BISEPARATING LINEAR MAPS BETWEEN CONTINUOUS VECTOR-VALUED FUNCTION SPACES

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Abstract. Let $X$, $Y$ be compact Hausdorff spaces and $E$, $F$ be Banach spaces. A linear map $T : C(X, E) \to C(Y, F)$ is separating if $Tf$, $Tg$ have disjoint cozeroes whenever $f$, $g$ have disjoint cozeroes. We prove that a biseparating linear bijection $T$ (i.e., $T$ and $T^{-1}$ are separating) is a weighted composition operator $Tf = h \cdot f \circ \varphi$, where $\varphi$ is a homeomorphism from $Y$ onto $X$; and $T$ is bounded if and only if $h(y)$ is a bounded operator from $E$ onto $F$ for all $y$ in $Y$.

1. Introduction

Let $X$ and $Y$ be compact Hausdorff spaces, $E$ and $F$ be Banach spaces, and $C(X, E)$ and $C(Y, F)$ be the Banach spaces of continuous $E$-valued and $F$-valued functions defined on $X$ and $Y$, respectively. In $C(X, E)$, we write $fg = 0$ for $\|f(x)\|\|g(x)\| = 0, \forall x \in X$. A linear operator $T : C(X, E) \to C(Y, F)$ is said to be separating, or (cozero) disjointness preserving, if $TfTg = 0$ whenever $fg = 0$. An invertible $T$ is biseparating if both $T$ and $T^{-1}$ are separating.

The notion of disjointness preserving operators seems to be used firstly in the 40’s (see e.g. [22, 23]). Since then many authors have developed this concept in different directions. Y. Abramovich, for example, made many contributions in the context of Banach and vector lattices (see e.g. [5, 3]). In the case of continuous scalar-valued functions, separating linear maps were studied by Beckenstein and Narici in [9, 10, 11] and further investigated in [7] and [16]. Separating linear bijections between spaces of continuous functions are automatically continuous. Indeed, a bijective linear operator $T$ from $C(X)$ onto $C(Y)$ is separating if and only if $T$ is an (automatically bounded) weighted composition operator (see e.g., [16, 13, 17]). This can be considered as a special case of the generalized Nakano’s theorem obtained in [5, 6], which asserts that $\pi$-isomorphic Banach lattices are order isomorphic and the $\pi$-isomorphism is continuous. In the context of vector-valued functions, however, a separating or even a biseparating linear operator is not necessarily continuous (see Example 2.4).

In [20], Jerison got the first vector-valued version of the Banach-Stone Theorem: If $E$ is a strictly convex Banach space then every surjective isometry $T$ from $C(X, E)$ onto $C(Y, E)$ can be written as a weighted composition operator $Tf = h \cdot f \circ \varphi$, that is,

$$Tf(y) = h(y)(f(\varphi(y))), \quad \forall y \in Y, \forall f \in C(X, E).$$
Here, $\varphi$ is a homeomorphism from $Y$ onto $X$ and $h$ is a continuous map from $Y$ into the space $(B(E, E), \text{SOT})$ of bounded linear operators from $E$ into $E$ equipped with the strong operator topology (SOT). Moreover, $h(y)$ is an isometry from $E$ onto $E$ for all $y$ in $Y$. The conclusion might not hold, however, if $E$ is not strictly convex (see e.g. [19]).

After Jerison [20], many authors work on different variants of the vector-valued Banach-Stone Theorem (cf. [20, 21, 15, 12, 14, 18, 19]). In particular, as an extension of the representation theorem of Abramovich [1], Hernandez, Beckenstein and Narici proved in [14] that if $T$ is an isometric biseparating linear map from $C(X, E)$ onto $C(Y, F)$ then $T$ is a weighted composition operator $Tf(y) = h(y)(f(\varphi(y)))$. In case $T$ is bounded but not necessarily invertible, $T$ can still be written as a weighted composition operator (see, e.g., [15, 12]). It is then possible to prove that every bounded invertible biseparating map provides a homeomorphism $\varphi$ from $Y$ onto $X$. However, these methods might not apply to unbounded biseparating linear maps.

In Section 2, we develop a new argument to prove that every biseparating linear bijection $T$ from $C(X, E)$ onto $C(Y, F)$ induces a homeomorphism $\varphi$ from $Y$ onto $X$. As expected, $T$ is a weighted composition operator

$$Tf(y) = h(y)(f(\varphi(y))), \quad \forall f \in C(X, E), \forall y \in Y.$$  

Here, $h(y)$ is an invertible linear map from $E$ onto $F$ for each $y$ in $Y$. However, $T$ is not necessarily bounded (see Example 2.4). In fact, $T$ is bounded if and only if $\|h(y)\| < \infty$ for all $y$ in $Y$. In this case, $h : Y \to (B(E, F), \text{SOT})$ is continuous and $\|T\| = \sup_{y \in Y} \|h(y)\| < \infty$.

