ATTRACTIVE POINT AND WEAK CONVERGENCE THEOREMS FOR NEW GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. In this paper we introduce a broad class of nonlinear mappings which contains the class of contractive mappings and the class of generalized hybrid mappings in a Hilbert space. Then we prove an attractive point theorem for such mappings in a Hilbert space. Furthermore, we prove a mean convergence theorem of Baillon’s type without convexity in a Hilbert space. Finally, we prove a weak convergence theorem of Mann’s type [12] without closedness. These results generalize attractive point, mean convergence and weak convergence theorems proved by Takahashi and Takeuchi [18], and Kohcurek, Takahashi and Yao [8].

1. Introduction

Throughout this paper, we denote by \( \mathbb{N} \) the set of positive integers and by \( \mathbb{R} \) the set of real numbers. Let \( H \) be a real Hilbert space and let \( C \) be a nonempty subset of \( H \). Let \( T \) be a mapping of \( C \) into \( H \). Then we denote by \( F(T) \) the set of fixed points of \( T \) and by \( A(T) \) the set of attractive points [18] of \( T \), i.e.,

\[
\begin{align*}
(i) \ F(T) &= \{ z \in C : Tz = z \}; \\
(ii) \ A(T) &= \{ z \in H : \|Tx - z\| \leq \|x - z\|, \forall x \in C \}.
\end{align*}
\]

We know from [18] that \( A(T) \) is closed and convex; see Lemma 2.3 in Section 2. This property is important. A mapping \( T \) of \( C \) into \( H \) is said to be contractive if there exists a real number \( \alpha \) with \( 0 < \alpha < 1 \) such that

\[
\|Tx - Ty\| \leq \alpha \|x - y\|
\]

for all \( x, y \in C \). From Banach [2] it is known that any contractive mapping of a closed subset \( C \) of \( H \) into itself has a unique fixed point. Let \( C \) be a nonempty subset of \( H \). A mapping \( T \) of \( C \) into \( H \) is said to be nonexpansive if

\[
\|Tx - Ty\| \leq \|x - y\|
\]

for all \( x, y \in C \). From Baillon [1] we know the following mean convergence theorem in a Hilbert space.

**Theorem 1.1.** Let \( H \) be a Hilbert space, let \( C \) be a nonempty closed convex subset of \( H \) and let \( T \) be a nonexpansive mapping of \( C \) into \( C \) with a fixed point. Then for any \( x \in C \),

\[
S_nx = \frac{1}{n} \sum_{k=0}^{n-1} T^kx
\]

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is weakly convergent to a fixed point of $T$.

Kohsaka and Takahashi [10], and Takahashi [17] introduced the following non-linear mappings. A mapping $T : C \to H$ is called nonspreading [10] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. A mapping $T : C \to H$ is called hybrid [17] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$; see also Iemoto and Takahashi [5] and Kohsaka and Takahashi [9]. Kocourek, Takahashi and Yao [8] introduced a broad class of nonlinear mappings which contains the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. A mapping $T : C \to H$ is called generalized hybrid [8] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. We call such a mapping an $(\alpha, \beta)$-generalized hybrid mapping. We know that $(1,0)$, $(2,1)$ and $(\frac{1}{2}, \frac{1}{2})$-generalized hybrid mappings are nonexpansive, nonspreading and hybrid mappings, respectively. Kocourek, Takahashi and Yao [8] proved a mean convergence theorem which generalizes the Baillon’s theorem (Theorem 1.1); see also Takahashi and Yao [20]. Recently, Takahashi and Takeuchi [18] proved the Kocourek, Takahashi and Yao’s mean convergence theorem without convexity.

In this paper, motivated by Kocourek, Takahashi and Yao [8], and Takahashi and Takeuchi [18], we introduce a broad class of nonlinear mappings of $C$ into $H$ which contains the class of contractive mappings and the class of generalized hybrid mappings. Then we prove an attractive point theorem for such mappings in a Hilbert space. Furthermore, we prove a mean convergence theorem of Baillon’s type without convexity in a Hilbert space. Finally, we prove a weak convergence theorem of Mann’s type [12] without closedness. These results generalize attractive point, mean convergence and weak convergence theorems proved by Takahashi and Takeuchi [18], and Kocourek, Takahashi and Yao [8].

