted to  $F$ . Now we consider the graph  $G'' = G - G'$ . Let  $L'$  be the list assignment defined as follows:

$$L'(v) = \begin{cases} L(v) - \{c(x)\} & \text{if } x \sim v \\ L(v) - \{c(y)\} & \text{if } y \sim v \\ L(v) - \{c(z)\} & \text{if } z \sim v \\ L(v), & \text{otherwise.} \end{cases}$$

Since any two triangles do not intersect, each vertex  $v$  in  $G''$  is adjacent to at most one of  $x, y, z$ . Moreover, since there is an integer  $m \in \{4, 5\}$  such that  $T$  is not adjacent to any  $m$ -cycle, for any edge  $uv$ , if  $u$  is adjacent to one of  $x, y, z$ , then  $v$  is

not adjacent to any of  $x, y, z$ . Therefore the list assignment  $L'$  of  $G''$  has no special edges and hence is a valid list assignment for  $G''$ . Hence  $G''$  has an  $L'$ -colouring  $c'$  adapted to  $F$ . Then the union of  $c$  and  $c'$  is an  $L$ -colouring of  $G$  adapted to  $F$ .

In the following, we assume that  $G$  has no separating triangles.

A *chord* of  $C$  is an edge  $v_i v_j$  such that  $v_i, v_j \in C$  and  $j \neq i + 1 \pmod{k}$ . A *long chord* of  $C$  is a path  $(v_i, x, v_j)$ , connecting two vertices  $v_i$  and  $v_j$  of  $C$ , where  $x \notin C$  and  $|L(v_i)| + |L(v_j)| = 5$  and  $j \neq i + 1 \pmod{k}$ . Let  $\mathcal{P}$  be the set consisting of all long chords, chords, and cut-vertices on  $C$ .

If  $P \in \mathcal{P}$  (i.e.,  $P$  is a chord or a long chord or a cut-vertex on  $C$ ), then  $C - P$  has two components. We denote by  $A_p$  and  $B_p$  the two components of  $C - P$ , and assume that  $|A_p| \leq |B_p|$ . Let  $\mathcal{B}$  be the set of chord  $P$  such that  $|A_P| = 1$ , i.e.,  $P$  is of the form  $v_i v_{i+2}$ . Let  $P^* \in \mathcal{P} \setminus \mathcal{B}$  such that  $|A_{P^*}|$  is the minimum, i.e., for any  $P' \in \mathcal{P} \setminus \mathcal{B}$ ,  $|A(P^*)| \leq |A(P')|$ .

**Claim A** For  $P^*$  chosen as above, one of the following holds:

- $A_{P^*}$  contains a triangle.
- $A_{P^*}$  contains a vertex  $v_t$  with  $|L(v_t)| = 3$  and  $v_t$  is not contained in any chord or long chord of  $C$  and is not a cut-vertex.

Assume  $A_{P^*}$  contains no triangles. We shall show that  $A_{P^*}$  contains a vertex  $v_t$  with  $|L(v_t)| = 3$  and  $v_t$  is not contained in any chord or long chord of  $C$  and is not a cut-vertex.

We first consider the case that  $P^* = (v_i, x, v_j)$  is a long chord. Without loss of generality, we may assume that  $|L(v_i)| = 2$  and  $|L(v_j)| = 3$  and  $A_{P^*} = \{v_{i+1}, v_{i+2}, \dots, v_{j-1}\}$ . If  $|A_{P^*}| = 1$ , then for  $t = i + 1$ ,  $|L(v_t)| = 3$ , for otherwise, the 4-cycle  $(v_i, v_{i+1}, v_{i+2}, x)$  contains a special edge  $v_i v_{i+1}$ , in contrary to our assumption. Assume  $|A_{P^*}| \geq 2$ . Then there is a  $t \in \{i+1, i+2\}$  such that  $|L(v_t)| = 3$ , for otherwise  $v_i v_{i+1}, v_{i+1} v_{i+2}$  are two intersecting special edges, in contrary to our assumption.