In the last section, we discuss when the inverse $T^{-1}$ of a disjointness preserving linear bijection $T : C(X, E) \to C(Y, F)$ is disjointness preserving. It is well-known that $T^{-1}$ is disjointness preserving when $E = F$ is the scaler field (see e.g. [8]). However, it is not the case even for finite dimensional $E$ and $F$ (see Example 5; see also [2, 3, 4]). We will present a new condition, so-called (support) containment preserving property of $T$, which is originally due to Abramovich and Kitover [4]. We prove that $T^{-1}$ preserves disjointness if $T$ preserves containment. In particular, $T$ preserves both disjointness and containment if and only if $T$ and $T^{-1}$ are both weighted composition operators. This can be considered as the vector-valued version of the results in [4].

2. Biseparating linear maps are weighted composition operators

In the following, we always assume $X$ and $Y$ are compact Hausdorff spaces, $E$ and $F$ are Banach spaces, and $B(E, F)$ is the space of bounded linear operators from $E$ into $F$ equipped with the strong operator topology. For each $x$ in $X$, let

$$I_x = \{f \in C(X, E) : f \text{ vanishes in a neighborhood of } x\}.$$  

Note that the linear manifold $I_x$ is not closed. But it is dense in the closed linear subspace $M_x = \{f \in C(X, E) : f(x) = 0\}$. Moreover, it is somehow ‘prime’ in the
sense that $f \in I_x$ whenever $fg = 0$ and $g(x) \neq 0$. In fact, $\|g(y)\| > 0$ for all $y$ in a neighborhood $V$ of $x$ and thus forces $f$ vanishes in $V$.

We start by observing that a biseparating linear bijection $T$ preserves $I_x$’s.

**Lemma 2.1.** Let $T : C(X, E) \rightarrow C(Y, F)$ be a biseparating linear bijection. Then for each $x$ in $X$ there is a unique $y$ in $Y$ such that

$$TI_x = I_y.$$ 

Moreover, this defines a bijection $\varphi$ from $Y$ onto $X$ by $\varphi(y) = x$.

**Proof.** For each $x$ in $X$, denote by $\ker T(I_x)$ the set $\bigcap_{f \in I_x} (Tf)^{-1}(0)$. We first claim that $\ker T(I_x)$ is non-empty. Suppose on the contrary that for each $y$ in $Y$, there were an $f_y$ in $I_x$ with $Tf_y(y) \neq 0$. Thus, an open neighborhood $U_y$ of $y$ exists such that $Tf_y$ is nonvanishing in $U_y$. Since $Y = \cup_{y \in Y} U_y$ and $Y$ is compact, $Y = U_{y_1} \cup U_{y_2} \cup \cdots \cup U_{y_n}$ for some $y_1, y_2, \ldots, y_n$ in $Y$. Let $V$ be an open neighborhood of $x$ such that $f_{y_i}|V = 0$ for all $i = 1, 2, \ldots, n$. Let $g \in C(X, E)$ such that $g(x) \neq 0$ and $g$ vanishes outside $V$. Then $f_y, g = 0$, and thus $Tf_y Tg = 0$ since $T$ preserves disjointness. This forces $Tg|U_i = 0$ for all $i = 1, 2, \ldots, n$. Therefore, $Tg = 0$ and hence $g = 0$ by the injectivity of $T$, a contradiction! We thus prove that $\ker T(I_x) \neq \emptyset$.

Let $y \in \ker T(I_x)$. For each $f \in I_x$, we want to show that $Tf \in I_y$. If there exists a $g$ in $C(X, E)$ such that $Tg(y) \neq 0$ and $fg = 0$, then we are done by the disjointness preserving property of $T$. Suppose there were no such $g$; that is, for any $g$ in $C(X, E)$ vanishing outside $V = f^{-1}(0)$, we have $Tg(y) = 0$. Let $W \subseteq V$ be a compact neighborhood of $x$ and $k \in C(X)$ such that $k|W = 1$ and $k$ vanishes outside $V$. Then for any $g$ in $C(X, E)$, $g = kg + (1 - k)g$. Since $(1 - k)|W = 0$, we have $(1 - k)g \in I_x$. This implies $T((1 - k)g)(y) = 0$ as $y \in \ker T(I_x)$. On the other hand, $kg$ vanishes outside $V$. Hence $T(kg)(y) = 0$ by the above assumption. It follows that $Tg(y) = Tk(g)g + T((1 - k)g)(y) = 0$ for all $g$ in $C(X, E)$. This conflicts with the surjectivity of $T$. Therefore, $TI_x \subseteq I_y$.