2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let $A$ be a nonempty subset of $H$. We denote by $\overline{\text{co}}A$ the closure of the convex hull of $A$. In a Hilbert space, it is known that

$$\|ax + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see [16]. Furthermore, in a Hilbert space, we have that

$$2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2$$

for all $x, y, z, w \in H$. Let $C$ be a nonempty subset of $H$ and let $T$ be a mapping from $C$ into $H$. We denote by $F(T)$ the set of fixed points of $T$. A mapping $T$ from $C$ into $H$ with $F(T) \neq \emptyset$ is called quasi-nonexpansive if $\|Tx - u\| \leq \|x - u\|$ for any $x \in C$ and $u \in F(T)$. It is well-known that if $T : C \to H$ is quasi-nonexpansive and $C$ is closed and convex, then $F(T)$ is closed and convex; see Ito and Takahashi [6]. It is not difficult to prove such a result in a Hilbert space. In fact, for proving
that \( F(T) \) is closed, take a sequence \( \{z_n\} \subset F(T) \) with \( z_n \to z \). Since \( C \) is weakly closed, we have \( z \in C \). Furthermore, from
\[
\|z - Tz\| \leq \|z - z_n\| + \|z_n - Tz\| \leq 2\|z - z_n\| \to 0,
\]
we have that \( z \) is a fixed point of \( T \) and so \( F(T) \) is closed. Let us show that \( F(T) \) is convex. For \( x, y \in F(T) \) and \( \alpha \in [0, 1] \), put \( z = \alpha x + (1-\alpha)y \). Then we have from (2.1) that
\[
\|z - Tz\|^2 = \|\alpha x + (1-\alpha)y - Tz\|^2
\]
\[
= \alpha\|x - Tz\|^2 + (1-\alpha)\|y - Tz\|^2 - \alpha(1-\alpha)\|x-y\|^2
\]
\[
\leq \alpha\|x - z\|^2 + (1-\alpha)\|y - z\|^2 - \alpha(1-\alpha)\|x-y\|^2
\]
\[
= \alpha(1-\alpha)^2\|x - y\|^2 + (1-\alpha)\alpha^2\|x - y\|^2 - \alpha(1-\alpha)\|x-y\|^2
\]
\[
= \alpha(1-\alpha)(1-\alpha + 1)\|x - y\|^2
\]
\[
= 0.
\]
This implies \( Tz = z \). Thus \( F(T) \) is convex. Let \( D \) be a nonempty closed convex subset of \( H \) and \( x \in H \). We know that there exists a unique nearest point \( z \in D \) such that \( \|x - z\| = \inf_{y \in D} \|x - y\| \). We denote such a correspondence by \( z = P_{D}x \).

The mapping \( P_{D} \) is called the metric projection of \( H \) onto \( D \). It is known that \( P_{D} \) is nonexpansive and
\[
\langle x - P_{D}x, P_{D}x - u \rangle \geq 0
\]
for all \( x \in H \) and \( u \in D \); see [16] for more details. For proving main results in this paper, we also need the following lemma proved by Takahashi and Toyoda [19].

**Lemma 2.1.** Let \( D \) be a nonempty closed convex subset of \( H \). Let \( P \) be the metric projection from \( H \) onto \( D \). Let \( \{u_n\} \) be a sequence in \( H \). If \( \|u_n+1 - u\| \leq \|u_n - u\| \) for all \( u \in D \) and \( n \in \mathbb{N} \), then \( \{P_{D}u_n\} \) converges strongly to some \( u_0 \in D \).

Let \( l^\infty \) be the Banach space of bounded sequences with supremum norm. Let \( \mu \) be an element of \( (l^\infty)^* \) (the dual space of \( l^\infty \)). Then, we denote by \( \mu(f) \) the value of \( \mu \) at \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \). Sometimes, we denote by \( \mu_n(x_n) \) the value \( \mu(f) \). A linear functional \( \mu \) on \( l^\infty \) is called a mean if \( \mu(e) = \|\mu\| = 1 \), where \( e = (1, 1, 1, \ldots) \). A mean \( \mu \) is called a Banach limit on \( l^\infty \) if \( \mu_n(x_n+1) = \mu_n(x_n) \) for all \( (x_n) \in l^\infty \). We know that there exists a Banach limit on \( l^\infty \). If \( \mu \) is a Banach limit on \( l^\infty \), then for \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \),
\[
\lim_{n \to \infty} \inf_{n} x_n \leq \mu_n(x_n) \leq \lim_{n \to \infty} \sup_{n} x_n.
\]

In particular, if \( f = (x_1, x_2, x_3, \ldots) \in l^\infty \) and \( x_n \to a \in \mathbb{R} \), then we have \( \mu(f) = \mu_n(x_n) = a \). See [15] for the proof of existence of a Banach limit and its other elementary properties. Using means and the Riesz theorem, we can obtain the following result; see [11], [13] and [15].