If  $v_t$  is a cut-vertex or  $v_t$  is contained in a long chord, then for  $P' = v_t$  (in case  $v_t$  is a cut-vertex) or  $P'$  is a long chord containing  $v_t$ , we have  $|A_{P'}| < |A_{P^*}|$ , in contrary to our choice of  $P^*$ . Thus we assume that  $v_t$  is not a cut-vertex and is not contained in a long chord. If  $v_t$  is not contained in a chord, then we are done. Assume to the contrary that  $v_t$  is contained in a chord  $P'$ . Since  $A_{P'} \subset A_{P^*}$ , which implies that  $|A_{P'}| < |A_{P^*}|$ , by the choice of  $P^*$ , we conclude that  $P' \in \mathcal{B}$ , i.e.,  $|A_{P'}| = 1$ . Assume  $P' = v_t v_{t+2}$ . Then  $T = \{v_t, v_{t+1}, v_{t+2}\}$  induces a triangle. Since no triangle is contained in  $A_{P^*}$ , we conclude that  $T$  contains a vertex which is not in  $A_{P^*}$ . This is possible only if  $\{v_t, v_{t+2}\} \cap \{v_i, v_j\} \neq \emptyset$ . If  $v_t = v_i$ , then since  $|L(v_i)| = 2$  we must have  $|L(v_{i+1})| = 3$ , for otherwise, the special edge  $v_i v_{i+1}$  intersect the triangle  $T$ , in

contrary to our assumption. It is obvious that  $v_{i+1}$  is not contained in a chord or a long chord and is not cut-vertex.

Assume  $v_{t+2} = v_j$ . Then  $v_t \neq v_{i+1}$ , for otherwise  $G$  has a 4-cycle  $(v_i, v_{i+1}, v_{i+3}, x)$ , which is adjacent to the triangle  $T$  and  $G$  also has a 5-cycle  $(v_i, v_{i+1}, v_{i+2}, v_{i+3}, x)$ , which is adjacent to the triangle  $T$ . If  $t \neq i+2$ , then there is a vertex  $v_{t'} \in \{v_{i+1}, v_{i+2}\}$  such that  $|L(v_{t'})| = 3$ , for otherwise, there are two intersecting special edges  $v_i v_{i+1}$  and  $v_{i+1} v_{i+2}$ . If  $t = i+2$ , then for  $t' = i+1$ ,  $|L(v_{t'})| = 3$ , for otherwise the triangle  $T$  has distance 1 to the special edge  $v_i v_{i+1}$  and is adjacent to the 5-cycle  $(v_i, v_{i+1}, v_{i+2}, v_j, x)$ . It is obvious that  $v_{t'}$  is not a cut-vertex, not contained in a chord or a long chord.

Next we consider the case that  $P^* = v_i v_j$  is a chord and  $A_{P^*} = \{v_{i+1}, v_{i+2}, \dots, v_{j-1}\}$ . Since  $P^* \notin \mathcal{B}$ , we know that  $|A_{P^*}| \geq 2$ . If  $|A_{P^*}| = 2$ , then there is a  $t \in \{i+1, i+2\}$  such that  $|L(v_t)| = 3$ , for otherwise,  $(v_i, v_{i+1}, v_{i+2}, v_{i+3})$  is a 4-cycle which contains a special edge. If  $|A_{P^*}| \geq 3$ , then there exist  $t \in \{i+1, i+2, i+3\}$  such that  $|L(v_t)| = 3$ , for otherwise  $v_{i+1} v_{i+2}, v_{i+2} v_{i+3}$  are two intersecting special edges, in contrary to our assumption. The same argument as above shows that  $v_t$  is not contained in a long chord and is not a cut-vertex. Assume  $v_t$  is contained in a chord. Then the same argument as above shows that the chord is of the form  $v_i v_{i+2}$  or  $v_{j-2} v_j$ . Without loss of generality, assume that  $v_i v_{i+2}$  is a chord and  $v_t = v_{i+2}$ . Let  $T$  be the triangle induced by  $\{v_i, v_{i+1}, v_{i+2}\}$ .

Assume first that for any  $A, B \in \mathcal{S} \cup \mathcal{T}$ ,  $d_G(A, B) \geq 2$ . Then  $v_{j-2} v_j$  is not a chord, for otherwise, let  $T' = \{v_{j-2}, v_{j-1}, v_j\}$ , then the two triangles  $T$  and  $T'$  have distance at most 1. Since there is an integer  $m \in \{4, 5\}$  such that  $T$  is not adjacent to any  $m$ -cycle, it follows that  $v_{i+3}, v_{i+4} \in A_{P^*}$ . Then there is a  $t' \in \{i+3, i+4\}$  such that  $|L(v_{t'})| = 3$ , for otherwise the special edge  $v_{i+3} v_{i+4}$  has distance 1 to the triangle  $T$ . Since  $v_{j-2} v_j$  is not a chord, the argument as above shows that  $v_{t'}$  is not a cut-vertex, is not contained in a chord or a long chord.

