ON C*-ALGEBRAS CUT DOWN BY CLOSED PROJECTIONS: CHARACTERIZING ELEMENTS VIA THE EXTREME BOUNDARY

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Abstract. Let $A$ be a C*-algebra. Let $z$ be the maximal atomic projection and $p$ a closed projection in $A^{**}$. It is known that $x$ in $A^{**}$ has a continuous atomic part, i.e. $zx = za$ for some $a$ in $A$, whenever $x$ is uniformly continuous on the set of pure states of $A$. Under some additional conditions, we shall show that if $x$ is uniformly continuous on the set of pure states of $A$ supported by $p$, or its weak* closure, then $pxp$ has a continuous atomic part, i.e. $zpxp = zpap$ for some $a$ in $A$.

1. Introduction

Let $A$ be a C*-algebra with Banach dual $A^*$ and double dual $A^{**}$. Let $Q(A) = \{ \varphi \in A^*: \varphi \geq 0 \text{ and } \|\varphi\| \leq 1 \}$ be the quasi-state space of $A$. When $A = C_0(X)$ for some locally compact Hausdorff space $X$, the weak* compact convex set $Q(C_0(X))$ consists of all positive regular Borel measures $\mu$ on $X$ with $\|\mu\| = \mu(X) \leq 1$. In this case, the extreme boundary of $Q(C_0(X)) \approx X \cup \{\infty\}$. The point $\infty$ at infinity is isolated if and only if $X$ is compact. For a non-abelian C*-algebra $A$, the extreme boundary of $Q(A)$ is the pure state space $P(A) \cup \{0\}$, in which $P(A)$ consists of pure states of $A$ and the zero functional $0$ is isolated if and only if $A$ is unital. In the Kadison function representation (see e.g. [16]), the self-adjoint part $A_{sa}^{**}$ of the W*-algebra $A^{**}$ is isometrically and order isomorphic to the ordered Banach space of all bounded affine real-valued functionals on $Q(A)$ vanishing at 0. Moreover, $x$ is in $A_{sa}$ if and only if in addition $x$ is weak* continuous on $Q(A)$.

Let $z$ be the maximal atomic projection in $A^{**}$. Note that $A^{**} = (1 - z)A^{**} \oplus zA^{**}$; in which $zA^{**}$ is the direct sum of type I factors and $(1 - z)A^{**}$ has no type I factor direct summand of $A^{**}$. In particular, $z$ is a central projection in $A^{**}$ supporting all pure states of $A$. In other words, $\varphi(x) = \varphi(zx)$ for all $x$ in $A^{**}$ and all pure states $\varphi$ of $A$. For an abelian C*-algebra $C_0(X)$, the enveloping W*-algebra $C_0(X)^{**} = \bigoplus_{\infty} \{ L^\infty(\mu) : \mu \in C \} \oplus_\infty \ell^\infty(X)$, where $C$ is a maximal family of mutually singular continuous measures on $X$. In this way, every $x$ in $C_0(X)^{**}$ can be written as a direct sum $x = x_d + x_a$ of the diffuse part $x_d$ and the atomic part $x_a$, and $zx = x_a \in \ell^\infty(X)$. Note that a measure $\mu$ on $X$ is atomic if $\langle x, \mu \rangle = \int x_a d\mu = \langle zx, \mu \rangle$, or equivalently, $\mu$ is supported by $z$. Alternatively, atomic measures are exactly countable linear sums of point masses. In general,
atomic positive functionals of a non-abelian C*-algebra $A$ are countable linear sums of pure states of $A$ ([13, 14]).

We call $zA^{**}$ the atomic part of $A^{**}$. An element $x$ of $A^{**}$ is said to have a continuous atomic part if $zx = za$ for some $a$ in $A$ (cf. [18]). In this case, $x$ and $a$ agree on $P(A) \cup \{0\}$ since $\varphi(x) = \varphi(zx) = \varphi(za) = \varphi(a)$ for all pure states $\varphi$ of $A$. In particular, $\varphi \mapsto \varphi(x)$ is uniformly continuous on $P(A) \cup \{0\}$. Shultz [18] showed that $x$ in $A^{**}$ has a continuous atomic part whenever $x, x^*x$ and $xx^*$ are uniformly continuous on $P(A) \cup \{0\}$. Later, Brown [7] proved:

**Theorem 1** ([7]). Let $x$ be an element of $A^{**}$. Then $x$ has a continuous atomic part (i.e. $zx \in zA$) if and only if $x$ is uniformly continuous on $P(A) \cup \{0\}$.