**Lemma 2.2.** Let \( H \) be a Hilbert space, let \( \{x_n\} \) be a bounded sequence in \( H \) and let \( \mu \) be a mean on \( l^\infty \). Then there exists a unique point \( z_0 \in \overline{C}(x_n | n \in \mathbb{N}) \) such that
\[
\mu_n(x_n, y) = \langle z_0, y \rangle, \quad \forall y \in H.
\]

The following result obtained by Takahashi and Takeuchi [18] is important in this paper.
Lemma 2.3. Let $H$ be a Hilbert space, let $C$ be a nonempty subset of $H$ and let $T$ be a mapping from $C$ into $H$. Then $A(T)$ is a closed and convex subset of $H$.

We also have the following result.

Lemma 2.4. Let $H$ be a Hilbert space, let $C$ be a nonempty subset of $H$ and let $T$ be a quasi-nonexpansive mapping from $C$ into $H$. Then $A(T) \cap C = F(T)$.

Proof. Let $z \in A(T) \cap C$. From $z \in A(T)$ we have that
\[ \|Tx - z\| \leq \|x - z\|, \quad \forall x \in C. \]
From $z \in C$ we have that $\|Tz - z\| = \|z - z\| = 0$ and hence $z \in F(T)$. Conversely, let $z \in F(T)$. Since $T : C \to H$ is quasi-nonexpansive, we have that
\[ \|Tx - z\| \leq \|x - z\|, \quad \forall x \in C. \]
This implies $z \in A(T)$. It is obvious that $z \in C$. Thus $z \in A(T) \cap C$. This completes the proof. $\square$

3. Attractive point theorems

Let $H$ be a real Hilbert space and let $C$ be a nonempty subset of $H$. A mapping $T$ from $C$ into $H$ is called normally generalized hybrid if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that
\begin{enumerate}
  \item $\alpha + \beta + \gamma + \delta \geq 0$;
  \item $\alpha + \gamma > 0$, or $\alpha + \beta > 0$;
  \item $\alpha \|Tx - Ty\|^2 + \beta \|x - Ty\|^2 + \gamma \|Tx - y\|^2 + \delta \|x - y\|^2 \leq 0$, \quad \forall x, y \in C.
\end{enumerate}
Such a mapping $T$ is called $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid. If $\alpha + \beta = -\gamma - \delta = 1$, then an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping is a generalized hybrid mapping in the sense of Kocourek, Takahashi and Yao [8]. A normally generalized hybrid mapping $T : C \to H$ with a fixed point is quasi-nonexpansive. In fact, if $y$ is a fixed point of $T$ in (3), then we have that
\[ \alpha \|Tx - y\|^2 + \beta \|x - y\|^2 + \gamma \|Tx - y\|^2 + \delta \|x - y\|^2 \leq 0 \]
and hence
\begin{equation}
(\alpha + \gamma) \|Tx - y\|^2 \leq (-\beta - \delta) \|x - y\|^2.
\end{equation}
Since $\alpha + \gamma \geq -\beta - \delta$ and $\alpha + \gamma > 0$, we have that
\[ \|Tx - y\|^2 \leq \frac{-\beta - \delta}{\alpha + \gamma} \|x - y\|^2 \leq \|x - y\|^2. \]
This implies that $T$ is quasi-nonexpansive. Similarly, we have the desired result in the case of $\alpha + \beta > 0$. We first prove an attractive fixed point theorem for normally generalized hybrid mappings in a Hilbert space.

Theorem 3.1. Let $H$ be a real Hilbert space, let $C$ be a nonempty subset of $H$ and let $T$ be an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping from $C$ into itself. Then $T$ has an attractive point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded. Additionally, if $C$ is closed and convex, then $T$ has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded. In particular, a fixed point of $T$ is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the condition (1).
Proof. Suppose that $T$ has an attractive point $z$. Then $\|Tx - z\| \leq \|x - z\|$ for all $x \in C$. Therefore $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded. Conversely, suppose that there exists $z \in C$ such that $\{T^n z \mid n = 0, 1, \ldots\}$ is bounded. Since $T$ is an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping of $C$ into itself, we have that