Assume that some  $A, B \in \mathcal{S} \cup \mathcal{T}$  have distance less than 2. Then as  $L$  is a valid list assignment, we know the following hold:

- For  $A, B \in \mathcal{T} \cup \mathcal{S}$  and  $A \neq B$ ,  $V(A) \cap V(B) = \emptyset$ .
- For  $A \in \mathcal{T} \cup \mathcal{S}$  and  $B \in \mathcal{C}_4 \cup \mathcal{C}_5$ ,  $A, B$  are not adjacent.
- For  $A_1, A_2, A_3 \in \mathcal{T} \cup \mathcal{S}$  ( $A_i \neq A_j$  if  $i \neq j$ ) and  $B \in \mathcal{C}_6$ , there is an  $i \in \{1, 2, 3\}$  such that  $A_i$  is not adjacent to  $B$ .

Since the triangle  $T$  is not adjacent to any 4-cycle or 5-cycle, we conclude that  $|A_{P^*}| \geq 5$ . There is a  $t' \in \{i+3, i+4, i+5\}$  such that  $|L(v_{t'})| = 3$ , for otherwise the special edges  $v_{i+3} v_{i+4}, v_{i+4} v_{i+5}$  intersect. The argument as above shows that  $v_{t'}$  is not a cut-vertex and is not contained in a long chord. Assume  $v_{t'}$  is contained in a chord. By

the minimality of  $A_{P^*}$ , we know that the chord is of the form  $v_{t'}v_{t'+2}$  and  $t' + 2 = j$ . This is in contrary to the assumption that  $L$  is a valid assignment, because

- If  $t' = i + 3$ , then  $(v_i, v_{i+2}, v_{i+3}, v_j)$  is 4-cycle adjacent to a triangle.
- If  $t' = i + 4$ , then  $(v_i, v_{i+2}, v_{i+3}, v_{i+4}, v_j)$  is 5-cycle adjacent to a triangle.
- If  $t' = i + 5$ , then  $(v_i, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_j)$  is 6-cycle adjacent to two triangles and a special edge.

Finally we consider the case that  $P^* = v_i$  is a cut-vertex. If  $|A_{P^*}| = 2$ , then  $\{v_i, v_{i+1}, v_{i+2}\}$  induces a triangle. Hence there is a  $t \in \{i + 1, i + 2\}$  such that  $|L(v_t)| = 3$ , and  $v_t$  is not a cut-vertex, not contained in a chord or a long chord. Assume  $|A_{P^*}| \geq 3$ . Then there exist  $t \in \{i + 1, i + 2, i + 3\}$  such that  $|L(v_t)| = 3$ . The same argument as above shows that  $v_t$  is not a cut-vertex, and is not contained in a long chord. Moreover, if  $v_t$  is contained in a chord, then by symmetry, we may assume that the chord is  $v_i v_{i+2}$ . In this case,  $|A_{P^*}| \geq 5$ , for otherwise,  $G$  have either two intersecting triangles or the triangle  $T$  induced by  $\{v_i, v_{i+1}, v_{i+2}\}$  is adjacent to a 4-cycle and a 5-cycle, in contrary to our assumption. Then there is a  $t' \in \{i + 3, i + 4\}$  such that  $|L(v_{t'})| = 3$ , and  $v_{t'}$  is not a cut-vertex, not contained in a chord or a long chord. This completes the proof of Claim A.  $\blacksquare$

**Case 1**  $A_{P^*}$  contains a triangle.

Since the triangle is not separating, the three vertices of the triangle are consecutive vertices of  $C$ , say the triangle is  $T = \{v_k, v_{k+1}, v_{k+2}\}$ , and moreover,  $v_{k+1}$  has degree 2. If  $|L(v_{k+1})| = 3$ , we can color  $G - v_{k+1}$  by induction, then color  $v_{k+1}$ . Assume  $|L(v_{k+1})| = 2$ . Then  $|L(v_k)| = |L(v_{k+2})| = 3$ , otherwise the distances between special edges will less than 3.

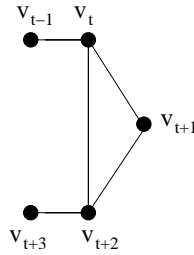


Figure 1:  $A_{P^*}$  contains a triangle

Choose a color  $\alpha \in L(v_k)$  such that  $\alpha \neq F(v_k v_{k-1}), F(v_k v_{k+2})$ , a color  $\gamma \in L(v_{k+2})$  such that  $\gamma \neq F(v_{k+2} v_{k+3}), F(v_{k+2} v_{k+1})$ , and a color  $\beta \in L(v_{k+1})$  such that  $\beta \neq F(v_{k+1} v_k)$ . Observe that  $v_k$  and  $v_{k+2}$  are not contained in any chord or long chord of  $C$ , otherwise it will contradict the minimality of  $A_{P^*}$ .

