Southeast Asian Bulletin of Mathematics (0000) 00: 000–9

Southeast Asian Bulletin of Mathematics

Sums of Angles of Star Polygons and the Eulerian Numbers

Meng-Han Li and Ngai-Ching Wong

Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan, R.O.C.

E-mail: wong@math.nsysu.edu.tw

AMS Mathematical Subject Classification (2000): 05A05, 11B68, 52B05

Abstract. Every (convex) star polygon with n vertices can be associated with a permutation σ on $\{1, 2, \ldots, n\}$. We give an exact formula to compute the sum of (interior) angles in term of σ. In particular, the sum of angles of the polygon is solely determined by σ. We make use of this formula to derive a recurrence relation concerning the number of star polygons having a particular value of sums of angles. The results are summarized in a Pascal type triangle. By observing the relation of such numbers and the Eulerian numbers, we obtain a closed formula. A possible application to quantum physics is presented.

Keywords: sum of angles, star polygons, Eulerian numbers, combinatorially equivalence

1. Introduction

Let $V_1, V_2, \ldots, V_n$ be n points in the plane $\mathbb{R}^2$ in convex position, that is, each V_i is an extreme point of the convex hull $\operatorname{co}\{V_1, V_2, \ldots, V_n\}$ for $i = 1, 2, \ldots, n$. Denoted by $\mathbf{P}=\{V_1V_2 \ldots V_n\}$ the (convex) star polygon formed by joining V_i to V_{i+1} as edges for $i = 1, 2, \ldots, n$. Here we set $V_{n+k} = V_k$. The sum of (interior) angles of $\mathbf{P}$ is defined to be the algebraic sum

$$\operatorname{ang}(\mathbf{P}) = \sum_{i=1}^{n} \angle V_{i-1}V_iV_{i+1}.$$

In this paper, we consider the counterclockwise orientation as positive.

We associate to $\mathbf{P}$ a permutation σ on $\{1, 2, \ldots, n\}$ such that $\sigma(k)$ is the position of the vertex V_k in the list $\{V_1V_2 \ldots V_n\}$ in their counterclockwise ordering in the plane $\mathbb{R}^2$. Two star polygons of n vertices are said to be *combinatorially equivalent* if they define the same permutation up to rotation. In particular, the permutations are fixed if we assume $\sigma(1) = 1$ always. In this paper, we show that the sum of angles of $\mathbf{P}$ is given by

$$\operatorname{ang}(\sigma) = \pi \sum_{i=1}^{n} \operatorname{sign}[\sigma(i-1) - \sigma(i)],$$

where we set $\sigma(n+k) = \sigma(k)$ for convenience. This includes, in particular, a result of Bezdek and Fodor in [1] asserting that if two star polygons are combinatorially equivalent then the sums of angles of them are equal. We also discuss the question about the number n_k of combinatorially non-equivalent star polygons with n vertices and sum of angles $k\pi$. The results is summarized in a Pascal type triangle. It is interesting to see that these numbers, whenever nonzero, are exactly the Eulerian numbers. Thus a closed formula is obtained:

$$n_k = \sum_{j=0}^{\frac{n+k}{2}-1} (-1)^j \left(\frac{n+k}{2} - j\right)^{n-1} \binom{n+1}{k},$$

whenever $n+k$ is even; and $n_k = 0$ if $n+k$ is odd.

We include a possible application to quantum physics at the end of the paper. The problem can also be formulated into the one discussing the probability a random star polygon having a particular sum of angles.