$$\alpha\|Tx - T^{n+1}z\|^2 + \beta\|x - T^{n+1}z\|^2 + \gamma\|Tx - T^nz\|^2 + \delta\|x - T^nz\|^2 \leq 0$$

for all $n \in \mathbb{N} \cup \{0\}$ and $x \in C$. Since $\{T^n z\}$ is bounded, we can apply a Banach limit $\mu$ to both sides of the inequality. Thus we have that

$$\mu_n\|T x - T^n z\|^2 \leq 0.$$ 

From $\|T x - T^n z\|^2 = \|T x - x\|^2 + 2\langle T x - x, x - T^n z \rangle + \|x - T^n z\|^2$, we also have that

$$\mu_n\|T x - x\|^2 + 2(\alpha + \gamma)\mu_n\langle T x - x, x - T^n z \rangle + (\alpha + \gamma + \beta + \delta)\mu_n\|x - T^n z\|^2 \leq 0.$$ 

From (1) $\alpha + \gamma + \beta + \delta \geq 0$, we have that

$$(\alpha + \gamma)\|T x - x\|^2 + 2(\alpha + \gamma)\mu_n\langle T x - x, x - T^n z \rangle \leq 0.$$ 

Since there exists $p \in C$ from Lemma 2.2 such that

$$\mu_n\langle y, T^n z \rangle = \langle y, p \rangle$$

for all $y \in H$, we have from (3.2) that

$$(\alpha + \gamma)\|T x - x\|^2 + 2(\alpha + \gamma)\mu_n\langle T x - x, x - T^n z \rangle = 0.$$ 

From (3.3) and (2.2) we obtain that

$$(\alpha + \gamma)\|T x - x\|^2 + (\alpha + \gamma)(\|T x - p\|^2 - \|T x - x\|^2 - \|x - p\|^2) \leq 0$$

and hence

$$(\alpha + \gamma)(\|T x - p\|^2 - \|x - p\|^2) \leq 0.$$ 

Since $\alpha + \gamma > 0$, we have that

$$\|T x - p\|^2 \leq \|x - p\|^2$$

for all $x \in C$. This implies $p \in A(T)$. In the case of $\alpha + \beta > 0$, we can obtain the result by replacing the variables $x$ and $y$. Additionally, if $C$ is closed and convex, then we have from $\{T^n x\} \subset C$ that

$$p \in \overline{\text{co}}\{T^n x : n \in \mathbb{N}\} \subset C.$$ 

Since $p \in A(T)$ and $p \subset C$, we have that

$$\|Tp - p\| \leq \|p - p\| = 0$$

and hence $p \in F(T)$. Conversely, if $z \in F(T)$, then it is obvious that $\{T^n z\} = \{z\}$ is bounded.

Next suppose that $\alpha + \beta + \gamma + \delta > 0$. Let $p_1$ and $p_2$ be fixed points of $T$. Then we have that

$$\alpha\|Tp_1 - Tp_2\|^2 + \beta\|p_1 - Tp_2\|^2 + \gamma\|Tp_1 - p_2\|^2 + \delta\|p_1 - p_2\|^2 \leq 0$$

and hence $p_1 = p_2$. Therefore a fixed point of $T$ is unique. This completes the proof. \qed
Remark 3.1. We can also prove Theorem 3.1 by using the following condition instead of the condition (2):

\[(2)' \beta + \delta < 0, \text{ or } \gamma + \delta < 0.\]

In the case of the condition \(\beta + \delta < 0\), we obtain from (1) that

\[\beta + \delta \geq -\alpha - \gamma.\]

Thus we obtain the desired result by Theorem 3.1. Similarly, for the case of \(\gamma + \delta < 0\), we can obtain the result by using the case of \(\alpha + \beta > 0\).

As a direct consequence of Theorem 3.1, we obtain the following theorem.

**Theorem 3.2.** Let \(H\) be a Hilbert space, let \(C\) be a nonempty bounded subset of \(H\) and let \(T\) be an \((\alpha, \beta, \gamma, \delta)\)-normally generalized hybrid mapping from \(C\) into itself. Then \(T\) has an attractive point. Additionally, if \(C\) is closed and convex, then \(T\) has a fixed point. In particular, a fixed point of \(T\) is unique in the case of \(\alpha + \beta + \gamma + \delta > 0\) on the condition (1).