Similarly, $T^{-1}(I_y) \subseteq I_x'$ for some $x'$ in $X$ since $T^{-1}$ is also separating. It follows that $I_x \subseteq T^{-1}(I_y) \subseteq I_x'$. Consequently, $x = x'$ and $T(I_x) = I_y$. The bijectivity of $\varphi$ is also clear now.

**Theorem 2.2.** Two compact Hausdorff spaces $X$ and $Y$ are homeomorphic whenever there is a biseparating linear bijection $T$ from $C(X, E)$ onto $C(Y, F)$.

**Proof.** We show that the bijection $\varphi$ given in Lemma 2.1 is a homeomorphism. It suffices to verify the continuity of $\varphi$ since $Y$ is compact and $X$ is Hausdorff. Suppose that there exists a net $\{y_\lambda\}_\lambda$ in $Y$ converging to $y$ but $\varphi(y_\lambda) \rightarrow x \neq \varphi(y)$, and we want to derive a contradiction.

Let $U_x$ and $U_{\varphi(y)}$ be disjoint open neighborhoods of $x$ and $\varphi(y)$, respectively. Now for any $f$ in $C(X, E)$ vanishing outside $U_{\varphi(y)}$, we shall show that $Tf(y) = 0$. In fact, $\varphi(y_\lambda)$ belongs to $U_x$ for large $\lambda$. Since $f|_{U_x} = 0$ and $U_x$ is also a neighborhood of
Every biseparating linear bijection

By Theorem 2.2, we have a homeomorphism \( \varphi \) such that \( \varphi = \phi \) that \( \text{ker} \, T \) is a compact neighborhood of \( \varphi(y) \). Then \( g = kg + (1 - k)g \) for every \( g \) in \( C(X, E) \). Since \( kg \) vanishes outside \( U_{\varphi(y)} \), we have \( T(kg)(y) = 0 \). On the other hand, we have \( (1 - k)g \in I_{\varphi(y)} \) since \( (1 - k)|_V = 0 \). By Lemma 2.1, \( T((1 - k)g) \in I_y \) and thus \( T((1 - k)g)(y) = 0 \). It follows that \( Tg(y) = T(kg)(y) + T((1 - k)g)(y) = 0 \). This is a contradiction since \( T \) is onto. Hence \( \varphi \) is a homeomorphism.

**Theorem 2.3.** Every biseparating linear bijection \( T : C(X, E) \rightarrow C(Y, F) \) is a weighted composition operator

\[
Tf(y) = h(y)(f(\varphi(y))), \quad \forall f \in C(X, E), \forall y \in Y.
\]

Here \( \varphi \) is a homeomorphism from \( Y \) onto \( X \) and \( h(y) \) is an invertible linear map from \( E \) onto \( F \) for each \( y \) in \( Y \). Moreover, \( T \) is bounded if and only if \( h(y) \) is bounded for all \( y \) in \( Y \). In this case, \( h \) is a continuous map from \( Y \) into \( (B(E, F), \text{SOT}) \), and \( \|T\| = \sup_{y \in Y} \|h(y)\| \).

**Proof.** By Theorem 2.2, we have a homeomorphism \( \varphi \) from \( Y \) onto \( X \) such that \( T(I_x) = I_y \) where \( \varphi(y) = x \).

**Claim.** \( TM_x \subseteq M_y \).

If the claim is verified then \( TM_x = M_y \) by the same argument for \( T^{-1} \). It follows that \( \ker \delta_x = \ker \delta_y \circ T \). Consequently, there is an invertible linear operator \( h(y) \) from \( E \) onto \( F \) such that \( \delta_y \circ T = h(y) \circ \delta_x \). Equivalently, \( Tf(y) = h(y)(f(\varphi(y))) \) for all \( f \) in \( C(X, E) \) and \( y \) in \( Y \).

