NONLINEAR ERGODIC THEOREM FOR POSITIVELY
HOMOGENEOUS NONEXPANSIVE MAPPINGS
IN BANACH SPACES

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Abstract. Recently, two retractions (projections) which are different from
the metric projection and the sunny nonexpansive retraction in a Banach space
were found. In this paper, using nonlinear analytic methods and new retrac-
tions, we prove a nonlinear ergodic theorem for positively homogeneous and
nonexpansive mappings in a uniformly convex Banach space. The limit points
are characterized by using new retractions.

1. Introduction

Let $E$ be a real Banach space and let $C$ be a nonempty subset of $E$. Let $\mathbb{N}$
and $\mathbb{R}$ be the sets of positive integers and real numbers, respectively. A mapping
$T : C \to C$ is called nonexpansive if

\begin{equation}
||Tx - Ty|| \leq ||x - y||, \quad \forall x, y \in C.
\end{equation}

We denote by $F(T)$ the set of fixed points of $T$. In 1938, Yosida [28] proved the
following strong convergence theorem for linear continuous operators in a Banach
space.

**Theorem 1.1 (Yosida [28]).** Let $E$ be a Banach space and let $T$ be a linear operator
of $E$ into itself. Suppose that there exists a constant $C$ with $\|T^n\| \leq C$ for $n \in \mathbb{N}$
and $T$ is weakly completely continuous, i.e., $T$ maps the closed unit ball of $E$ into
a weakly compact subset of $E$. Then, for each $x \in E$, the Cesàro means

$$S_n x = \frac{1}{n} \sum_{k=1}^{n} T^k x$$

converge strongly as $n \to \infty$ to $z \in F(T)$.

On the other hand, Baillon [2] proved the first nonlinear ergodic theorem for
nonexpansive mappings in a Hilbert space.

**Theorem 1.2 (Baillon [2]).** Let $H$ be a Hilbert space and let $C$ be a nonempty,
closed and convex subset of $H$. Let $T : C \to C$ be a nonexpansive mapping with
$F(T) \neq \emptyset$. Then, for any $x \in C$, the Cesàro means

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly as $n \to \infty$ to $z \in F(T)$.

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Bruck [5] extended Baillon’s result to Banach spaces as follows:

**Theorem 1.3 (Bruck [5]).** Let $E$ be a uniformly convex Banach space whose norm is a Fréchet differentiable and let $C$ be a nonempty, closed and convex subset of $E$. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Then, for any $x \in C$, the Cesàro means

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converge weakly as $n \to \infty$ to $z \in F(T)$.

However, the limit points $z \in F(T)$ in Theorems 1.1 and 1.3 are not characterized. Recently, two retractions (projections) which are different from the metric projection and the sunny nonexpansive retraction in a Banach space were found; see, for instance, Alber [1], and Ibaraki and Takahashi [11]. Such retractions are called the generalized projection and the sunny generalized nonexpansive retraction.

In this paper, using nonlinear analytic methods and new retractions which were found recently, we prove a nonlinear ergodic theorem for positively homogeneous and nonexpansive mappings in a uniformly convex Banach space. The limit points are characterized by new retractions.

2. Preliminaries

Let $E$ be a real Banach space and let $E^*$ be the dual space of $E$. For a sequence $\{x_n\}$ of $E$ and a point $x \in E$, the weak convergence of $\{x_n\}$ to $x$ and the strong convergence of $\{x_n\}$ to $x$ are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. Let $A$ be a nonempty subset of $E$. We denote by $\text{co}A$ the closure of the convex hull of $A$.