Note that an \((\alpha, \beta, \gamma, \delta)\)-normally generalized hybrid mapping \(T\) above with \(\alpha = 1, \beta = \gamma = 0\) and \(-1 < \delta < 0\) is a contractive mapping. Using Theorem 3.1, we can show an attractive point theorem for contractive mappings in a Hilbert space.

**Theorem 3.3.** Let \(H\) be a Hilbert space, let \(C\) be a nonempty subset of \(H\) and let \(T\) be a contractive mapping from \(C\) into \(C\), that is, there exists a real number \(\alpha\) with \(0 < \alpha < 1\) such that

\[\|Tx - Ty\| \leq \alpha\|x - y\|\]

for all \(x, y \in C\). Then \(T\) has an attractive point.

**Proof.** Let \(x \in C\). We have that

\[\|T^n x - x\| \leq \|T^n x - T^n-1 x\| + \|T^n-1 x - T^{n-2} x\| + \cdots + \|Tx - x\| \leq (\alpha^{n-1} + \alpha^{n-2} + \cdots + 1)\|Tx - x\| \leq \frac{1}{1 - \alpha}\|Tx - x\|.

Then \(\{T^n x \mid n = 0, 1, \ldots\}\) is bounded. By Theorem 3.1 \(T\) has an attractive point. \(\square\)

Using Theorem 3.1, we can show the following attractive point theorem for generalized hybrid mappings in a Hilbert space.

**Theorem 3.4** (Takahashi and Takeuchi [18]). Let \(H\) be a Hilbert space, let \(C\) be a nonempty subset of \(H\) and let \(T\) be a generalized hybrid mapping from \(C\) into \(C\), that is, there exist real numbers \(\alpha\) and \(\beta\) such that

\[\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2\]

for all \(x, y \in C\). Then \(T\) has an attractive point if and only if there exists \(z \in C\) such that \(\{T^n z \mid n = 0, 1, \ldots\}\) is bounded.

**Proof.** An \((\alpha, \beta)\)-generalized hybrid mapping \(T\) is an \((\alpha, 1 - \alpha, -\beta, -(1 - \beta))\)-normally generalized hybrid mapping such that \(\alpha + (1 - \alpha) - \beta - (1 - \beta) = 0 \geq 0\) and \(\alpha + (1 - \alpha) = 1 > 0\). Then we have the desired result from Theorem 3.1. \(\square\)
4. Mean convergence theorems

In this section, using the technique developed by Takahashi [13], we prove a mean convergence theorem of Baillon’s type without convexity for normally generalized hybrid mappings in a Hilbert space.

Theorem 4.1. Let $H$ be a Hilbert space, let $C$ be a nonempty subset of $H$ and let $T$ be an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping from $C$ into $C$ which has an attractive point. Let $P$ be the metric projection from $H$ onto $A(T)$. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to an attractive point $p$ of $T$, where $p = \lim_{n \to \infty} P T^n x$.

Proof. Since $A(T)$ is nonempty, we have that $\{T^n x\}$ is bounded for all $x \in C$. Since

$$\|S_n x - y\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \|T^k x - y\| \leq \|x - y\|$$

for all $n \in \mathbb{N} \cup \{0\}$ and $y \in A(T)$, we have that $\{S_n x \mid n = 0, 1, \ldots\}$ is bounded. Then there exists a strictly increasing sequence $\{n_i\}$ and $p \in H$ such that $\{S_{n_i} x \mid i = 0, 1, \ldots\}$ is weakly convergent to $p$. We first show that $p \in A(T)$. Indeed, since $T$ is an $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid mapping of $C$ into itself, we have that

$$\alpha \|Tz - T^{k+1} x\|^2 + \beta \|z - T^{k+1} x\|^2 + \gamma \|Tz - T^k x\|^2 + \delta \|z - T^k x\|^2 \leq 0$$

for all $k \in \mathbb{N} \cup \{0\}$ and $z \in C$. We also have that

$$\gamma \|Tz - T^k x\|^2 = (\alpha + \gamma)(\|Tz - z\|^2 + \|z - T^k x\|^2 + 2(Tz - z, z - T^k x)) - \alpha \|Tz - T^k x\|^2.$$

Since $-\alpha - \beta - \gamma \leq \delta$ from (1), we obtain that

$$\alpha \|Tz - T^{k+1} x\|^2 - \|Tz - T^k x\|^2 + \beta (\|z - T^{k+1} x\|^2 - \|z - T^k x\|^2) + 2(\alpha + \gamma)(Tz - z, z - T^k x) + (\alpha + \gamma)\|z - Tz\|^2 \leq 0.$$