Suppose \( T \) is bounded. For any \( e \) in \( E \), let \( f \in C(X, E) \) such that \( f(x) = e, \forall x \in X \). Since \( \|h(y)e\| = \|h(y)(f(\varphi(y)))\| = \|Tf(y)\| \leq \|Tf\| \leq \|T\|\|f\| = \|T\|\|e\| \), we conclude that \( \|h(y)\| \leq \|T\| \) for all \( y \) in \( Y \). On the other hand, if \( \|h(y)\| < \infty \) for all \( y \) in \( Y \) then \( h \) is continuous on \( Y \). In fact, let \( \{y_\lambda\}_\lambda \) be a net convergent to \( y \) in \( Y \). Then \( \|h(y_\lambda)e - h(y)e\| = \|h(y_\lambda)(f(\varphi(y_\lambda)) - h(y)(f(\varphi(y)))\| = \|Tf(y_\lambda) - Tf(y)\| \rightarrow 0 \) as \( T \in C(Y, F) \). Consequently, \( \sup_{y \in Y} \|h(y)\| < \infty \) since the map \( y \mapsto \|h(y)\| \) is continuous on the compact space \( Y \). Finally, for any \( g \) in \( C(X, E) \) and \( y \) in \( Y \), we have \( \|Tg(y)\| = \|h(y)(g(\varphi(y)))\| \leq \|h(y)\||\|g\||. \) Hence \( \|T\| \leq \sup_{y \in Y} \|h(y)\| \).

To verify the claim, suppose on the contrary \( f \in M_x \) but \( Tf(y) \neq 0 \). If \( x \) belongs to the interior of \( f^{-1}(0) \), then \( f \in I_x \) and thus \( Tf(y) = 0 \). Therefore, we may assume there is a net \( \{x_\lambda\}_\lambda \) in \( X \) converging to \( x \) and \( f(x_\lambda) \) is never zero. Let \( y_\lambda \) be in \( Y \) such that \( \varphi(y_\lambda) = x_\lambda \). Clearly, \( y_\lambda \) converges to \( y \) and we may assume there is a constant \( \epsilon \) such that \( \|Tf(y_\lambda)\| \geq \epsilon > 0 \) for all \( \lambda \). For \( n = 1, 2, \ldots, \) set

\[
V_n = \{z \in X : \frac{1}{2n + 1} \leq \|f(z)\| \leq \frac{1}{2n}\}
\]

and

\[
W_n = \{z \in X : \frac{1}{2n} \leq \|f(z)\| \leq \frac{1}{2n - 1}\}.
\]
Then at least one of the unions $V = \bigcup_{n=1}^{\infty} V_n$ and $W = \bigcup_{n=1}^{\infty} W_n$ contains a subnet of $\{x_\lambda\}_\lambda$. Without loss of generality, we assume that all $x_\lambda$ belong to $V$. Let $V'_n$ be an open set containing $V_n$ such that $V'_n \cap V'_m = \emptyset$ if $n \neq m$. Let $g_n$ in $C(X, E)$ be of norm at most $1/2n$ such that $g_n$ agrees with $f$ on $V_n$ and vanishes outside $V'_n$ for each $n$. Then $g_ng_m = 0$ for all $m \neq n$. Let $g = \sum_{n=1}^{\infty} 2ng_n \in C(X, E)$. Note that $g$ agrees with $2nf$ on each $V_n$. Moreover, each $x_\lambda$ belongs to a unique $V_n$ and $n \to \infty$ as $\lambda \to \infty$. Therefore, $g - 2nf \in I_{x_\lambda}$. This implies $T(g - 2nf) \in I_{y_\lambda}$ and thus $\|Tg(y_\lambda)\| = 2n\|Tf(y_\lambda)\| \geq 2nc \to \infty$ as $\lambda \to \infty$. But the limit should be $\|Tg(y)\|$, a contradiction. This completes the proof. 

In the following example, we see that the invertible linear operator $h(y)$ in (1) can be unbounded.

Example 2.4. Let $X = \{0\}$ and $\psi$ be an unbounded linear functional of $c_0$ such that $\psi((1, 0, 0, \cdots)) = 1$. Define an unbounded linear bijection $H$ from $c_0$ onto $c_0$ by

$$H(\lambda) = (\lambda_1 + \psi(\lambda), \lambda_2, \lambda_3, \cdots), \quad \forall \lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots) \in c_0.$$ 

Set $h(0) = H$ and define the biseparating linear bijection $T : C(X, c_0) \to C(X, c_0)$ by

$$Tf(0) = h(0)((f(0))), \quad \forall f \in C(X, c_0).$$ 

Then $T$ is an unbounded weighted composition operator. Note that $\varphi : X \to X$ with $\varphi(0) = 0$ is a homeomorphism.

3. Containment and disjointness preserving operators

In above results, we have to assume $T$ is biseparating, namely both $T$ and $T^{-1}$ are separating. It is known that the inverse of a separating linear bijection between Banach lattices (in particular, $C(X)$’s) always preserves disjointness (see e.g., [2, Theorem 1]). Recently, Abramovich and Kitover [2, 3] showed that $T^{-1}$ need not be separating in the general vector lattice setting.

