An upper bound on adaptable choosability of graphs

Mickaël Montassier, André Raspaud
Université Bordeaux I
LaBRI UMR CNRS 5800
33405 Talence Cedex, France
{montassi, raspaud@labri.fr}

Xuding Zhu
National Sun Yat-sen University
Kaohsiung, Taiwan
National Center for Theoretical Sciences
{zhu@math.nsysu.edu.tw}

Abstract

Given a (possibly improper) edge-colouring $F$ of a graph $G$, a vertex colouring $c$ of $G$ is adapted to $F$ if no colour appears at the same time on an edge and on its two endpoints. If for some integer $k$, a graph $G$ is such that given any list assignment $L$ of $G$, with $|L(v)| \geq k$ for all $v$, and any edge-colouring $F$ of $G$, there exists a vertex colouring $c$ of $G$ adapted to $F$ such that $c(v) \in L(v)$ for all $v$, then $G$ is said to be adaptably $k$-choosable. The smallest $k$ such that $G$ is adaptably $k$-choosable is called the adaptable choice number and is denoted by $ch_{ad}(G)$. This note proves that $ch_{ad}(G) \leq \lceil \text{Mad}(G)/2 \rceil + 1$, where $\text{Mad}(G)$ is the maximum of $2|E(H)|/|V(H)|$ over all subgraphs $H$ of $G$. As a consequence, we give bounds for classes of graphs embeddable into surfaces of non-negative Euler characteristics.

Keywords: Adapted colouring, list colouring, planar graphs.

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1 Introduction

Suppose $G$ is a multigraph and let $F : E(G) \rightarrow \mathbb{N}$ be a (possibly improper) colouring of the edges of $G$. A $k$-colouring $c : V(G) \rightarrow \{1, \ldots, k\}$ of the vertices of $G$ is adapted to $F$ if for every $uv \in E(G)$, $c(u) \neq c(v)$ or $c(v) \neq F(uv)$. In other words, there is no monochromatic edge i.e. an edge whose two ends are coloured with the same colour as the edge itself. If there is an integer $k$ such that for any edge colouring $F$ of $G$, there exists a vertex $k$-colouring of $G$ adapted to $F$, we say that $G$ is adaptably $k$-colourable. The smallest $k$ such that $G$ is adaptably $k$-colourable is called the adaptable chromatic number of $G$ and is denoted by $\chi_{ad}(G)$. The concept of adapted colouring of a graph was introduced by Hell and Zhu in [10], and has connections with matrix partitions of graphs, graph homomorphisms, and full constraint satisfaction problems [4, 5, 6].

Let $L : V(G) \rightarrow 2^{\mathbb{N}}$ be a list assignment that assigns to each vertex $v$ of $G$ a set $L(v)$ of permissible colours. Let $F$ be a (possibly improper) edge colouring of $G$. A vertex colouring $c$ of $G$ adapted to $F$ is an $L$-colouring adapted to $F$ if for any vertex $v \in V(G)$, we have $c(v) \in L(v)$. If for any edge colouring $F$ of $G$ and any list assignment $L$ with $|L(v)| \geq k$ for all $v \in V(G)$ there exists an $L$-colouring of $G$ adapted to $F$, we say that $G$ is adaptably $k$-choosable. The smallest $k$ such that $G$ is adaptably $k$-choosable is called the adaptable choice number (or the adaptable choosability) of $G$ and is denoted by $ch_{ad}(G)$. The concept of adapted list colouring of graphs and hypergraphs was introduced by Kostochka and Zhu in [11].

Adapted list colouring can be used as a model for scheduling problems. Compared to the original list colouring model, the adapted list colouring allows different constraints for different colours. For example, suppose there is a set of basketball games that need to be scheduled into a set of time slots. The time slots are the colours. The constraints are (1): each game has a list of permissible time slots, and (2): some pairs of games cannot be assigned to the same time slot. This problem is modeled as a list colouring problem. It may happen that two games $a, b$ cannot be both assigned to time slot $i$, however, they can be both assigned to time slot $j$. The adapted list colouring of graphs provides a model for this problem.

Since a proper vertex $k$-colouring of a graph $G$ is adapted to any edge colouring of $G$, we have $\chi_{ad}(G) \leq \chi(G)$ and $ch_{ad}(G) \leq ch(G)$ for any graph $G$, where $\chi(G)$ is the usual chromatic number of $G$, and $ch(G)$ is the usual choice number of $G$.

The adaptable choosability of planar graphs was studied in [3, 8]. It is known that planar graphs are adaptably 4-choosable. Moreover, a planar graph $G$ is adaptably 3-choosable if one of the following holds:

1. $G$ is triangle-free.
2. No two triangles intersect, and no triangle is adjacent to a 5-cycle, and each...
6-cycle is adjacent to at most two triangles.