Summing up these inequalities with respect to $k = 0, 1, \ldots, n - 1$ and dividing by $n$, we obtain that

$$\frac{\alpha}{n} \|Tz - T^n x\|^2 - \|Tz - x\|^2 + \frac{\beta}{n} (\|z - T^n x\|^2 - \|z - x\|^2) + 2(\alpha + \gamma)(Tz - z, z - S_n x) + (\alpha + \gamma)\|z - Tz\|^2 \leq 0.$$

Replacing $n$ by $n_i$, we have that

$$\frac{\alpha}{n_i} \|Tz - T^{n_i} x\|^2 - \|Tz - x\|^2 + \frac{\beta}{n_i} (\|z - T^{n_i} x\|^2 - \|z - x\|^2) + 2(\alpha + \gamma)(Tz - z, z - S_{n_i} x) + (\alpha + \gamma)\|z - Tz\|^2 \leq 0.$$

Letting $i \to \infty$, we obtain that

$$2(\alpha + \gamma)(Tz - z, z - p) + (\alpha + \gamma)\|z - Tz\|^2 \leq 0.$$

As in the proof of Theorem 3.1, we obtain that

$$(\alpha + \gamma)(\|Tz - p\|^2 - \|z - Tz\|^2 + \|z - p\|^2) + (\alpha + \gamma)\|z - Tz\|^2 \leq 0.$$
From $\alpha + \gamma > 0$ we have that for all $z \in C$,
\[
\|p - Tz\|^2 \leq \|p - z\|^2.
\]
This implies $p \in A(T)$. Similarly, we can obtain the desired result for the case of $\alpha + \beta > 0$.

Since $A(T)$ is nonempty, closed and convex from Lemma 2.3, the metric projection $P$ from $H$ onto $A(T)$ is well-defined. We also obtain that
\[
\|T^{n+1}x - y\| \leq \|T^nx - y\|
\]
for all $n \in \mathbb{N} \cup \{0\}$ and $y \in A(T)$. By Lemma 2.1, there exists $q \in A(T)$ such that \{PT^n x \mid n = 0, 1, \ldots\} is convergent to $q$. To complete the proof, we show that $q = p$. Note that the metric projection $P$ satisfies
\[
\langle z - Pz, Pz - u \rangle \geq 0
\]
for all $z \in H$ and $u \in A(T)$; see [15]. Therefore
\[
\langle T^k x - PT^k x, PT^k x - y \rangle \geq 0
\]
for all $k \in \mathbb{N} \cup \{0\}$ and $y \in A(T)$. Since $P$ is the metric projection from $H$ onto $A(T)$, we obtain that
\[
\|T^nx - PT^n x\| \leq \|x - y\|
\]
that is, \{\|T^nx - PT^n x\| \mid n = 0, 1, \ldots\} is non-increasing. Therefore we obtain
\[
\langle T^k x - PT^k x, y - q \rangle \leq \|T^k x - PT^k x\| \cdot \|PT^k x - q\|
\]
\[
\leq \|x - y\| \cdot \|PT^k x - q\|
\]
for all $k \in \mathbb{N} \cup \{0\}$ and $y \in A(T)$. Since $\{S_n x \mid i = 0, 1, \ldots\}$ is weakly convergent to $p$ and \{PT^n x \mid n = 0, 1, \ldots\} is convergent to $q$, we obtain that
\[
\langle p - q, y - q \rangle \leq 0.
\]
Putting $y = p$, we obtain
\[
\|p - q\|^2 \leq 0
\]
and hence $q = p$. This completes the proof.

As in the proof of Theorem 3.4, from Theorem 4.1 we can prove the following mean convergence theorem for generalized hybrid mappings in a Hilbert space.

**Theorem 4.2** (Takahashi and Takeuchi [18]). Let $H$ be a Hilbert space, let $C$ be a nonempty subset of $H$ and let $T$ be a generalized hybrid mapping from $C$ into $C$, that is, there exist $\alpha, \beta \in \mathbb{R}$ such that
\[
\alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta)\|x - y\|^2
\]
for all \( x, y \in C \). Suppose \( A(T) \neq \emptyset \) and let \( P \) be the metric projection from \( H \) onto \( A(T) \). Then for any \( x \in C \),

\[
S_nx = \frac{1}{n} \sum_{k=0}^{n-1} T^k x
\]

is weakly convergent to an attractive point \( p \) of \( T \), where \( p = \lim_{n \to \infty} P T^n x \).