2. The Results

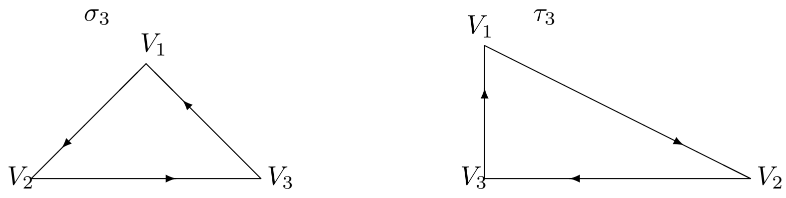
Theorem 2.1. *Let σ be the permutation associated to a star polygon $\mathbf{P} = \{V_1\ V_2 \ldots V_n\}$. Then the sum of angles of $\mathbf{P}$ is*

$$\operatorname{ang}(\sigma) = \pi \sum_{i=1}^{n} \operatorname{sign}[\sigma(i-1) - \sigma(i)], \tag{$\dagger$}$$

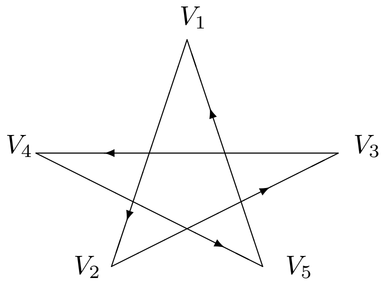
where we set $\sigma(n+k) = \sigma(k)$ for convenience.

Before we present the proof, let's look at some examples.

Example 2.2. It is clear that all triangles have sum of angles $\pm\pi$. More precisely, they are either associated with the permutation $\sigma_3 = \begin{pmatrix} 1\ 2\ 3 \\ 1\ 2\ 3 \end{pmatrix}$ or $\tau_3 = \begin{pmatrix} 1\ 2\ 3 \\ 1\ 3\ 2 \end{pmatrix}$. Note that $\operatorname{ang}(\sigma_3) = -\pi$ and $\operatorname{ang}(\tau_3) = \pi$.



Example 2.3. For the star pentagon, a *5-star*,



the permutation it defines is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 5 & 2 & 4 \end{pmatrix}.$$

For instance, $\sigma(4) = 2$ since the vertex V_4 is in the second position in the list V_1, V_4, V_2, V_5, V_3 in their counterclockwise ordering. Applying the formula (†), the sum of angles of the 5-star is

$$[\mathrm{sign}(1-3) + \mathrm{sign}(3-5) + \mathrm{sign}(5-2) + \mathrm{sign}(2-4) + \mathrm{sign}(4-1)]\pi = -\pi.$$

Proof. [Proof of Theorem 2.1] The conclusion will follow from induction. When $n = 3$, the result is clear (see Example 2.2). Suppose that it holds for star polygons of up to $n-1$ vertices for $n \geq 4$.

Given a star polygon $\mathbf{P}$ with n vertices and associated with a permutation σ_n. Without loss of generality, we can assume that $\sigma_n(1) = 1$. Join the two vertices V_{n-1} and V_1 by a pair of parallel but opposite edges. Then we divide the star polygon $\mathbf{P}$ into two new star polygons. One of them has $n-1$ vertices V_1, V_2, $\ldots, V_{n-1}$ and the other one is a triangle with vertices V_1, V_{n-1}, V_n. The new star polygon and the triangle give rise to permutations σ_{n-1} and σ'_3, respectively, where

$$\sigma_{n-1}(k) = \begin{cases} \sigma_n(k) & \text{if } \sigma_n(k) < \sigma_n(n), \\ \sigma_n(k) - 1 & \text{if } \sigma_n(k) > \sigma_n(n), \end{cases}$$

and

$$\sigma_3' = \begin{cases} \begin{pmatrix} 1\ 2\ 3 \\ 1\ 2\ 3 \end{pmatrix} & \text{if } \ \sigma_n(n-1) < \sigma_n(n), \\ \begin{pmatrix} 1\ 2\ 3 \\ 1\ 3\ 2 \end{pmatrix} & \text{if } \ \sigma_n(n-1) > \sigma_n(n). \end{cases}$$

It is plain that the sum of angles of the star polygon $\mathbf{P}$ is equal to the sum of the sums of angles of the new star polygon and the triangle. By the induction hypothesis, the sum of angles of $\mathbf{P}$ is the sum $\operatorname{ang}(\sigma_{n-1}) + \operatorname{ang}(\sigma_3')$.