The duality mapping $J$ from $E$ into $E^*$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in E.$$

Let $S(E)$ be the unit sphere centered at the origin of $E$. Then the space $E$ is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for all $x, y \in S(E)$. The norm of $E$ is said to be Fréchet differentiable if for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$. A Banach space $E$ is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. It is said to be uniformly convex if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that $\|\frac{x+y}{2}\| \leq 1 - \delta$ whenever $x, y \in S(E)$ and $\|x - y\| \geq \varepsilon$. Furthermore, we know from [23] that

(i) if $E$ is smooth, then $J$ is single-valued;
(ii) if $E$ is reflexive, then $J$ is onto;
(iii) if $E$ is strictly convex, then $J$ is one-to-one;
(iv) if $E$ is strictly convex, then $J$ is strictly monotone, i.e.,

$$(x - y, Jx - Jy) > 0, \quad \forall x, y \in E, \ x \neq y;$$

(v) if $E$ has a Fréchet differentiable norm, then $J$ is norm-to-norm continuous.

Let $E$ be a smooth Banach space and let $J$ be the duality mapping on $E$. Throughout this paper, define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2 \langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$
Observe that, in a Hilbert space $H$, $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. We also know that for each $x, y, z, w \in E$,

$$
(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2;
$$

$$
\phi(x, y) = \phi(x, z) + \phi(z, y) + 2(x - z, Jz - Jy);

$$
2(x - y, Jz - Jw) = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w).
$$

If $E$ is additionally assumed to be strictly convex, then

$$
\phi(x, y) = 0 \text{ if and only if } x = y.
$$

The following results were proved by Xu [27] and Kamimura and Takahashi [17].

**Lemma 2.1** (Xu [27]). Let $E$ be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and

$$
\|ax + (1 - a)y\|^2 \leq a\|x\|^2 + (1 - a)\|y\|^2 - a(1 - a)g(\|x - y\|)
$$

for all $x, y \in B_r$ and $a \in [0, 1]$, where $B_r = \{z \in E : \|z\| \leq r\}$.

**Lemma 2.2** (Kamimura and Takahashi [17]). Let $E$ be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and

$$
g(\|x - y\|) \leq \phi(x, y)
$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let $E$ be a Banach space and let $C$ be a nonempty subset of $E$. A mapping $T : C \to C$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. We know the following results.

**Lemma 2.3** (Bruck [6]). Let $E$ be a uniformly convex Banach space and let $C$ be a bounded, closed and convex subset of $E$. Let $T$ be a nonexpansive mapping of $C$ into itself. Define

$$
S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x, \quad \forall x \in C, \ n \in \mathbb{N}.
$$

Then,

$$
\lim_{n \to \infty} \sup_{x \in C} \|S_n x - TS_n x\| = 0.
$$

**Lemma 2.4** (Browder [4]). Let $E$ be a uniformly convex Banach space and let $C$ be a bounded, closed and convex subset of $E$. Let $T$ be a nonexpansive mapping of $C$ into itself. If $x_n \to z$ and $x_n - Tx_n \to 0$, then $z \in F(T)$.

**Lemma 2.5** (Itoh and Takahashi [16]). Let $E$ be a strictly convex Banach space and let $C$ be a nonempty, closed and convex subset of $E$. Let $T$ be a quasi-nonexpansive mapping of $C$ into itself. Then $F(T)$ is closed and convex.

Let $E$ be a smooth Banach space and let $C$ be a nonempty subset of $E$. A mapping $T : C \to C$ is called generalized nonexpansive [11] if $F(T) \neq \emptyset$ and

$$
\phi(Tx, y) \leq \phi(x, y), \quad \forall x \in C, \ y \in F(T).
$$

Let $E$ be a Banach space and let $C$ be a closed and convex cone of $E$. A mapping $T : C \to C$ is called positively homogeneous if $T(\alpha x) = \alpha T(x)$ for all $x \in C$ and $\alpha \geq 0$. 
Lemma 2.6 (Takahashi and Yao [26]). Let $E$ be a Banach space and let $C$ be a closed and convex cone of $E$. Let $T : C \to C$ be a positively homogeneous nonexpansive mapping. Then, for any $x \in C$ and $m \in F(T)$, there exists $j \in Jm$ such that
\[ \langle x - Tx, j \rangle \leq 0, \]
where $J$ is the duality mapping of $E$ into $E^\ast$.

