An upper bound on adaptable choosability of graphs*

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Abstract

Given a (possibly improper) edge-colouring F of a graph G, a vertex colouring c of G is *adapted to* F if no colour appears at the same time on an edge and on its two endpoints. If for some integer k, a graph G is such that given any list assignment L of G, with $|L(v)| \geq k$ for all v, and any edge-colouring F of G, there exists a vertex colouring c of G adapted to F such that $c(v) \in L(v)$ for all v, then G is said to be *adaptably k-choosable*. The smallest k such that G is adaptably k-choosable is called the *adaptable choice number* and is denoted by $ch_{ad}(G)$. This note proves that $ch_{ad}(G) \leq \lceil Mad(G)/2 \rceil + 1$, where $Mad(G)$ is the maximum of $2|E(H)|/|V(H)|$ over all subgraphs H of G. As a consequence, we give bounds for classes of graphs embeddable into surfaces of non-negative Euler characteristics.

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1 Introduction

Suppose G is a multigraph and let $F : E(G) \to \mathbb{N}$ be a (possibly improper) colouring of the edges of G. A k-colouring $c : V(G) \to \{1, \ldots, k\}$ of the vertices of G is *adapted* to F if for every $uv \in E(G)$, $c(u) \neq c(v)$ or $c(v) \neq F(uv)$. In other words, there is no monochromatic edge i.e. an edge whose two ends are coloured with the same colour as the edge itself. If there is an integer k such that for any edge colouring F of G, there exists a vertex k-colouring of G adapted to F, we say that G is *adaptably k-colourable.* The smallest k such that G is adaptably k-colourable is called the *adaptable chromatic number* of G and is denoted by $\chi_{ad}(G)$. The concept of adapted colouring of a graph was introduced by Hell and Zhu in [10], and has connections with matrix partitions of graphs, graph homomorphisms, and full constraint satisfaction problems [4, 5, 6].

Let $L : V(G) \to 2^{\mathbb{N}}$ be a list assignment that assigns to each vertex v of G a set $L(v)$ of permissible colours. Let F be a (possibly improper) edge colouring of G. A vertex colouring c of G adapted to F is an *L-colouring adapted to F* if for any vertex $v \in V(G)$, we have $c(v) \in L(v)$. If for any edge colouring F of G and any list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$ there exists an L-colouring of G adapted to F, we say that G is *adaptably k-choosable.* The smallest k such that G is adaptably k-choosable is called the *adaptable choice number* (or the *adaptable choosability*) of G and is denoted by $ch_{ad}(G)$. The concept of adapted list colouring of graphs and hypergraphs was introduced by Kostochka and Zhu in [11].

Adapted list colouring can be used as a model for scheduling problems. Compared to the original list colouring model, the adapted list colouring allows different constraints for different colours. For example, suppose there is a set of basketball games that need to be scheduled into a set of time slots. The time slots are the colours. The constraints are (1): each game has a list of permissible time slots, and (2): some pairs of games cannot be assigned to the same time slot. This problem is modeled as a list colouring problem. It may happen that two games a, b cannot be both assigned to time slot i, however, they can be both assigned to time slot j. The adapted list colouring of graphs provides a model for this problem.

Since a proper vertex k-colouring of a graph G is adapted to any edge colouring of G, we have $\chi_{ad}(G) \leq \chi(G)$ and $ch_{ad}(G) \leq ch(G)$ for any graph G, where $\chi(G)$ is the usual chromatic number of G, and $ch(G)$ is the usual choice number of G.

The adaptable choosability of planar graphs was studied in [3, 8]. It is known that planar graphs are adaptably 4-choosable. Moreover, a planar graph G is adaptably 3-choosable if one of the following holds:

1. G is triangle-free.

2. No two triangles intersect, and no triangle is adjacent to a 5-cycle, and each

6-cycle is adjacent to at most two triangles.

3. Any two triangles have distance at least 2 and no triangle is adjacent to a 4-cycle.

On the other hand, there are C_4-free planar graphs that are not adaptably 3-colourable; and for any integer $k \geq 5$, there are planar graphs that are C_t-free for all $5 \leq t \leq k$ and not adaptably 3-colourable; and for any integer k, there are planar graphs G in which any two triangles have distance at least k and G is not adaptably 3-choosable.

