LEFT QUOTIENTS OF A C*-ALGEBRA, III: OPERATORS ON LEFT QUOTIENTS

LAWRENCE G. BROWN AND NGAI-CHING WONG

ABSTRACT. Let $L$ be a norm closed left ideal of a C*-algebra $A$. Then the left quotient $A/L$ is a left $A$-module. In this paper, we shall implement Tomita’s idea about representing elements of $A$ as left multiplications: $\pi_p(a)(b + L) = ab + L$. A complete characterization of bounded endomorphisms of the $A$-module $A/L$ is given. The double commutant $\pi_p(A)'''$ of $\pi_p(A)$ in $B(A/L)$ is described. Density theorems of von Neumann and Kaplansky type are obtained. Finally, a comprehensive study of relative multipliers of $A$ is carried out.

1. Introduction

Let $A$ be a C*-algebra with Banach dual $A^*$ and double dual $A^{**}$. We also consider $A^{**}$ as the enveloping W*-algebra of $A$, as usual. Let $L$ be a norm closed left ideal of $A$. The quotient $A/L$ of $A$ by $L$ is a Banach space. Let $B(A/L) = B(A/L, A/L)$ be the Banach algebra of bounded linear operators from $A/L$ into $A/L$. In [17, 18], Tomita initiated a program to study the left regular representation $\pi_p$ of $A$ on the Banach space $A/L$. More precisely, he considered the Banach algebra representation of $A$,

$$\pi_p : A \longrightarrow B(A/L),$$

defined by

$$\pi_p(a)(b + L) = ab + L, \quad a, b \in A.$$

The objective of this paper is to answer the following three questions raised by Tomita [18].

Q1: How do we describe $\pi_p(A)$? In other words, which properties of an operator $T$ in $B(A/L)$ characterize that $T = \pi_p(t)$ for some $t$ in $A$?

Q2: How do we describe the commutant $\pi_p(A)'$ and the double commutant $\pi_p(A)'''$ of $\pi_p(A)$ in $B(A/L)$? Note that $\pi_p(A)' = \{T \in B(A/L) : T\pi_p(a) = \pi_p(a)T, \forall a \in A\}$ is the Banach algebra of bounded $A$-module maps when we consider $A/L$ as a left $A$-module.

Q3: Do we have density theorems of von Neumann and Kaplansky type in this context? In other words, is it true that $\pi_p(A)$ (resp. its unit ball) is dense in $\pi_p(A)'''$ (resp. its unit ball)?

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In [17, 18], Tomita tried to represent elements of \( A/L \) as vector sections (he called them “vector fields”) over a compact subset of the state space \( S(A) \) (assuming that the C*-algebra \( A \) has an identity). In [17], he defined the notion of a “vector field” as “a mapping of a state space into the dual space of the algebra which satisfies a suitable norm condition”. However, due to insufficient tools, “unlike in abelian case, even in a compact space of pure states, the corresponding quotient space of non-commutative algebra \( A \) may not generally be represented as the totality of continuous fields on that space”. Thus, his treatment in [18] of the left regular representation \( \pi_p \) based on his vector section representation does not work in general.

In Part I [20] of this series of papers, the second author offered another approach. It is well-known that closed left ideals \( L \) of a C*-algebra \( A \) are in one-to-one correspondence with closed projections \( p \) in \( A^\ast \ast \) such that \( A/L \) is isometrically isomorphic to \( Ap \) as Banach spaces and also as left \( A \)-modules (see Section 3). For an arbitrary closed projection \( p \) in \( A^\ast \ast \) (and thus for an arbitrary closed left ideal \( L \) of \( A \)), we use the weak* closed face \( F(p) \) of the quasi-state space \( Q(A) \) of \( A \) supported by \( p \) as the base space. We implement, in addition to the norm conditions of Tomita, an affine structure of vector sections. Then it was established that the quotient space \( A/L \) \((\cong Ap)\) is isometrically isomorphic to the Banach space of all continuous admissible vector sections over \( F(p) \) (see Theorem 3.4). Based on these new techniques, we are able to provide in this paper more satisfactory answers to the above three questions.

We begin with the W*-algebra version in Section 2 in which we shall completely answer all three questions stated above. For example, if \( p \) is a (necessarily closed) projection in a W*-algebra \( M \) then \( \pi_p(M)' \) consists of right multiplications induced by elements of \( pMp \) and \( \pi_p(M)^\prime\prime = \pi_p(M) \) (Theorem 2.3). In particular, all \( M \)-module maps \( T \) in \( B(Mp) \) are of the form \( T(xp) = xptp \) for some \( t \) in \( M \).