Using Lemma 2.6, Takahashi and Yao [26] proved the following result.

Lemma 2.7 (Takahashi and Yao [26]). Let $E$ be a smooth Banach space and let $C$ be a closed and convex cone of $E$. Let $T : C \to C$ be a positively homogeneous nonexpansive mapping. Then, $T$ is a generalized nonexpansive mapping.

Let $D$ be a nonempty subset of a Banach space $E$. A mapping $R : E \to D$ is said to be sunny if
\[ R(Rx + t(x - Rx)) = Rx, \quad \forall x \in E, \ t \geq 0. \]
A mapping $R : E \to D$ is said to be a retraction or a projection if $Rx = x$ for all $x \in D$. A nonempty subset $D$ of a smooth Banach space $E$ is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of $E$ if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) $R$ from $E$ onto $D$; see [10, 12, 11] for more details. The following results are in Ibaraki and Takahashi [11].

Lemma 2.8 (Ibaraki and Takahashi [11]). Let $C$ be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space $E$. Then the sunny generalized nonexpansive retraction from $E$ onto $C$ is uniquely determined.

Lemma 2.9 (Ibaraki and Takahashi [11]). Let $C$ be a nonempty closed subset of a smooth and strictly convex Banach space $E$ such that there exists a sunny generalized nonexpansive retraction $R$ from $E$ onto $C$ and let $(x, z) \in E \times C$. Then the following hold:
\begin{enumerate}
  \item [(i)] $z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;
  \item [(ii)] $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.
\end{enumerate}

In 2007, Kohsaka and Takahashi [18] also proved the following results:

Lemma 2.10 (Kohsaka and Takahashi [18]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty closed subset of $E$. Then the following are equivalent:
\begin{enumerate}
  \item [(a)] $C$ is a sunny generalized nonexpansive retract of $E$;
  \item [(b)] $C$ is a generalized nonexpansive retract of $E$;
  \item [(c)] $JC$ is closed and convex.
\end{enumerate}

Lemma 2.11 (Kohsaka and Takahashi [18]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a nonempty closed sunny generalized nonexpansive retract of $E$. Let $R$ be the sunny generalized nonexpansive retraction from $E$ onto $C$ and let $(x, z) \in E \times C$. Then the following are equivalent:
\begin{enumerate}
  \item [(i)] $z = Rx$;
  \item [(ii)] $\phi(x, z) = \min_{y \in C} \phi(x, y)$.
\end{enumerate}
Lemma 2.12 (Inthakon, Dhompongsa and Takahashi [15]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a closed subset of $E$ such that $J(C)$ is closed and convex. Let $T$ be a generalized nonexpansive mapping from $C$ into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.

The following is a direct consequence of Lemmas 2.10 and 2.12.

Lemma 2.13 (Inthakon, Dhompongsa and Takahashi [15]). Let $E$ be a smooth, strictly convex and reflexive Banach space and let $C$ be a closed subset of $E$ such that $J(C)$ is closed and convex. Let $T$ be a generalized nonexpansive mapping from $C$ into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of $E$.

Let $l^\infty$ be the Banach space of bounded sequences with supremum norm. Let $\mu$ be an element of $(l^\infty)^*$ (the dual space of $l^\infty$). Then, we denote by $\mu(f)$ the value of $\mu$ at $f = (x_1, x_2, x_3, \ldots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional $\mu$ on $l^\infty$ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \ldots)$. A mean $\mu$ is called a Banach limit on $l^\infty$ if $\mu_n(x_n+1) = \mu_n(x_n)$. We know that there exists a Banach limit on $l^\infty$. If $\mu$ is a Banach limit on $l^\infty$, then for $f = (x_1, x_2, x_3, \ldots) \in l^\infty$,

$$\lim \inf_{n \to \infty} x_n \leq \mu_n(x_n) \leq \lim \sup_{n \to \infty} x_n.$$ 

In particular, if $f = (x_1, x_2, x_3, \ldots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other elementary properties, see [23, 24]. Using means and the Riesz theorem, we can obtain the following result; see [21] and [8, 9].