Let  $G^* = G - \{v_k, v_{k+1}, v_{k+2}\}$ . Let  $L'$  be the list assignment of  $G^*$  obtained from  $L$  as follows:

$$L'(x) = \begin{cases} L(x) - \{\alpha\}, & \text{if } x \sim v_k \text{ and } x \neq v_{k-1}, \\ L(x) - \{\gamma\}, & \text{if } x \sim v_{k+2} \text{ and } x \neq v_{k+3}, \\ L(x), & \text{otherwise.} \end{cases}$$

**Claim B** If  $e = xy$  is a special edge of  $G^*$ , then  $e$  is a special edge of  $G$ .

Assume to the contrary that  $e$  is not a special edge of  $G$ . Since  $v_k, v_{k+2}$  are not contained in any triangle other than  $T$ , each of  $v_k, v_{k+2}$  is adjacent to at most one of  $x, y$ . Since there is an integer  $m \in \{4, 5\}$  such that  $T$  is not adjacent to any  $m$ -cycles, at most one of  $x, y$  is adjacent to a vertex in  $v_k, v_{k+2}$ . As  $e$  is a special edge of  $G^*$  and not a special edge of  $G$ , we conclude that exactly one of  $x, y$  is adjacent to one of  $v_k, v_{k+2}$ , and the other vertex of  $x, y$  has only 2 permissible colours in the list assignment  $L$ . By symmetry, we may assume that  $x$  is adjacent to  $v_k$  and  $|L(y)| = 2$ . If  $x$  is a vertex of  $C$ , then  $P = v_kx$  is a chord of  $C$ . By the choice of  $P^*$ , we conclude that  $P \in \mathcal{B}$ . Hence  $|A_P| = 1$  and  $A_P \cup P$  induces a triangle  $T'$ . But  $T$  and  $T'$  intersect each other, in contrary to our assumption.

Assume  $x$  is not a vertex of  $C$ . Then  $P' = v_kxy$  is a long chord of  $C$ , and  $A_{P'}$  and is contained  $A_{P^*}$ , in contrary to the choice of  $P^*$ . This completes the proof of Claim B.  $\blacksquare$

It follows from the claim above that  $L'$  is valid list assignment of  $G^*$ . Therefore there is an  $L'$ -colouring  $c$  of  $G^*$  adapted to  $F$ . By letting  $c(v_k) = \alpha$ ,  $c(v_{k+1}) = \beta$  and  $c(v_{k+2}) = \gamma$ , we obtain an  $L$ -colouring of  $G$  adapted to  $F$ .

**Case 2**  $A_{P^*}$  contains no triangle.

By Claim A, there is a vertex  $v_t \in A_{P^*}$  such that  $|L(v_t)| = 3$  and  $v_t$  is not a cut-vertex, not contained in a chord or a long chord. Let  $\alpha \in L(v_t)$  such that  $\alpha$  is not equal to the colours of the edges  $v_{t-1}v_t$  and  $v_tv_{t+1}$ .

Let  $G^* = G - v_t$  and let  $\alpha \in L(v_k)$  be a colour such that  $\alpha \neq F(v_{t-1}v_t), F(v_tv_{t+1})$ . Let  $L'$  be the list assignment of  $G^*$  obtained from  $L$  as follows:

$$L'(x) = \begin{cases} L(x) - \{\alpha\}, & \text{if } x \sim v_t \text{ and } x \neq v_{t-1}, v_{t+1}, \\ L(x), & \text{otherwise.} \end{cases}$$

Since  $v_t$  is not contained in a long chord of  $G$ , it follows that if  $e = xy$  is a special edge of  $G^*$ , then either  $e$  is a special edge of  $G$  or  $\{x, y, v_t\}$  induces a triangle. This implies that  $L'$  is a valid list assignment of  $G^*$ . By induction hypothesis  $G^*$  has an  $L'$ -colouring  $c$  adapted to  $F$ . We can extend  $c$  to an  $L$ -colouring of  $G$  adapted to  $F$  by colouring  $v_t$  with colour  $\alpha$ .  $\blacksquare$

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