We claim that the correspondence $\sigma_n(i) \longrightarrow \sigma_{n-1}(i)$ between the finite sequences

$$\{\sigma_n(1), \sigma_n(2), \ldots, \sigma_n(n-1)\} \quad \text{and} \quad \{\sigma_{n-1}(1), \sigma_{n-1}(2), \ldots, \sigma_{n-1}(n-1)\}$$

is order preserving. In fact, if $\sigma_n(k)$ and $\sigma_n(k+1)$ are both less than $\sigma_n(n)$, then $\sigma_{n-1}(k) = \sigma_n(k)$ and $\sigma_{n-1}(k+1) = \sigma_n(k+1)$. Hence they have the same ordering. If they are both greater than $\sigma_n(n)$, then $\sigma_{n-1}(k) = \sigma_n(k) - 1$ and $\sigma_{n-1}(k+1) = \sigma_n(k+1) - 1$. This also gives the same ordering. There are two other cases, where the value of $\sigma_n(n)$ is strictly in between the values of $\sigma_n(k)$ and $\sigma_n(k+1)$, then the value of $\sigma_n(n)$ is also in between (but not necessarily strictly) the values of $\sigma_{n-1}(k)$ and $\sigma_{n-1}(k+1)$. Hence the ordering does not change, either.

Now the sum of angles of $\mathbf{P}$ is

$$\operatorname{ang}(\sigma_{n-1}) + \operatorname{ang}(\sigma_3') = \pi \sum_{i=1}^{n-1} \operatorname{sign}[\sigma_{n-1}(i-1) - \sigma_{n-1}(i)] + \pi \operatorname{sign}[\sigma_3'(2) - \sigma_3'(3)].$$

Here we note $\sigma_3'(1) = 1$. But the sum

$$\sum_{i=1}^{n-1} \operatorname{sign}[\sigma_{n-1}(i-1) - \sigma_{n-1}(i)] = \sum_{i=1}^{n-1} \operatorname{sign}[\sigma_n(i-1) - \sigma_n(i)]$$

and,

$$\operatorname{sign}[\sigma_3'(2) - \sigma_3'(3)] = \operatorname{sign}[\sigma_n(n-1) - \sigma_n(n)].$$

Hence the sum of angles of $\mathbf{P}$ equals

$$\begin{aligned} &\pi \sum_{i=1}^{n-1} \operatorname{sign}[\sigma_n(i-1) - \sigma_n(i)] + \pi \operatorname{sign}[\sigma_n(n-1) - \sigma_n(n)] \\ &= \pi \sum_{i=1}^{n} \operatorname{sign}[\sigma_n(i-1) - \sigma_n(i)] \\ &= \operatorname{ang}(\sigma_n), \end{aligned}$$

as asserted. ■

As a direct consequence of Theorem 2.1, we obtain a new proof of the following result in [1].

Corollary 2.4. *If two star polygons are combinatorially equivalent then their sums of angles are equal.*

We now discuss the question how many star polygons with n vertices having sums of angles $k\pi$. More precisely, it is $\sharp\{\sigma \in S_n : \mathrm{ang}(\sigma) = k\pi\}$ by Theorem 2.1. Here, S_n denotes the nth symmetric group, as usual. Since rotating a star polygon does not change the sum of angles, we have

$$\sharp\{\sigma \in S_n : \mathrm{ang}(\sigma) = k\pi\} = n \times \sharp\{\sigma \in S_n : \sigma(1) = 1 \text{ and } \mathrm{ang}(\sigma) = k\pi\}.$$

Set

$$\mathbf{P}_{n,k} = \{\sigma \in S_n : \sigma(1) = 1 \text{ and } \mathrm{ang}(\sigma) = k\pi\},$$

and

$$n_k = \sharp\mathbf{P}_{n,k}.$$

Note that $\mathbf{P}_{n,k}$ consists of all combinatorially non-equivalent star polygons with n vertices and sum of angles $k\pi$.