In this note we give a new upper bound for the adaptable choice number of graphs. Given a graph G, the *maximum average degree* of G, denoted by $Mad(G)$, is the maximum average degree of the subgraphs of G, i.e.,

$$Mad(G) = \max\{2|E(H)|/|V(H)| : H \text{ is a subgraph of } G\}.$$

We shall prove that for any graph G, its adaptable choice number is at most $\lceil Mad(G)/2 \rceil + 1$.

We denote by $\mathbb{S}_h$ the orientable surface of genus h, i.e., the surface obtained from the sphere by adding h handles, and denote by $\mathbb{N}_h$ the non-orientable surface of genus h, i.e., the surface obtained from the sphere by adding h crosscaps. The *Euler characteristic* $\chi(\mathbb{S})$ of a surface $\mathbb{S}$ is defined as

$$\chi(\mathbb{S}) = \begin{cases} 2 - 2h, & \text{if } \mathbb{S} = \mathbb{S}_h, \\ 2 - h, & \text{if } \mathbb{S} = \mathbb{N}_h. \end{cases}$$

Two cycles C_1 and C_2 in a graph G are said to be *adjacent* if they have at least one edge in common. As a consequence of the above upper bound for $ch_{ad}(G)$, we shall show that if G is a simple graph which can be embedded in a surface S of non-negative Euler characteristic, then G is adaptably 4-choosable. Moreover, if G is simple, embedded in a surface of non-negative Euler characteristic and no triangle of G is adjacent to a triangle or a C_4, and each C_5 is adjacent to at most three triangles, then G is adaptably 3-choosable.

In 1976, Steinberg [12] conjectured that planar graphs without cycles of length 4 and 5 are 3-colourable. This conjecture remains unsolved. The corresponding question for adaptable choosability and adaptable colourability was asked in [3]: Are simple planar graphs without 4-cycles and 5-cycles adaptably 3-colourable (or even adaptably 3-choosable)? This question is answered in positive, because if a planar graph G has no 4-cycle and no 5-cycle, then no two adjacent triangles of G are adjacent, and no 5-cycle is adjacent to more than three triangles, and hence G is adaptably 3-choosable.

Finally we give a new proof of the fact that every K_5-minor free graph is adaptably 4-choosable [3] based on the relationship between adaptable choice number and maximum average degree.

2 Upper bounds for $ch_{ad}(G)$

Theorem 2.1 *For any graph G (parallel edges are allowed),*

$$ch_{ad}(G) \leq \lceil Mad(G)/2 \rceil + 1.$$

Proof. To prove this Theorem we will use the following result of Hakimi [9]. A graph G on vertices $x_1, x_2, \cdots, x_n$ has an orientation in which x_i has out-degree $d^+(x_i) = k_i$ if and only if the following hold:

1. For each subset X of $V(G)$, $\sum_{x_i \in X} k_i \geq |E(G[X])|$.
2. $\sum_{i=1}^{n} k_i = |E(G)|$.

An easy consequence of this result is that if for each subset X of $V(G)$, $\sum_{x_i \in X} k_i \geq |E(G[X])|$, then G has an orientation in which $d^+(x_i) \leq k_i$ for each x_i (see also [7]).

If $Mad(G) \leq k$ for an integer k, then for any subgraph H of G, $|E(H)| \leq \frac{k}{2}|V(H)|$. It follows from the above result that G has an orientation in which each vertex x_i has $d^+(x_i) \leq \lceil \frac{k}{2} \rceil$. Assume each vertex x_i is given a list $L(x_i)$ of $\lceil \frac{k}{2} \rceil + 1$ colours and F is an edge colouring of G. Let $c(x_i)$ be any colour in $L(x_i)$ which does not appear in any outgoing edges of x_i. Then it is obvious that c is an L-colouring of G adapted to F. This completes the proof of Theorem 2.1. ∎

Corollary 2.1 *If G is a simple graph which can be embedded in a surface S of non-negative Euler characteristic, then G is adaptably 4-choosable. If, moreover, no triangle of G is adjacent to a triangle or a C_4, and each C_5 is adjacent to at most three triangles, then G is adaptably 3-choosable.*

Proof. Assume G is a simple graph embedded in a surface $\mathbb{S}$ of Euler characteristic $\chi(\mathbb{S}) \geq 0$. Let H be a subgraph of G. Then H is also a simple graph embedded in $\mathbb{S}$. Let V, F, E be the sets of vertices, faces and edges of H, respectively. By Euler's formula,

$$|V| + |F| - |E| = \chi(\mathbb{S}) \geq 0.$$

Let f_i be the number of i-faces, i.e., faces whose boundary is a walk of length i. Since H is simple, each face is an i-face for some $i \geq 3$. Therefore

$$3|F| \leq \sum_{i \geq 3} i \cdot f_i = 2|E|.$$

It follows that

$$|E| \leq 3|V|.$$

Hence $Mad(G) \leq 6$, and by Theorem 2.1, G is adaptably 4-choosable.