Lemma 2.14. Let $E$ be a reflexive Banach space, let $\{x_n\}$ be a bounded sequence in $E$ and let $\mu$ be a mean on $l^\infty$. Then there exists a unique point $z_0 \in \overline{C\{x_n : n \in \mathbb{N}\}}$ such that

$$\mu_n(x_n, y^*) = \langle z_0, y^* \rangle, \quad \forall y^* \in E^*.$$ 

Such a point $z_0$ in Lemma 2.14 is called the mean vector of $\{x_n\}$ for $\mu$. This point $z_0$ plays a crucial role in this paper. The following result is in Hirano, Kido and Takahashi [8].

Lemma 2.15. Let $E$ be a uniformly convex Banach space and let $C$ be a nonempty, closed and convex subset of $E$. Let $T$ be a nonexpansive mapping of $C$ into $C$ such that $F(T) \neq \emptyset$. Let $\mu$ be a Banach limit on $l^\infty$. Then the mean vector of $\{x_n\}$ for $\mu$ is in $F(T)$.

The following result is in Lin, Takahashi and Yu [20].

Lemma 2.16 (Lin, Takahashi and Yu [20]). Let $E$ be a smooth, strictly convex and reflexive Banach space with the duality mapping $J$ and let $D$ be a nonempty, closed and convex subset of $E$. Let $\{x_n\}$ be a bounded sequence in $D$ and let $\mu$ be a mean on $l^\infty$. If $g : D \to \mathbb{R}$ is defined by

$$g(z) = \mu_n(\phi(x_n, z)), \quad \forall z \in D,$$ 

then the mean vector $z_0$ of $\{x_n\}$ for $\mu$ is a unique minimizer in $D$ such that

$$g(z_0) = \min\{g(z) : z \in D\}.$$
3. Lemmas

In the section, we first prove the following lemma which plays an important role for proving our main theorem.

**Lemma 3.1.** Let $E$ be a uniformly convex and smooth Banach space and let $T$ be a positively homogeneous nonexpansive mapping of $E$ into itself. Then for any $x \in C$, the sequence $\{T^n x\}$ is bounded and the set

$$\cap_{k=1}^{\infty} \text{co} \{T^{k+n} x : n \in \mathbb{N}\} \cap F(T)$$

consists of one point $z_0$, where $z_0$ is a unique minimizer of $F(T)$ such that

$$\lim_{n \to \infty} \phi(T^n x, z_0) = \min \left\{ \lim_{n \to \infty} \phi(T^n x, z) : z \in F(T) \right\}.$$  

Proof. Since $T : E \to E$ is positively homogeneous and nonexpansive, it follows from Lemma 2.7 that $T$ is generalized nonexpansive. Thus we have that for any $z \in F(T)$ and $x \in C$,

$$\phi(T^{n+1} x, z) \leq \phi(T^n x, z) \leq \cdots \leq \phi(x, z), \quad \forall n \in \mathbb{N}.$$  

Then $\{T^n x\}$ is bounded. Let $\mu$ be a Banach limit on $\ell^\infty$. From Lemma 2.16, the mean vector $z_0 \in E$ of $\{T^n x\}$ for $\mu$ is a unique minimizer $z_0 \in E$ such that

$$\mu_n \phi(T^n x, z_0) = \min \{\mu_n \phi(T^n x, y) : y \in E\}.$$  

We also know from Lemma 2.15 that $z_0 \in F(T)$. Furthermore, this $z_0 \in F(T)$ satisfies that

$$\mu_n \phi(T^n x, z_0) = \min \{\mu_n \phi(T^n x, y) : y \in F(T)\}.$$  

Let us show that $z_0 \in \cap_{k=1}^{\infty} \text{co} \{T^{k+n} x : n \in \mathbb{N}\}$. If not, there exists some $k \in \mathbb{N}$ such that $z_0 \notin \text{co} \{T^{k+n} x : n \in \mathbb{N}\}$. By the separation theorem, there exists $y_0^* \in E^*$ such that

$$\langle z_0, y_0^* \rangle < \inf \{\langle z, y_0^* \rangle : z \in \text{co} \{T^{k+n} x : n \in \mathbb{N}\} \}.$$  