However, the C*-algebra case is much more difficult (due to lack of projections) and we need to develop some new tools. In [20], elements \( bp \) of the Banach space \( Ap \) are interpreted as Hilbert space vector sections over \( F(p) \). The main idea in this paper is to represent Banach space operators \( \pi_p(a) \) in \( B(Ap) \) as Hilbert space operator sections (Definition 3.7), which is developed in Section 3. In particular, an operator \( T \) in \( B(Ap) \) is said to be decomposable if \( T \) can be represented by an operator section (Definition 3.10). A simple way to verify the decomposability of \( T \) is to check if the condition \( \varphi(a^*a) = 0 \) ensures \( \varphi((TAp)^*TAp)) = 0 \) whenever \( \varphi \) is a pure state supported by \( p \) and \( a \in A \) (Theorem 3.13). In this case, \( T \) has to be a \( \pi_p(t) \) for some \( t \) in \( LM(A,p) = \{ x \in A^\ast : xAp \subseteq Ap \} \) (Corollary 3.14). This answers our first question Q1.

Various relative multipliers of \( A \) associated to \( p \) play important roles in the theory of left regular representations. Beside \( LM(A,p) \), we shall introduce and study \( RM(A,p) \), \( M(A,p) \) and \( QM(A,p) \) in Section 4. They behave in a similar way as the sets \( LM(A) \), \( RM(A) \), \( M(A) \) and \( QM(A) \) of classical multipliers of \( A \). For example, they are closures of \( A \) in \( A^\ast \ast \) under
corresponding relative strict topologies (Theorem 4.3). The object studied by Tomita in [18] is essentially the closure of \( \pi_p(A) \) in \( B(Ap) \) with respect to the so-called quotient-(double) strong topology, or \( Q^* \)-topology. In fact, the \( Q^* \)-topology is induced by the relative strict topology of \( A^{**} \). Thus, the closure of the Banach algebra \( \pi_p(A) \) in \( B(Ap) \) in the \( Q^* \)-topology is the image of the \( C^*-algebra \) \( M(A,p) = \{ x \in A^{**} : xAp \subseteq Ap, pAx \in pA \} \) under \( \pi_p \) (see Remark 4.5). Tomita expected that the double commutant \( \pi_p(A)'' \) of \( \pi_p(A) \) in \( B(Ap) \) coincides with \( \pi_p(M(A,p)) \). This is, however, not always true for an arbitrary projection \( p \).

In some important cases, we have \( \pi_p(A)'' = \pi_p(LM(A,p)) \) (Theorem 4.8). A counter example is Example 4.9. This partially answers our second question Q2.

The classical density theorems of von Neumann and Kaplansky have counterparts in this context. Also in Section 4, we show that \( \pi_p(A) \) (resp. its unit ball) is dense in \( \pi_p(LM(A,p)) \) (resp. its unit ball) in the strong operator topology (SOT) as well as the weak operator topology (WOT) of \( B(Ap) \) (Theorem 4.4). This answers our last question Q3.

It is then interesting and useful to find a \( C^*-subalgebra \) \( \mathcal{A} = \text{Alg}(A,p) \) of \( A^{**} \) such that \( \text{LM}(A,p) = \text{LM}(A), \text{RM}(A,p) = \text{RM}(A), M(A,p) = M(A) \) and \( \text{QM}(A,p) = \text{QM}(A) \), and thus all good tools of multipliers apply (see e.g. [5]). Several examples and results are provided in Section 5 for the investigation of what \( \mathcal{A} \) should consist of (especially Theorem 5.3).

Finally, we remark that the atomic part of \( Ap \) is studied in Part II [9] of this series of papers. Some interesting and new results in this direction are obtained in Section 6. For example, we show that if \( x \) is in \( A^{**} \) and \( \pi_p(x) \) preserves continuous atomic parts, i.e., \( z_{at}xAp \subseteq z_{at}Ap \), then \( z_{at}xc(p) \in z_{at}\text{LM}(A,p) \), where \( z_{at} \) is the maximal atomic projection in \( A^{**} \) and \( c(p) \) is the central support of \( p \) in \( A^{**} \) (Theorem 6.2). In particular, when \( p = 1 \), we have \( z_{at}x = z_{at}l \) for some left multiplier \( l \) of \( A \) whenever \( z_{at}xA \subseteq z_{at}A \) (Corollary 6.3). This supplements results of Shultz [16] and Brown [7]. Similar results are obtained for other relative multipliers as well.

This paper, together with [20, 9], is based on the doctoral dissertation [19] of the second author under the supervision of the first author. We would like to thank Edward Effors for his suggestion to study a paper of Tomita [18], based on his success on working with its predecessor [17].

2. The left regular representation of a \( W*-algebra \)

We provide a new elementary proof of the following result of Tomita [18].