3. Any two triangles have distance at least 2 and no triangle is adjacent to a 4-cycle.

On the other hand, there are $C_4$-free planar graphs that are not adaptably 3-colourable; and for any integer $k \geq 5$, there are planar graphs that are $C_t$-free for all $5 \leq t \leq k$ and not adaptably 3-colourable; and for any integer $k$, there are planar graphs $G$ in which any two triangles have distance at least $k$ and $G$ is not adaptably 3-choosable.

In this note we give a new upper bound for the adaptable choice number of graphs. Given a graph $G$, the maximum average degree of $G$, denoted by $\text{Mad}(G)$, is the maximum average degree of the subgraphs of $G$, i.e.,

$$\text{Mad}(G) = \max\{2|E(H)|/|V(H)| : H \text{ is a subgraph of } G\}.$$ 

We shall prove that for any graph $G$, its adaptable choice number is at most $\lceil \text{Mad}(G)/2 \rceil + 1$.

We denote by $S_h$ the orientable surface of genus $h$, i.e., the surface obtained from the sphere by adding $h$ handles, and denote by $N_h$ the non-orientable surface of genus $h$, i.e., the surface obtained from the sphere by adding $h$ crosscaps. The Euler characteristic $\chi(S)$ of a surface $S$ is defined as

$$\chi(S) = \begin{cases} 
2 - 2h, & \text{if } S = S_h, \\
2 - h, & \text{if } S = N_h.
\end{cases}$$

Two cycles $C_1$ and $C_2$ in a graph $G$ are said to be adjacent if they have at least one edge in common. As a consequence of the above upper bound for $\text{ch}_{\text{ad}}(G)$, we shall show that if $G$ is a simple graph which can be embedded in a surface $S$ of non-negative Euler characteristic, then $G$ is adaptably 4-choosable. Moreover, if $G$ is simple, embedded in a surface of non-negative Euler characteristic and no triangle of $G$ is adjacent to a triangle or a $C_4$, and each $C_5$ is adjacent to at most three triangles, then $G$ is adaptably 3-choosable.

In 1976, Steinberg [12] conjectured that planar graphs without cycles of length 4 and 5 are 3-colourable. This conjecture remains unsolved. The corresponding question for adaptable choosability and adaptable colourability was asked in [3]: Are simple planar graphs without 4-cycles and 5-cycles adaptably 3-colourable (or even adaptably 3-choosable)? This question is answered in positive, because if a planar graph $G$ has no 4-cycle and no 5-cycle, then no two adjacent triangles of $G$ are adjacent, and no 5-cycle is adjacent to more than three triangles, and hence $G$ is adaptably 3-choosable.

Finally we give a new proof of the fact that every $K_5$-minor free graph is adaptably 4-choosable [3] based on the relationship between adaptable choice number and maximum average degree.
2 Upper bounds for $ch_{ad}(G)$

**Theorem 2.1** For any graph $G$ (parallel edges are allowed),

$$ch_{ad}(G) \leq \lceil \text{Mad}(G)/2 \rceil + 1.$$ 

**Proof.** To prove this Theorem we will use the following result of Hakimi [9]. A graph $G$ on vertices $x_1, x_2, \cdots, x_n$ has an orientation in which $x_i$ has out-degree $d^+(x_i) = k_i$ if and only if the following hold:

1. For each subset $X$ of $V(G)$, $\sum_{x_i \in X} k_i \geq |E(G[X])|.$
2. $\sum_{i=1}^{n} k_i = |E(G)|.$

An easy consequence of this result is that if for each subset $X$ of $V(G)$, $\sum_{x_i \in X} k_i \geq |E(G[X])|$, then $G$ has an orientation in which $d^+(x_i) \leq k_i$ for each $x_i$ (see also [7]).

If $\text{Mad}(G) \leq k$ for an integer $k$, then for any subgraph $H$ of $G$, $|E(H)| \leq \frac{k}{2}|V(H)|.$ It follows from the above result that $G$ has an orientation in which each vertex $x_i$ has $d^+(x_i) \leq \lceil \frac{k}{2} \rceil$. Assume each vertex $x_i$ is given a list $L(x_i)$ of $\lceil \frac{k}{2} \rceil + 1$ colours and $F$ is an edge colouring of $G$. Let $c(x_i)$ be any colour in $L(x_i)$ which does not appear in any outgoing edges of $x_i$. Then it is obvious that $c$ is an $L$-colouring of $G$ adapted to $F$. This completes the proof of Theorem 2.1. 

**Corollary 2.1** If $G$ is a simple graph which can be embedded in a surface $S$ of non-negative Euler characteristic, then $G$ is adaptably 4-choosable. If, moreover, no triangle of $G$ is adjacent to a triangle or a $C_4$, and each $C_5$ is adjacent to at most three triangles, then $G$ is adaptably 3-choosable.