Example 3.1 ([14]). Let $X = \{0\}$ and $E = \mathbb{R}^2$ with sup norm, and let $Y = \{1, 2\}$ and $F = \mathbb{R}$ with its usual norm. Define $T : C(X, E) \to C(Y, F)$ by $T(\frac{g}{g}) = g$ with $g(1) = a$ and $g(2) = b$. Then the surjective linear isometry $T$ is separating, but its inverse $T^{-1}$ is not.

Recall that for an $f$ in $C(X, E)$, the cozero of $f$ is $\text{coz}(f) = \{x \in X : f(x) \neq 0\}$ and the support $\text{supp}(f)$ of $f$ is the closure of $\text{coz}(f)$ in $X$. The following definition modifies the one given by Abramovich [4].

Definition 3.2. A linear map $T : C(X, E) \to C(Y, F)$ is said to be (support) containment preserving if

$$\text{supp}(f) \subseteq \text{supp}(g) \text{ implies } \text{supp}(Tf) \subseteq \text{supp}(Tg).$$ 

For any injective map $T : C(X, E) \to C(Y, F)$ we denote by $T^{-1}$ the inverse operator defined on $\text{ran}T$, the range space of $T$. 
Lemma 3.3. Let $T : C(X, E) \rightarrow C(Y, F)$ be a linear injection. If $T$ is containment preserving, then $T^{-1}$ is disjointness preserving.

Proof. Suppose, on the contrary, that there exist $f$ and $g$ in $C(X, E)$ such that $TfTg = 0$ but $\|f(x)\|\|g(x)\| \neq 0$ for some $x$ in $X$. Then we can find an open neighborhood $V$ of $x$ such that $V \subseteq \text{coz}(f) \cap \text{coz}(g)$. Let $h \in C(X, E)$ such that $h(x) \neq 0$ and $h|_V = 0$. It is clear that $\text{supp}(h) \subseteq \text{supp}(f) \cap \text{supp}(g)$. Since $T$ preserves containment, we have $\text{supp}(Th) \subseteq \text{supp}(Tf) \cap \text{supp}(Tg)$. Consequently, $\text{coz}(Th) \subseteq \text{supp}(Tg)$. On the other hand, $TfTg = 0$ implies $\text{coz}(Tf) \cap \text{coz}(Tg) = \emptyset$. It follows that $\text{coz}(Th) \subseteq \text{supp}(Tf) \subseteq Y \setminus \text{coz}(Tg)$. Since $\text{coz}(Th)$ is open, it forces that $\text{coz}(Th) \cap \text{supp}(Tg) = \emptyset$. Therefore, $Th = 0$ and hence $h = 0$ by the injectivity of $T$, a contradiction! We thus prove that $T^{-1}$ is disjointness preserving.

Combining Theorem 2.3 and Lemma 3.3, we will have the following

Corollary 3.4. Let $T : C(X, E) \rightarrow C(Y, F)$ be a linear bijection. Then the following statements are equivalent:

(a) $T$ preserves disjointness and containment;
(b) $T$ and $T^{-1}$ preserve disjointness;
(c) $T$ and $T^{-1}$ preserve containment;
(d) $T$ and $T^{-1}$ are weighted composition operators.

Proof. We need only to prove (b) $\Rightarrow$ (a). Suppose $\text{supp}(f) \subseteq \text{supp}(g)$, we want to show that $\text{supp}(Tf) \subseteq \text{supp}(Tg)$. Suppose on the contrary that there exists a $y$ in $Y$ such that $y \in \text{supp}(Tf) \setminus \text{supp}(Tg)$. Thus, there exists an open neighborhood $V$ of $y$ such that $V \cap \text{supp}(Tg) = \emptyset$. Choose a $y' \in V \cap \text{coz}(Tf)$ and let $h \in C(Y, F)$ such that $h(y') \neq 0$ and $h|_V = 0$. Since $T$ is surjective, say $Tk = h$ for some $k$ in $C(X, E)$. Then $TkTg = 0$, and thus $kg = 0$ since $T^{-1}$ preserves disjointness. This forces $kf = 0$, because $\text{supp}(f) \subseteq \text{supp}(g)$. Therefore, $hTf = TkTf = 0$. But $\|h(y')\|\|Tf(y')\| \neq 0$, a contradiction!

In Example 3.1, the surjective isometry $T$ preserves disjointness but not containment while its inverse $T^{-1}$ preserves containment but not disjointness. Moreover, $T$ is a weighted composition operator but $T^{-1}$ is not.

References


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