Theorem 2.5. *For all $n \in \mathbb{N}$, all $k \in \mathbb{Z}$, we have*

(1) $n_k = 0$ *if* $k < -n+2$ *or* $k > n-2$*;*

(2) $n_k = n_{-k}$*;*

(3) $(2n)_{2k+1} = 0$*;*

(4) $(2n+1)_{2k} = 0$*;*

(5) $n_k = \frac{n+k}{2}(n-1)_{k+1} + \frac{n-k}{2}(n-1)_{k-1}$.

Proof. The assertions (1) and (2) are obvious. For (3) and (4), recall the sum of angles is given by

$$\mathrm{ang}(\sigma) = \pi \sum_{i=1}^{n} \mathrm{sign}[\sigma(i-1) - \sigma(i)].$$

The above summation is an algebraic sum of n "$+1$" and "-1". In case n is even, the number of "$+1$" and "-1" have the same parity and thus their difference must be even. When n is odd, the number of "$+1$" and "-1" are of different parity and their difference must be odd. This forces $(2n)_{2k+1} = (2n+1)_{2k} = 0$ for all $n \in \mathbb{N}$ and $k \in \mathbb{Z}$.

For (5), let a (resp. b) be the number of "$+1$" (resp. "-1") in the above algebraic sum. Then $a = \frac{n+k}{2}$ and $b = \frac{n-k}{2}$ are constants for all σ_n in $\mathbf{P}_{n,k}$.

Thus we may write $a(n,k)$ and $b(n,k)$ to indicate they are functions of n and k. Consider the following n different ways to obtain a permutation σ_{n+1} in S_{n+1} by modifying σ_n in S_n with $\sigma_n(1) = \sigma_{n+1}(1) = 1$: we set

$$\sigma_{n+1}(k) = \begin{cases} \sigma_n(k) & \text{if } 1 \le k < k_0 \ , \\ n+1 & \text{if } k = k_0 \ , \\ \sigma_n(k-1) & \text{if } k > k_0 \ , \end{cases} \tag{‡}$$

where k_0 can be any one of 2, 3, $\ldots$, $n+1$. Note that if $\sigma_n \in \mathbf{P}_{n,k}$ then either

$$\sigma_{n+1} \in \mathbf{P}_{n+1,k+1} \quad \text{or} \quad \sigma_{n+1} \in \mathbf{P}_{n+1,k-1}.$$

Indeed, since $\sigma_{n+1}(k_0) = n+1$ we have

$$\begin{aligned}
\operatorname{ang}(\sigma_{n+1}) &= \pi \sum_{i=1}^{n+1} \operatorname{sign}[\sigma_{n+1}(i-1) - \sigma_{n+1}(i)] \\
&= \pi \left\{ \sum_{i=1}^{k_0-1} \operatorname{sign}[\sigma_{n+1}(i-1) - \sigma_{n+1}(i)] \right. \\
&\quad + \operatorname{sign}[\sigma_{n+1}(k_0-1) - \sigma_{n+1}(k_0)] + \operatorname{sign}[\sigma_{n+1}(k_0) - \sigma_{n+1}(k_0+1)] \\
&\quad \left. + \sum_{i=k_0+2}^{n+1} \operatorname{sign}[\sigma_{n+1}(i-1) - \sigma_{n+1}(i)] \right\} \\
&= \pi \left\{ \sum_{i=1}^{k_0-1} \operatorname{sign}[\sigma_n(i-1) - \sigma_n(i)] + (-1) + (+1) \right. \\
&\quad \left. + \sum_{i=k_0+2}^{n+1} \operatorname{sign}[\sigma_n(i-2) - \sigma_n(i-1)] \right\} \\
&= \pi \sum_{i=1}^{n} \operatorname{sign}[\sigma_n(i-1) - \sigma_n(i)] - \pi \operatorname{sign}[\sigma_n(k_0-1) - \sigma_n(k_0)] \\
&= \operatorname{ang}(\sigma_n) - \pi \operatorname{sign}[\sigma_n(k_0-1) - \sigma_n(k_0)] \\
&= k\pi - \pi \operatorname{sign}[\sigma_n(k_0-1) - \sigma_n(k_0)].
\end{aligned}$$