Using the property of the Banach limit $\mu$, we have that

$$\langle z_0, y_0^* \rangle < \inf \{\langle z, y_0^* \rangle : z \in \text{co} \{T^{k+n} x : n \in \mathbb{N}\} \} \leq \inf \{\langle T^{k+n} x, y_0^* \rangle : n \in \mathbb{N}\} \leq \mu_n \langle T^{k+n} x, y_0^* \rangle = \mu_n \langle T^n x, y_0^* \rangle = \langle z_0, y_0^* \rangle.$$  

This is a contradiction. Thus we have that $z_0 \in \cap_{k=1}^{\infty} \text{co} \{T^{k+n} x : n \in \mathbb{N}\}$. Next we show that $\cap_{k=1}^{\infty} \text{co} \{T^{k+n} x : n \in \mathbb{N}\} \cap F(T)$ consists of one point $z_0$. Assume that $z_1 \in \cap_{k=1}^{\infty} \text{co} \{T^{k+n} x : n \in \mathbb{N}\} \cap F(T)$. Since $z_1 \in F(T) = B(T)$, we have that

$$\phi(T^{n+1} x, z_1) \leq \phi(T^n x, z_1), \quad \forall n \in \mathbb{N}.$$  

Then $\lim_{n \to \infty} \phi(T^n x, z_1)$ exists. Furthermore, we know from the property of a Banach limit $\mu$ that

$$\mu_n \phi(T^n x, z_1) = \lim_{n \to \infty} \phi(T^n x, z_1).$$  

In general, since $\lim_{n \to \infty} \phi(T^n x, z)$ exists for every $z \in F(T)$, we define a function $g : F(T) \to \mathbb{R}$ as follows:

$$g(z) = \lim_{n \to \infty} \phi(T^n x, z), \quad \forall z \in F(T).$$
Since
\[ \phi(z_0, z_1) = \phi(T^n x, z_1) - \phi(T^n x, z_0) - 2(T^n x - z_0, Jz_0 - Jz_1) \]
for every \( n \in \mathbb{N} \), we have
\[ \phi(z_0, z_1) + 2 \lim_{n \to \infty} \langle T^n x - z_0, Jz_0 - Jz_1 \rangle = \lim_{n \to \infty} \phi(T^n x, z_1) - \lim_{n \to \infty} \phi(T^n x, z_0) \geq 0. \]

Let \( \epsilon > 0 \). Then we have that
\[ 2 \lim_{n \to \infty} \langle T^n x - z_0, Jz_0 - Jz_1 \rangle > -\phi(z_0, z_1) - \epsilon. \]

Hence there exists \( n_0 \in \mathbb{N} \) such that
\[ 2 \langle z_1 - z_0, Jz_0 - Jz_1 \rangle \geq -\phi(z_0, z_1) - \epsilon. \]

We have from (2.3) that
\[ \phi(z_1, z_1) + \phi(z_0, z_0) - \phi(z_1, z_0) - \phi(z_0, z_1) \geq -\phi(z_0, z_1) - \epsilon \]
and hence \( \phi(z_1, z_0) \leq \epsilon \). Since \( \epsilon > 0 \) is arbitrary, we have \( \phi(z_1, z_0) = 0 \). Since \( E \) is strictly convex, we have \( z_0 = z_1 \). Therefore
\[ \{ z_0 \} = \cap_{k=1}^{\infty} \overline{co}\{ T^{k+n} x : n \in \mathbb{N} \} \cap F(T). \]

This completes the proof. \( \square \)

For proving our main theorem (Theorem 4.1), we also need the following two lemmas.