The Stone-Weierstrass problem for C*-algebras conjectures that if $B$ is a C*-subalgebra of a C*-algebra $A$ separating points in $P(A) \cup \{0\}$ then $A = B$ (see e.g. [11]). The facial structure of the compact convex set $Q(A)$ sheds some light on solving the Stone-Weierstrass problem. The classical papers of Tomita [19, 20], Effros [12], Prosser [17], and Akemann, Andersen and Pedersen [5], among others, have been exploring the interrelationship among weak* closed faces of $Q(A)$, closed projections in $A^{**}$ and norm closed left ideals of $A$, in the hope that this will help to solve the Stone-Weierstrass problem.

Recall that a projection $p$ in $A^{**}$ is closed if the face

$$F(p) = \{ \varphi \in Q(A) : \varphi(1 - p) = 0 \}$$

of $Q(A)$ supported by $p$ is weak* closed (and thus weak* compact). In the abelian case $A = C_0(X)$, closed projections arise exactly from characteristic functions of closed subsets of $X$. Closed projections $p$ in $A^{**}$ are also in one-to-one correspondence with norm closed left ideals $L$ of $A$ via

$$L = A^{**}(1 - p) \cap A.$$ 

Note also that the Banach double dual $L^{**}$ of $L$, identified with the weak* closure of $L$ in $A^{**}$, is a weak* closed left ideal of the W*-algebra $A^{**}$. More precisely, we have $L^{**} = A^{**}(1 - p)$. Moreover, we have isometrical isomorphisms $a + L \mapsto ap$ and $x + L^{**} \mapsto xp$ under which

$$A/L \cong Ap \quad \text{and} \quad (A/L)^{**} \cong A^{**}/L^{**} \cong A^{**}p$$

as Banach spaces, respectively [12, 17, 1]. Similarly, we have Banach space isomorphisms between $A/(L + L')$ and $pAp$, and $A^{**}/(L^{**} + L^{**'})$ and $pA^{**}p$, respectively, where $B'$ denotes the set $\{ b^* : b \in B \}$. The significance of these objects arises from the following local versions of the Kadison function representation for $pAp$ and $Ap$.

**Theorem 2** ([6, 3.5], [21]).

1. $pA_{sa}p$ (resp. $pA^{**}_{sa}p$) is isometrically order isomorphic to the Banach space of all continuous (resp. bounded) affine functions on $F(p)$ which vanish at zero.

2. Let $xp$ be an element of $A^{**}p$. Then $xp \in Ap$ if and only if the affine functions $\varphi \mapsto \varphi(x^*x)$ and $\varphi \mapsto \varphi(a^*x)$ are continuous on $F(p)$, $\forall a \in A$. Consequently, $xp \in Ap \Leftrightarrow px^*xp \in pAp$ and $pa^*xp \in pAp$, $\forall a \in A$. 


Denote the extreme boundary of $F(p)$ by $X_0 = (P(A) \cup \{0\}) \cap F(p)$, which consists of all pure states of $A$ supported by $p$ together with the zero functional. Motivated by Theorem 1, we shall attack the following

**Problem 3.** Suppose that $pxp$ in $pA^{**}p$ is uniformly continuous on $X_0$, or continuous on its weak* closure, when we consider $pxp$ as an affine functional on $F(p)$ (Theorem 2). Can we infer that $pxp$ has a continuous atomic part as a member of $pA^{**}p$, i.e., $zpxp = zpap$ for some $a$ in $A$?

A quite satisfactory and affirmative answer for a similar question for elements $xp$ of the left quotient $A^{**}p$ was obtained in [10]. Utilizing the technique and repeating parts of the argument provided in [10], we will achieve positive results here as well. We will impose conditions on the closed projection $p$ (or equivalently, geometric conditions on $F(p)$) to ensure an affirmative answer to Problem 3. We note that the counter examples in [10] indicate that our results are sharp and Problem 3 does not always have an appropriate solution in general. For the convenience of the readers, we borrow an example from [10] and present it at the end of this note.