Hence there are exactly $b(n,k)$ of such σ_{n+1}'s in $\mathbf{P}_{n+1,k+1}$ and $a(n,k)$ of such σ_{n+1}'s in $\mathbf{P}_{n+1,k-1}$.

In view of the above argument, each σ_n in $\mathbf{P}_{n,k}$ can be uniquely obtained from a σ_{n-1} in either $\mathbf{P}_{n-1,k-1}$ or $\mathbf{P}_{n-1,k+1}$ as in (‡). We thus have

$$\begin{aligned}
n_k &= b(n-1,k-1)(n-1)_{k-1} + a(n-1,k+1)(n-1)_{k+1} \\
&= \frac{n-k}{2}(n-1)_{k-1} + \frac{n+k}{2}(n-1)_{k+1}.
\end{aligned}$$

■

We summarize the results in Theorem 2.5 into the following Pascal type triangle.

	$k=-8$	-7	-6	-5	-4	-3	-2	-1	$k=0$	1	2	3	4	5	6	7	$8=k$
$n=2$									1								
$n=3$								1	0	1							
$n=4$							1	0	4	0	1						
$n=5$						1	0	11	0	11	0	1					
$n=6$					1	0	26	0	66	0	26	0	1				
$n=7$				1	0	57	0	302	0	302	0	57	0	1			
$n=8$			1	0	120	0	1191	0	2416	0	1191	0	120	0	1		
$n=9$		1	0	247	0	4293	0	15619	0	15619	0	4293	0	247	0	1	
$n=10$	1	0	502	0	14608	0	88234	0	156190	0	88234	0	14608	0	502	0	1

Table of the numbers n_k of combinatorically non-equivalent star n-polygons with sum of angles $k\pi$

Each entry in the triangle represents n_k. The first row for $n = 2$ is added for convenience. Note that on each of the right to left diagonals, the function $a(n,k) = \frac{n+k}{2}$ is constant and indicated above the edges. Similarly, $b(n,k) = \frac{n-k}{2}$ is constant on each of the left to right diagonals and indicated above the edges, too. They are weights to bring two consecutive nonzero entries to the one direct down in the next row. For example, the left "26" in the fifth row is obtained by

$$26 = 4 \times 1 + 2 \times 11,$$

or

$$6_{-2} = b(5,-3) \times 5_{-3} + a(5,-1) \times 5_{-1}.$$

The information the fifth row represents is that the total number among those $\frac{6!}{6} = 120$ combinatorially non-equivalent star hexagons with sum of angles $\pm 4\pi$ is 1, $\pm 3\pi$ is 0, $\pm 2\pi$ is 26, $\pm \pi$ is 0, and 0 is 66. In particular, the probability that the sum of angles of a random star hexagon is zero is 66/120, or 11/20.

One might notice the nonzero entries in the table of n_k coincide with those in the one of the Eulerian numbers. Recall that the descent set of a permutation σ in S_n is

$$D(\sigma) = \{1 \le i \le n-1 : \sigma(i) > \sigma(i+1)\} \cup \{n\}.$$

The Eulerian numbers are defined by

$$E(n,a) = \sharp\{\sigma \in S_n : d(\sigma) = a\}, \quad a = 1, \dots n,$$

where $d(\sigma) = \sharp D(\sigma)$ is the number of decents in σ. It is well known that the Eulerian numbers satisfy the equation

$$\sum_{r,s} E(r,s) \frac{x^r y^s}{(r+s+1)!} = \frac{e^x - e^y}{xe^y - ye^x},$$

as well as the recurrence relations $E(1,1) = 1$, $E(1,a) = 0$ for $a \ne 1$, and in general,

$$E(n,a) = aE(n-1,a) + (n-a+1)E(n-1,a-1). \qquad (*)$$

Indeed, we have

$$E(n,a) = \sum_{j=0}^{a-1} (-1)^j (a-j)^n \binom{n+1}{k}$$

for $n = 1, 2, \dots$ and $a = 1, \dots, n$. For general properties of the Eulerian numbers, the readers are referred to [3].