**Proof.** Assume $G$ is a simple graph embedded in a surface $S$ of Euler characteristic $\chi(S) \geq 0$. Let $H$ be a subgraph of $G$. Then $H$ is also a simple graph embedded in $S$. Let $V, F, E$ be the sets of vertices, faces and edges of $H$, respectively. By Euler’s formula,

$$|V| + |F| - |E| = \chi(S) \geq 0.$$ 

Let $f_i$ be the number of $i$-faces, i.e., faces whose boundary is a walk of length $i$. Since $H$ is simple, each face is an $i$-face for some $i \geq 3$. Therefore

$$3|F| \leq \sum_{i \geq 3} i \cdot f_i = 2|E|.$$ 

It follows that

$$|E| \leq 3|V|.$$
Hence $Mad(G) \leq 6$, and by Theorem 2.1, $G$ is adaptably 4-choosable.

Assume moreover that no triangle in $G$ is adjacent to a triangle or a $C_4$, and each $C_5$ is adjacent to at most three triangles. Then each 3-face of $H$ is adjacent to three faces of degree at least 5. Each 5-face is adjacent to at most three 3-faces, and for $i \geq 6$, each $i$-face is adjacent to at most $i$ 3-faces. Therefore

$$3f_3 \leq 3f_5 + \sum_{i \geq 6} i \cdot f_i.$$ 

It follows that

$$4|F| = 3f_3 + 4f_4 + 5f_5 + (f_3 - f_5) + \sum_{i \geq 6} f_i$$

$$\leq 3f_3 + 4f_4 + 5f_5 + \sum_{i \geq 6} \left( \frac{i}{3} + 4 \right) f_i$$

$$\leq 2|E|.$$ 

By Euler formula, $|V| + |F| - |E| = \chi(S) \geq 0$. By replacing $|F|$ with $|E|/2$, we obtain the inequality that $|E| \leq 2|V|$. So $Mad(G) \leq 4$. By Theorem 2.1, $ch_{ad}(G) \leq 3$. \hfill \blacksquare

The following result was proved in [3].

**Corollary 2.2** Every $K_5$-minor free simple graph is adaptably 4-choosable.

**Proof.** It suffices to prove that any maximal $K_5$-minor free graph $G$ has $|E(G)| \leq 3|V(G)| - 6$. It is known that a maximal $K_5$-minor free graph is constructed recursively, by pasting along $K_2$’s and $K_3$’s, from plane triangulations and copies of the Wagner’s graph (the graph obtained from $C_8$ by adding four diagonal edges). Assume $G$ is obtained from the union of $G_1, G_2$ by pasting along a $K_2$ or $K_3$, and $|E(G_1)| \leq 3|V(G_1)| - 6$. Then $|E(G)| = |E(G_1)| + |E(G_2)| - t$, where $t = 1$ or 3, respectively, and $|V(G)| = |V(G_1)| + |V(G_2)| - s$, where $s = 2$ or 3, respectively. Now a straightforward calculation shows that $|E(G)| \leq 3|V(G)| - 6$. \hfill \blacksquare

Corollaries 2.1 and 2.2 show that the upper bound for $ch_{ad}(G)$ in Theorem 2.1 is very useful. In fact for graphs embedded in surface of non-negative Euler characteristic, the upper bounds for $ch_{ad}(G)$ in Corollary 2.1 are sharp. Theorem 2.1 is also sharp in the sense that for any integer $g, d$, there are $d$-regular graphs $G$ of girth at least $g$ for which $ch_{ad}(G) = \chi_{ad}(G) = d + 1$ [11]. However, for random graphs, the upper bound given in Theorem 2.1 is usually far from sharp. As an example, we consider random $d$-regular graphs $G$, which have $Mad(G) = d$. Let $k_d$ be the smallest integer $k$ such that $d < 2k \log k$. It is known that with high probability, $\chi(G) = k_d$ or $k_d + 1$ or $k_d + 2$ [1], and that $ch_{ad}(G) \leq \sqrt{8d}$ [11]. It is likely that for most graphs, $ch_{ad}(G)$ is much less than $ch(G)$ and $Mad(G)/2$. Question 2.1 below concerns the adaptable chromatic number of graphs. It was asked in [10] and remains open.
**Question 2.1** Let $f(n) = \min\{\chi_{ad}(G) : \chi(G) = n\}$. Is it true that $f(n) = \chi_{ad}(K_n)$? Is it true that $\lim_{n \to \infty} f(n) = \infty$? If so, what is the order of $f(n)$?

Similar questions can be asked for adaptable choosability of graphs.

**Question 2.2** Let $g(n) = \min\{ch_{ad}(G) : ch(G) = n\}$. Is it true that $\lim_{n \to \infty} g(n) = \infty$? If so, what is the order of $g(n)$?

It follows from a result of Alon [2] that there is a function $h(d)$ goes to infinity with $d$ such that if $Mad(G) \geq d$ then $ch(G) \geq h(d)$.

**Question 2.3** Let $\phi(t) = \min\{ch_{ad}(G) : Mad(G) = t\}$. Is it true that $\lim_{t \to \infty} \phi(t) = \infty$?

**References**