**2. The results**

Let $A$ be a C*-algebra and $p$ a closed projection in $A^{**}$. Recall that $A_{sa}^{m}$ consists of all limits in $A_{sa}^{**}$ of monotone increasing nets in $A_{sa}$ and $(A_{sa})_{m} = -A_{sa}^{m}$. While $A_{sa}$ consists of continuous affine real-valued functions of $Q(A)$ vanishing at 0 (the Kadison function representation), the norm closure $(A_{sa})^{-}$ of $A_{sa}$ consists of lower semicontinuous elements and the norm closure $(A_{sa})_{m}$ of $(A_{sa})_{m}$ consists of upper semicontinuous elements in $A^{**}$. An element $x$ of $A_{sa}^{**}$ is said to be universally measurable if for each $\varphi$ in $Q(A)$ and $\varepsilon > 0$ there exist a lower semicontinuous element $l$ and an upper semicontinuous element $u$ in $A^{**}$ such that $u \leq x \leq l$ and $\varphi(l - u) < \varepsilon$ [15].

We note that $pA_{sa}p$ consists of continuous affine real-valued functions on $F(p)$. It was shown in [9] that every lower (resp. upper) semicontinuous bounded affine real-valued function on $F(p)$ vanishing at 0 is the restriction of a lower (resp. upper) semicontinuous element in $A_{sa}^{**}$ to $F(p)$; namely it is of the form $pxp$ for some $x$ in $(A_{sa})^{-}$ or $(A_{sa})_{m}$. Analogously, $pxp$ in $pA_{sa}^{**}p$ is said to be universally measurable on $F(p)$ if for each $\varphi$ in $F(p)$ and $\varepsilon > 0$, there exist an $l$ in $(A_{sa})^{-}$ and a $u$ in $(A_{sa})_{m}$ such that $pup \leq pxp \leq plp$ and $\varphi(l - u) < \varepsilon$. And $pxp$ in $pA^{**}p$ is said to be universally measurable on $F(p)$ if both the real and imaginary parts of $pxp$ are.

A Borel measure on $F(p)$ is a boundary measure if it is supported by the closure of the extreme boundary $X_0$ of $F(p)$. A boundary measure $m$ of $F(p)$ with $\|m\| = m(F(p)) = 1$ represents a unique point $\phi$ in $F(p)$, where $\phi(a) = \int \psi(a)dm(\psi), \forall a \in A$. An element $pxp$ of $pA_{sa}^{**}p$ is said to satisfy the barycenter formula if $\phi(x) = \int \psi(x)dm(\psi)$ whenever $m$ is a boundary measure of $F(p)$ representing $\phi$. Semicontinuous affine elements in $pA_{sa}^{**}p$ satisfy the barycenter formula, and so do universally measurable elements.

**Lemma 4.** Let $x$ be an element of $A_{sa}^{**}$ and let $\overline{X}$ be the weak* closure of $X = F(p) \cap P(A)$ in $F(p)$. If $pxp$ satisfies the barycenter formula and is continuous on $\overline{X}$ then $pxp \in pAp$. 
Proof. We give a sketch of the proof here, and refer the readers to [10] in which a similar result is given in full detail. In view of Theorem 2, we need only verify that \( \varphi \mapsto \varphi(x) \) is weak* continuous on \( F(p) \). Suppose \( \varphi_\lambda \) and \( \varphi \) are in \( F(p) \) and \( \varphi_\lambda \to \varphi \) weak*. Since the norm of an element of \( pA_{sa}p \) is determined by the pure states supported by \( p \), we can embed \( pA_{sa}p \) as a closed subspace of the Banach space \( C_\mathbb{R}(X) \) of continuous real-valued functions defined on \( X \). Let \( m_\lambda \) be any positive extension of \( \varphi_\lambda \) from \( pA_{sa}p \) to \( C_\mathbb{R}(X) \) with \( \|m_\lambda\| = \|\varphi_\lambda\| \leq 1 \). Hence, \( (m_\lambda)_\lambda \) is a bounded net in \( M(X) \), the Banach dual space of \( C_\mathbb{R}(X) \), consisting of regular finite Borel measures on the compact Hausdorff space \( X \). Then, by passing to a subnet if necessary, we have \( m_\lambda \to m \) in the weak* topology of \( M(X) \). Clearly, \( m \geq 0 \) and \( m|_{pA_{sa}p} = \varphi \). Since \( pxp \) satisfies the barycenter formula and is continuous on \( X \), we have

\[
\varphi_\lambda(x) = \int_X \psi(x) \, dm_\lambda(\psi) = \int_X \psi(pxp) \, dm_\lambda(\psi) \to \int_X \psi(pxp) \, dm(\psi) = \int_X \psi(x) \, dm = \varphi(x).
\]

\[\square\]