**Lemma 3.2.** Let \( E \) be a uniformly convex Banach space and let \( C \) be a bounded, closed and convex subset of \( E \). Let \( T \) be a nonexpansive mapping of \( C \) into itself. For any \( x \in S \), define
\[ S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x, \quad \forall n \in \mathbb{N}. \]

If a subsequence \( \{ S_n x \} \) of \( \{ S_n x \} \) converges weakly to a point \( u \), then \( u \in F(T) \).

**Proof.** We know from Lemma 2.3 that
\[ \lim_{n \to \infty} \sup_{x \in C} \| S_n x - T S_n x \| = 0. \]
Since a subsequence \( \{ S_n x \} \) of \( \{ S_n x \} \) converges weakly to a point \( u \), we have from Lemma 2.4 that \( u \in F(T) \). This completes the proof. \( \square \)

**Lemma 3.3.** Let \( E \) be a uniformly convex and smooth Banach space and let \( T : E \to E \) be a positively homogeneous nonexpansive mapping. Then, there exists a unique sunny generalized nonexpansive retraction \( R \) of \( E \) onto \( F(T) \). Furthermore, for any \( x \in E \), \( \lim_{n \to \infty} R T^n x \) exists in \( F(T) \).
Lemma 3.1 that the set $E \rightarrow J$.

Theorem 4.1. Let $\{\text{ }\}$ be a positively homogeneous nonexpansive mapping in a Banach space. Furthermore, since $D$ is invariant under $\phi$, $g$ is a strictly increasing, continuous and convex real-valued function with $\phi(T^n x, RT^n x)$ is nonincreasing. Putting $u = RT^n x$ and $v = T^n x$ with $n \leq m$ in (3.2), we have from Lemma 2.2 that

\begin{align*}
g(||RT^n x - RT^m x||) &\leq \phi(RT^n x, RT^m x) \\
&\leq \phi(T^n x, RT^n x) - \phi(T^m x, RT^m x) \\
&\leq \phi(T^n x, RT^n x) - \phi(T^m x, RT^m x),
\end{align*}

where $g$ is a strictly increasing, continuous and convex real-valued function with $g(0) = 0$. From the properties of $g$, $\{RT^n x\}$ is a Cauchy sequence. Therefore $\{RT^n x\}$ converges strongly to a point $q \in F(T)$. This completes the proof. \hfill \Box

4. **Nonlinear Ergodic Theorem**

Using Lemmas 3.1, 3.2 and 3.3, we now prove the following nonlinear ergodic theorem for positively homogeneous nonexpansive mappings in a Banach space.

**Theorem 4.1.** Let $E$ be a uniformly convex and smooth Banach space. Let $T : E \rightarrow E$ be a positively homogeneous nonexpansive mapping. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to $z_0 \in F(T)$. Additionally, if the norm of $E$ is a Fréchet differentiable, then $z_0 = \lim_{n \rightarrow \infty} R_{F(T)} T^n x$, where $R_{F(T)}$ is the sunny generalized nonexpansive retraction of $E$ onto $F(T)$.

**Proof.** Let $x \in E$ and define $D = \{z \in E : \|z\| \leq \|x\]\}$. Then $D$ is nonempty, bounded, closed and convex. Furthermore, since $T$ is nonexpansive and $0 \in F(T)$, $D$ is invariant under $T$ and hence $\{T^n x\}$ and $\{S_n x\}$ are in $D$. We know from Lemma 3.1 that the set

$$\bigcap_{k=1}^{\infty} \text{cl} \{T^{k+n} x : n \in \mathbb{N}\} \cap F(T)$$

Proof. Let $x \in E$ and define $D = \{z \in E : \|z\| \leq \|x\]\}$. Then $D$ is nonempty, bounded, closed and convex. Furthermore, since $T$ is nonexpansive and $0 \in F(T)$, $D$ is invariant under $T$ and hence $\{T^n x\}$ and $\{S_n x\}$ are in $D$. We know from Lemma 3.1 that the set

$$\bigcap_{k=1}^{\infty} \text{cl} \{T^{k+n} x : n \in \mathbb{N}\} \cap F(T)$$
consists of one point \( z_0 \). To prove that \( \{S_n x\} \) converges weakly to \( z_0 \) in \( F(T) \), it is sufficient to show that for any subsequence \( \{S_{n_k} x\} \) of \( \{S_n x\} \) such that \( S_{n_k} x \to v \), \( v \in F(T) \) and

\[
\{v \in \cap_{k=1}^{\infty} \overline{T} \{T^{k+n} x : n \in \mathbb{N}\} : \}.
\]