Comparing (*) with Theorem 2.5(5), we have

Theorem 2.6. *The relation between n_k and $E(n,a)$ is given by*

$$E(n,a) = (n+1)_{2a-n-1}.$$

Therefore, we have

$$n_k = \begin{cases} \displaystyle\sum_{j=0}^{\frac{n+k}{2}-1} (-1)^j \left(\frac{n+k}{2} - j\right)^{n-1} \binom{n+1}{k}, & \textit{if } n+k \textit{ is even;} \\ 0, & \textit{if } n+k \textit{ is odd.} \end{cases}$$

3. A possible application

To end this paper, we present a possible application. We imagine that the data in the triangle of n_k might be useful in some physics experiments. For example, suppose an electron is ejected into a black box through a window. Assume that there are a number of moving attractor-reflectors installed in the box, that the electron is attracted and rejected from each of them one by one without going back, and that it finally escapes from the box through the same window where it submerged. Then the locus of the electron is a random star polygon. If the measurement of the magnetic field changes above the box is neutral approximately every 11 times out of 20, then one can conclude that there are exactly 6 attractor-reflectors in the box. In other setups, the data in the triangle can help to determine both the number and the position of the attractor-reflectors.

One particular interesting related problem is about the probability that the sum of angles of a random star m-polygon being zero. In case $m = 2n - 1$, the probability is clearly zero, and that for $m = 2n$, by an integral representation for the Eulerian numbers [5], is

$$(2n)_0 = \frac{E(2n-1, n)}{(2n-1)!} = \frac{2}{\pi} \int_0^\infty \left(\frac{\sin t}{t}\right)^{2n} dt.$$

Consequently, the chance of getting a random star polygon of zero sum of angles tends to zero as the number of vertices approaches to infinity. Indeed, these numbers $E(2n-1, n)/(2n-1)!$ are decreasing to zero [4]. In general, one can find the probability that a random star $(n+1)$-polygon having sum of angles $(2a - n - 1)\pi$ by the asymptotic formula

$$\frac{(n+1)_{2a-n-1}}{n!} = \frac{E(n, a)}{n!} = \frac{\varphi(a - m_n)}{{s_n}^2} + O(n^{-3/4}).$$

Here, $m_n = (n+1)/2$ and ${s_n}^2 = (n+1)/12$ are the mean and variance of the distribution $E(n, a)/n!$ which converges to the normality as $n \to \infty$, and φ is the normal density function [2].

References

[1] A. Bezdek and F. Fodor, "On convex polygons of maximal width", *Arch. Math.*, **74** (2000), 75-80.

[2] L. Carlitz, D. C. Kurtz, R. Scoville and O. P. Stackelberg, "Asymptotic properties of Eulerian numbers", *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **23** (1972), 47–54.

[3] L. Comtet, *Advanced combinatorics. The art of finite and infinite expansions*, D. Reidel Publishing Co., Dordrecht, 1974.

[4] L. Lesieur and J.-L. Nicolas, "On the Eulerian numbers $M_n = \max_{1 \leq k \leq n} A(n,k)$", *European J. Math.*, **13** (1992), no. 5, 379–399.

[5] J.-L. Nicolas, "An integral representation for Eulerian numbers", in *Sets, graphs and numbers* (Budapest, 1991), 513–527, Colloq. Math. Soc. Janos Bolyai, **60**, North-Holland, Amsterdam, 1992.