2.1. The case where \( p \) has MSQC. Let \( A \) be a C*-algebra. Recall that a projection \( p \) in \( A^{**} \) is closed if the face \( F(p) = \{ \varphi \in Q(A) : \varphi(1 - p) = 0 \} \) is weak* closed. Analogously, \( p \) is said to be compact [2] (see also [6]) if \( F(p) \cap S(A) \) is weak* closed, where \( S(A) = \{ \varphi \in Q(A) : \|\varphi\| = 1 \} \) is the state space of \( A \). Let \( p \) be a closed projection in \( A^{**} \). Then \( h \) in \( pA_{sa}^{**}p \) is said to be q–continuous [3] on \( p \) if the spectral projection \( E_F(h) \) (computed in \( pA^{**}p \)) is closed for every closed subset \( F \) of \( \mathbb{R} \). Moreover, \( h \) is said to be strongly q–continuous [6] on \( p \) if, in addition, \( E_F(h) \) is compact whenever \( F \) is closed and \( 0 \notin F \). It is known from [6, 3.43] that \( h \) is strongly q–continuous on \( p \) if and only if \( h = pa = ap \) for some \( a \) in \( A_{sa} \). In general, \( h \) in \( pA^{**}p \) is said to be strongly q–continuous on \( p \) if both \( \text{Re} \) and \( \text{Im} \) are.

Denote by \( SQC(p) \) the C*-algebra of all strongly q–continuous elements on \( p \). We say that \( p \) has MSQC (“many strongly q–continuous elements”) if \( SQC(p) \) is \( \sigma \)-weakly dense in \( pA^{**}p \). Brown [8] showed that \( p \) has MSQC if and only if \( pAp = SQC(p) \) if and only if \( pAp \) is an algebra. In particular, every central projection \( p \) (especially, \( p = 1 \)) has MSQC. We provide a partial answer to Problem 3 by the following:

**Theorem 5.** Let \( p \) have MSQC and \( x \) be in \( A^{**} \). Let \( X_0 = (F(p) \cap P(A)) \cup \{0\} \) be the extreme boundary of \( F(p) \). Then \( zpxp \in zpAp \) if and only if \( pzp \) is uniformly continuous on \( X_0 \).

**Proof.** The necessities are obvious and we check the sufficiency. Note that \( pAp \) is now a C*-algebra with the pure state space \( P(pAp) = F(p) \cap P(A) \). The maximal atomic projection of \( pAp \) is \( zp \). By Theorem 1, \( zpxp \) belongs to \( zpAp \) whenever it is uniformly continuous on \( X_0 \). \[\square\]

**Corollary 6.** Let \( p \) have MSQC and \( x \) be in \( A^{**} \). If \( pzp \) is continuous on \( X = F(p) \cap P(A) \) then \( zpxp \in zpAp \).

**Proof.** We simply note that either 0 belongs to \( X \) or 0 is isolated from \( X = F(p) \cap P(A) \) in \( X_0 = (F(p) \cap P(A)) \cup \{0\} \). Consequently, continuity on the compact set \( X \) ensures uniform continuity on \( X_0 \). \[\square\]
2.2. The case where \( p \) is semiatomic. Let \( A \) be a C*-algebra and \( p \) a closed projection in \( A^{**} \). Recall that \( A \) is said to be scattered [13, 14] if \( Q(A) \subseteq zQ(A) \) and \( p \) is said to be atomic [8] if \( F(p) \subseteq zF(p) \). If \( A \) is scattered then every closed projection in \( A^{**} \) is atomic. Moreover, \( A \) is said to be semiscattered [4] if \( \overline{F(A)} \subseteq zQ(A) \). Analogously, we say that a closed projection \( p \) is semiatomic if the weak* closure of \( F(p) \cap P(A) \) contains only atomic positive linear functionals of \( A \), i.e., \( \overline{F(p) \cap P(A)} \subseteq zF(p) \). It is easy to see that if \( A \) is semiscattered then every closed projection in \( A^{**} \) is semiatomic.