From Lemma 3.2, we have that \( v \in F(T) \). Next, we show that

\[
v \in \cap_{k=1}^{\infty} \overline{T} \{T^{k+n} x : n \in \mathbb{N}\}.
\]

Fix \( k \in \mathbb{N} \). We have that for any \( n_i \in \mathbb{N} \) with \( n_i > k \),

\[
S_{n_i} x = \frac{1}{n_i} (x + T x + \cdots + T^{k} x)
+ \frac{n_i - (k + 1)}{n_i} \frac{1}{n_i - (k + 1)} (T^{k+1} x + \cdots + T^{n_i-1}).
\]

Thus from \( S_{n_i} x \to v \), we have

\[
\frac{1}{n_i - (k + 1)} (T^{k+1} x + \cdots + T^{n_i-1}) \to v
\]

and hence \( v \in \overline{T} \{T^{k+n} x : n \in \mathbb{N}\} \). Since \( k \in \mathbb{N} \) is arbitrary, we have that

\[
v \in \cap_{k=1}^{\infty} \overline{T} \{T^{k+n} x : n \in \mathbb{N}\}.
\]

Therefore \( \{S_n x\} \) converges weakly to a point \( z_0 \) of \( F(T) \).

Additionally, assume that the norm of \( E \) is a Fréchet differentiable. We have from Lemma 3.3 that there exists the sunny generalized nonexpansive retraction \( R = R_{F(T)} \) of \( E \) onto \( F(T) \) and \( \{R^n T x\} \) converges strongly to a point \( q \in F(T) \).

Rewriting the characterization of the retraction \( R \), we have that

\[
0 \leq \langle T^k x - RT^k x, JRT^k x - Ju \rangle, \quad \forall u \in F(T)
\]

and hence

\[
\langle T^k x - RT^k x, J u - J q \rangle \leq \langle T^k x - RT^k x, JRT^k x - J q \rangle
\]

\[
\leq || T^k x - RT^k x || \cdot || JRT^k x - J q ||
\]

\[
\leq K || JRT^k x - J q ||,
\]

where \( K \) is an upper bound for \( || T^k x - RT^k x || \). Summing up these inequalities for \( k = 0, 1, \ldots, n - 1 \) and dividing by \( n \), we arrive to

\[
\langle S_n x - \frac{1}{n} \sum_{k=0}^{n-1} RT^k x, J u - J q \rangle \leq \frac{K}{n} \sum_{k=0}^{n-1} || JRT^k x - J q ||.
\]

Letting \( n \to \infty \) and remembering that \( J \) is continuous because the norm of \( E \) is a Fréchet differentiable, we get that

\[
\langle z_0 - q, J u - J q \rangle \leq 0.
\]

This holds for any \( u \in F(T) \). Putting \( u = z_0 \), we have \( \langle z_0 - q, J z_0 - J q \rangle \leq 0 \). Since \( J \) is monotone, we have \( \langle z_0 - q, J z_0 - J q \rangle = 0 \). Since \( E \) is strictly convex, we have \( z_0 = q \). Thus \( z_0 = \lim_{n \to \infty} R_{F(T)} T^n x \).

\[\square\]

Compare Theorem 4.1 with Theorem 1.3. Though the assumption of a mapping in Theorem 4.1 is stronger than that of Theorem 1.3, the assumption of a Banach space is weaker. Furthermore, the limit points are characterized by sunny generalized nonexpansive retractions.
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