5. Weak convergence theorems of Mann’s type

In this section, we prove a weak convergence theorem of Mann’s type [12] for normally generalized hybrid mappings in a Hilbert space. Before proving the result, we need the following lemma.

**Lemma 5.1.** Let \( H \) be a Hilbert space and let \( C \) be a nonempty subset of \( H \). Let \( T : C \to H \) be a normally generalized hybrid mapping. If \( x_n \to z \) and \( x_n - T x_n \to 0 \), then \( z \in A(T) \).

**Proof.** Since \( T : C \to H \) is a normally generalized hybrid mapping, there exist \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \) such that (1) \( \alpha + \beta + \gamma + \delta \geq 0 \), (2) \( \alpha + \gamma > 0 \), or \( \alpha + \beta > 0 \) and

\[
\alpha \| tx - ty \|^2 + \beta \| x - ty \|^2 + \gamma \| tx - y \|^2 + \delta \| x - y \|^2 \leq 0
\]

for all \( x, y \in C \). Suppose \( x_n \to z \) and \( x_n - T x_n \to 0 \). Replacing \( x \) by \( x_n \) in (5.1), we have that

\[
\alpha \| T x_n - Ty \|^2 + \beta \| x_n - Ty \|^2 + \gamma \| T x_n - y \|^2 + \delta \| x_n - y \|^2 \leq 0.
\]

From this inequality, we have that

\[
\alpha (\| T x_n - x_n \|^2 + \| x_n - Ty \|^2 + 2 \langle T x_n - x_n, x_n - Ty \rangle) + \beta \| x_n - Ty \|^2
\]

\[
+ \gamma (\| T x_n - x_n \|^2 + \| x_n - y \|^2 + 2 \langle T x_n - x_n, x_n - y \rangle) + \delta \| x_n - y \|^2 \leq 0.
\]

We apply a Banach limit \( \mu \) to both sides of this inequality. We have that

\[
\alpha \mu_n (\| T x_n - x_n \|^2 + \| x_n - Ty \|^2 + 2 \langle T x_n - x_n, x_n - Ty \rangle) + \beta \mu_n \| x_n - Ty \|^2
\]

\[
+ \gamma \mu_n (\| T x_n - x_n \|^2 + \| x_n - y \|^2 + 2 \langle T x_n - x_n, x_n - y \rangle) + \delta \mu_n \| x_n - y \|^2 \leq 0.
\]

and hence

\[
\alpha \mu_n \| x_n - Ty \|^2 + \beta \mu_n \| x_n - Ty \|^2
\]

\[
+ \gamma \mu_n \| x_n - y \|^2 + \delta \mu_n \| x_n - y \|^2 \leq 0.
\]

Thus we have

\[
(\alpha + \beta) \mu_n \| x_n - Ty \|^2 + (\gamma + \delta) \mu_n \| x_n - y \|^2 \leq 0.
\]

From \( \| x_n - Ty \|^2 = \| x_n - y \|^2 + \| y - Ty \|^2 + 2 \langle x_n - y , y - Ty \rangle \), we also have

\[
(\alpha + \beta) (\mu_n \| x_n - y \|^2 + \| y - Ty \|^2 + 2 \mu_n \langle x_n - y , y - Ty \rangle) + (\gamma + \delta) \mu_n \| x_n - y \|^2 \leq 0.
\]

From \( \alpha + \beta + \gamma + \delta \geq 0 \) we obtain that

\[
(\alpha + \beta) (\| y - Ty \|^2 + 2 \mu_n \langle x_n - y , y - Ty \rangle) \leq 0.
\]

Since \( x_n \to z \), we have that

\[
(\alpha + \beta) (\| y - Ty \|^2 + 2 (z - y , y - Ty)) \leq 0.
\]

Using (2.2), we have that

\[
(\alpha + \beta) (\| y - Ty \|^2 + \| z - Ty \|^2 - \| z - y \|^2 - \| y - Ty \|^2) \leq 0.
\]

It follows that

\[
(\alpha + \beta) (\| y - Ty \|^2 + \| z - Ty \|^2 - \| z - y \|^2 - \| y - Ty \|^2) = 0.
\]

Hence

\[
\| z - T y \|^2 = 0.
\]

Thus \( z \in A(T) \).
Since $\alpha + \beta > 0$, we have that
\[
\|z - Ty\| - \|z - y\| \leq 0
\]
for all $y \in C$. This implies $z \in A(T)$. Similarly, we can obtain the desired result for the case of $\alpha + \gamma > 0$. This completes the proof.