The following is a generalization of [7, Theorem 6] in which \( p = 1 \).

**Lemma 7** ([10]). Let \( x \) in \( zpA^{**}p \) be uniformly continuous on \( X_0 = (F(p) \cap P(A)) \cup \{0\} \). Then \( x \) is in the C*-algebra \( B \) generated by \( zpAp \). In particular, \( x = zy \) for some universally measurable element \( y \) of \( pA^{**}p \).

We provide another partial answer to Problem 3 by the following

**Theorem 8.** Let \( p \) be semiatomic and \( x \) be in \( A^{**} \). Let \( X = \overline{F(p) \cap P(A)} \). Then \( zpxp \in zpAp \) if and only if \( pxp \) is continuous on \( X \).

**Proof.** We prove the sufficiency only. Let \( x \) in \( A^{**} \) satisfy the stated condition. Since \( zpxp \) is uniformly continuous on \( X_0 = (P(A) \cap F(p)) \cup \{0\} \), by Lemma 7, there is a universally measurable element \( y \) of \( pA^{**}p \) such that \( zpxp = zy \). Since \( p \) is assumed to be semiatomic, each \( \varphi \) in \( X = \overline{F(A) \cap F(p)} \) is atomic and thus \( \varphi(x) = \varphi(zpxp) = \varphi(zy) = \varphi(y) \). In particular, the universally measurable element \( y \) is continuous on \( X \). It follows from Lemma 4 that \( y \in pAp \). As a consequence, \( zpxp \in zpAp \). \( \square \)

**Example 9** (The full version appeared in [10]). This example tells us that \( p \) having MSQC is necessary in Theorem 5 and continuity on \( \overline{X} \) is necessary in Theorem 8.

Let \( A \) be the scattered C*-algebra of sequences of \( 2 \times 2 \) matrices \( x = (x_n)_{n=1}^{\infty} \) such that

\[
\begin{pmatrix}
  a_n & b_n \\
  c_n & d_n
\end{pmatrix} \longrightarrow x_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}
\]

entrywise, and equipped with the \( \ell^\infty \)-norm. Note that the maximal atomic projection \( z = 1 \) in this case. Let

\[
p_n = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad n = 1, 2, \ldots, \quad \text{and} \quad p_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Then \( p = (p_n)_{n=1}^{\infty} \) is a closed projection in \( A^{**} \). We claim that \( p \) does not have MSQC. In fact, suppose \( x = (x_n)_{n=1}^{\infty} \) in \( A \) is given by

\[
x_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \quad n = 1, 2, \ldots, \quad \text{and} \quad x_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}
\]

such that \( x_n \to x_\infty \). Then \( (pxp)_n = \lambda_n p_n, \quad n = 1, 2, \ldots, \) and \( (pxp)_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \) where \( \lambda_n = \frac{a_n + b_n + c_n + d_n}{2} \to \frac{a + d}{2} \). Consequently, \( (pxp)^2_n = \lambda_n^2 p_n, \quad n = 1, 2, \ldots, \) and \( (pxp)^2_\infty = \begin{pmatrix} a^2 & 0 \\ 0 & d^2 \end{pmatrix} \). If
We must have $\lambda^2_n \to \frac{a^2 + d^2}{2}$. This occurs exactly when $a = d$. In particular, $pAp$ is not an algebra and thus $p$ does not have MSQC.

On the other hand, the set $X = P(A) \cap F(p)$ of all pure states in $F(p)$ consists exactly of $\varphi_n, \psi_1$ and $\psi_2$ which are given by

$$\varphi_n(x) = \text{tr}(x_n p_n), \quad n = 1, 2, \ldots,$$

and

$$\psi_1(x) = a, \quad \psi_2(x) = d,$$

where $x = (x_n)_{n=1}^\infty \in A$ and $x_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Since $\varphi_n \to \frac{1}{2}(\psi_1 + \psi_2) \neq 0$, $X_0 = X \cup \{0\}$ is discrete. Consider $y = (y_n)_{n=1}^\infty$ in $A^{**}$ given by

$$y_n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad n = 1, 2, \ldots, \quad \text{and} \quad y_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, the universally measurable element $pyp$ is uniformly continuous on $X_0$ but $pyp \notin pAp$. \hfill $\Box$

### References


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