Assume moreover that no triangle in G is adjacent to a triangle or a C_4, and each C_5 is adjacent to at most three triangles. Then each 3-face of H is adjacent to three faces of degree at least 5. Each 5-face is adjacent to at most three 3-faces, and for $i \geq 6$, each i-face is adjacent to at most i 3-faces. Therefore

$$3f_3 \leq 3f_5 + \sum_{i \geq 6} i \cdot f_i.$$

It follows that

$$\begin{aligned} 4|F| &= 3f_3 + 4f_4 + 5f_5 + (f_3 - f_5) + 4\sum_{i \geq 6} f_i \\ &\leq 3f_3 + 4f_4 + 5f_5 + \sum_{i \geq 6} (\frac{i}{3} + 4) f_i \\ &\leq 2|E|. \end{aligned}$$

By Euler formula, $|V| + |F| - |E| = \chi(\mathbb{S}) \geq 0$. By replacing $|F|$ with $|E|/2$, we obtain the inequality that $|E| \leq 2|V|$. So $Mad(G) \leq 4$. By Theorem 2.1, $ch_{ad}(G) \leq 3$. ∎

The following result was proved in [3].

Corollary 2.2 *Every K_5-minor free simple graph is adaptably 4-choosable.*

Proof. It suffices to prove that any maximal K_5-minor free graph G has $|E(G)| \leq 3|V(G)| - 6$. It is known that a maximal K_5-minor free graph is constructed recursively, by pasting along K_2's and K_3's, from plane triangulations and copies of the Wagner's graph (the graph obtained from C_8 by adding four diagonal edges). Assume G is obtained from the union of G_1, G_2 by pasting along a K_2 or K_3, and $|E(G_i)| \leq 3|V(G_i)| - 6$. Then $|E(G)| = |E(G_1)| + |E(G_2)| - t$, where $t = 1$ or 3, respectively, and $|V(G)| = |V(G_1)| + |V(G_2)| - s$, where $s = 2$ or 3, respectively. Now a straightforward calculation shows that $|E(G)| \leq 3|V(G)| - 6$. ∎

Corollaries 2.1 and 2.2 show that the upper bound for $ch_{ad}(G)$ in Theorem 2.1 is very useful. In fact for graphs embedded in surface of non-negative Euler characteristic, the upper bounds for $ch_{ad}(G)$ in Corollary 2.1 are sharp. Theorem 2.1 is also sharp in the sense that for any integer g, d, there are d-regular graphs G of girth at least g for which $ch_{ad}(G) = \chi_{ad}(G) = d + 1$ [11]. However, for random graphs, the upper bound given in Theorem 2.1 is usually far from sharp. As an example, we consider random d-regular graphs G, which have $Mad(G) = d$. Let k_d be the smallest integer k such that $d < 2k \log k$. It is known that with high probability, $\chi(G) = k_d$ or $k_d + 1$ or $k_d + 2$ [1], and that $ch_{ad}(G) \leq \sqrt{8d}$ [11]. It is likely that for most graphs, $ch_{ad}(G)$ is much less than $ch(G)$ and $Mad(G)/2$. Question 2.1 below concerns the adaptable chromatic number of graphs. It was asked in [10] and remains open.

Question 2.1 *Let* $f(n) = \min\{\chi_{ad}(G) : \chi(G) = n\}$*. Is it true that* $f(n) = \chi_{ad}(K_n)$*? Is it true that* $\lim_{n\to\infty} f(n) = \infty$*? If so, what is the order of* $f(n)$*?*

Similar questions can be asked for adaptable choosability of graphs.

Question 2.2 *Let* $g(n) = \min\{ch_{ad}(G) : ch(G) = n\}$*. Is it true that* $\lim_{n\to\infty} g(n) = \infty$*? If so, what is the order of* $g(n)$*?*

It follows from a result of Alon [2] that there is a function $h(d)$ goes to infinity with d such that if $Mad(G) \geq d$ then $ch(G) \geq h(d)$.

Question 2.3 *Let* $\phi(t) = \min\{ch_{ad}(G) : Mad(G) = t\}$*. Is it true that* $\lim_{t\to\infty} \phi(t) = \infty$*?*

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