We can prove the following theorem by using Lemma 5.1 and the technique developed by Ibaraki and Takahashi [3,4].

**Theorem 5.1.** Let $H$ be a Hilbert space and let $C$ be a convex subset of $H$. Let $T : C \to C$ be a normally generalized hybrid mapping with $A(T) \neq \emptyset$ and let $P$ be the metric projection of $H$ onto $A(T)$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N}.
\]
Then $\{x_n\}$ converges weakly to an element $v \in A(T)$, where $v = \lim_{n \to \infty} Px_n$.

**Proof.** Let $z \in A(T)$. Then we have that
\[
\|x_{n+1} - z\|^2 = \|\alpha_n x_n + (1 - \alpha_n)Tx_n - z\|^2
\]
\[
\leq \alpha_n \|x_n - z\|^2 + (1 - \alpha_n)\|Tx_n - z\|^2
\]
\[
\leq \alpha_n \|x_n - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2
\]
\[
= \|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2.
\]

Thus we have
\[
\alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2.
\]

Since $\lim_{n \to \infty} \|x_n - z\|^2$ exists and $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$, we have that
\[
(5.3) \quad \|Tx_n - x_n\| \to 0.
\]

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to v$. By Lemma 5.1 and (5.3), we obtain that $v \in A(T)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \to v_1$ and $x_{n_j} \to v_2$. To complete the proof, we show $v_1 = v_2$. We know $v_1, v_2 \in A(T)$ and hence $\lim_{n \to \infty} \|x_n - v_1\|^2$ and $\lim_{n \to \infty} \|x_n - v_2\|^2$ exist. Put
\[
a = \lim_{n \to \infty} (\|x_n - v_1\|^2 - \|x_n - v_2\|^2).
\]

Note that for $n = 1, 2, \ldots,$
\[
\|x_n - v_1\|^2 - \|x_n - v_2\|^2 = 2(x_n, v_2 - v_1) + \|v_1\|^2 - \|v_2\|^2.
\]

From $x_{n_i} \to v_1$ and $x_{n_j} \to v_2$, we have
\[
(5.4) \quad a = 2(v_1, v_2 - v_1) + \|v_1\|^2 - \|v_2\|^2
\]
and
\begin{equation}
(5.5) \quad a = 2(v_2, v_2 - v_1) + \|v_1\|^2 - \|v_2\|^2.
\end{equation}
Combining (5.4) and (5.5), we obtain $0 = 2(v_2 - v_1, v_2 - v_1)$. Thus we obtain $v_2 = v_1$. This implies that \{x_n\} converges weakly to an element $v \in A(T)$. Since $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $z \in A(T)$ and $n \in \mathbb{N}$, we obtain from Lemma 2.1 that \{Px_n\} converges strongly to an element $p \in A(T)$. On the other hand, we have from the property of $P$ that
\[
\langle x_n - Px_n, Px_n - u \rangle \geq 0
\]
for all $u \in A(T)$ and $n \in \mathbb{N}$. Since $x_n \rightharpoonup v$ and $Px_n \rightarrow p$, we obtain
\[
\langle v - p, p - u \rangle \geq 0
\]
for all $u \in A(T)$. Putting $u = v$, we obtain $p = v$. This means $v = \lim_{n \to \infty} Px_n$. This completes the proof. 

Using Theorem 5.1, we can show the following weak convergence theorem of Mann’s type for generalized hybrid mappings in a Hilbert space.

**Theorem 5.2** (Kocourek, Takahashi and Yao [8]). Let $H$ be a Hilbert space and let $C$ be a closed convex subset of $H$. Let $T : C \rightarrow C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$. Let \{\alpha_n\} be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\lim \inf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$. Suppose \{x_n\} is the sequence generated by $x_1 = x \in C$ and
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \in \mathbb{N}.
\]
Then the sequence \{x_n\} converges weakly to an element $v \in F(T)$.

**Proof.** As in the proof of Theorem 3.4, a generalized hybrid mapping is a normally generalized hybrid mapping. Since \{x_n\} $\subset C$ and $C$ is closed and convex, we have from Theorem 5.1 that $v \in A(T) \cap C$. A normally generalized hybrid mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive, we have from Lemma 2.4 that $A(T) \cap C = F(T)$. Thus \{x_n\} converges weakly to an element $v \in F(T)$. 

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