**Theorem 2.1 ([18]).** Let \( \pi \) be a bounded homomorphism from a \( C^*-algebra \) \( A \) into a Banach algebra \( B \). Then \( \pi(A) \) is topologically isomorphic to \( A/\ker \pi \). If \( \| \pi \| \leq 1 \), then \( \pi(A) \) is isometrically isomorphic to \( A/\ker \pi \).
Suppose Lemma 2.2. Clearly, \( \pi \) is an isometry. We shall just verify the necessity. Assume that \( \pi \) is an isometry by Theorem 2.1. In particular, \( \pi \) is a Banach algebra, \( \pi \) is the left regular representation of \( M \). First assume that \( \pi \) is an isometry. For any positive \( \lambda \), \( \| \lambda \| = \rho_\pi(\lambda) \) and \( \| \pi(\lambda) \| = \rho_\pi(\lambda) \) for all \( \lambda \). Let \( x = f(a) \) and \( y = g(a) \). We have \( x, y \in A \) and \( xy = yx \). It follows that \( \rho_\pi(x) \leq \rho_\pi(y) \). Therefore, \( \| \pi(x) \| \geq 1 \). Now, \( \| (\lambda - \lambda)x \| < \varepsilon \) implies \( \| \rho_\pi(\lambda)x \| = \| \rho_\pi((\lambda - \lambda)x) \| < k\varepsilon \). The fact that \( \varepsilon \) can be arbitrarily small ensures \( \pi \) is an isometry, as asserted. Hence, 

\[
\| \pi(a) \| \geq \rho_\pi(\rho_\pi(a)) \geq \rho_\pi(a) = \| a \|
\]

for all positive \( a \) in \( A \). In general, if \( a \in A \) and \( a \neq 0 \),

\[
\| \pi(a) \| \geq \frac{\| \pi(a^*a) \|}{\| \pi(a^*) \|} \geq \frac{\| a^*a \|}{\| \pi(a^*) \|} \geq \frac{\| a \|^2}{k\| a \|} = \frac{1}{k} \| a \|.
\]

Let \( p \) be a projection in a W*-algebra \( M \). Let \( c(p) \) be the central support of \( p \) in \( M \). In other words, \( c(p) \) is the minimum central projection in \( M \) such that \( pc(p) = c(p)p = p \). Recall that \( \pi_p \) is the left regular representation of \( M \) into \( B(Mp) \), i.e.,

\[
\pi_p(x)p = xyp, \quad y \in M.
\]

Clearly, \( \pi_p(c(p)) = 1 \) in \( B(Mp) \). Hence, \( \pi_p(t) = \pi_p(tc(p)) \) for all \( t \) in \( M \), and in fact \( \ker \pi_p = M(1 - c(p)) \).

**Lemma 2.2.** Suppose \( T \in B(Mp) \). \( T \) commutes with all right multiplications \( R_{xp} \) for \( x \) in \( M \) if and only if there is a \( t \) in \( M \) such that \( T = \pi_p(t) \). In this case, \( \| T \| = \| tc(p) \| \).

**Proof.** We shall just verify the necessity. Assume \( T \in B(Mp) \) such that \( TR_{xp} = R_{xp}T, \forall x \in M \). For every central projection \( z \) in \( M \), we have

\[
T(zp) = T(xp(pzp)) = T(R_{zp}(xp)) = R_{zp}(T(xp)) = (Tzp)zp = z(Txp), \quad x \in M.
\]

In particular, \( T(zMp) \subseteq zMp \). By passing to \( c(p)M \), we can assume \( c(p) = 1 \) and \( \pi_p \) is an isometry by Theorem 2.1.
Let
\[ S = \{ S \in B(Mp) : SR_{pxp} = R_{pxp}S, \forall x \in M \} \]
and
\[ Q = \{ q \in M : q \text{ is a projection and } S\pi_p(q) \in \pi_p(M), \forall S \in S \}. \]

**Claim 1.** \( p \in Q. \)

For \( S \) in \( S \), let \( s = S(p) \in Mp. \) We have
\[ \pi_p(s)(xp) = sxp = S(p)(pxp) = R_{pxp}S(p) = S(R_{pxp}(p)) = S(pxp) = S\pi_p(p)(xp), \]
for all \( xp \) in \( Mp. \) Therefore, \( S\pi_p(p) = \pi_p(s) \in \pi_p(M). \) Hence, \( p \in Q. \)

**Claim 2.** \( Q \) is hereditary under the quasi-ordering \( \lessdot \) of projections.

Suppose \( q \in Q \) and \( r \lessdot q. \) In other words, \( r = v^*v \) and \( vv^* \leq q \) for some partial isometry \( v \) in \( M. \) Note that \( r = v^*qv. \) Since \( \pi_p(v^*) \) is in \( S \), the operator \( S\pi_p(v^*) \) belongs to \( S \) whenever \( S \) does. As \( q \in Q, \) for each \( S \) in \( S \) there is an \( s' \) in \( M \) such that
\[ (S\pi_p(v^*))\pi_p(q) = \pi_p(s'). \]
Consequently,
\[ S(xxp) = S(v^*qvxp) = S\pi_p(v')(\pi_p(q)(vxp)) = \pi_p(s'xp), \quad \forall x \in M. \]
Set \( s'' = s'v. \) We have
\[ S\pi_p(r) = \pi_p(s'') \in \pi_p(M). \]
Hence \( r \in Q. \) Therefore, \( Q \) is hereditary under \( \lessdot \) and, in particular, \( Q \) contains all projections \( q \) such that \( q \lessdot p \) by Claim 1.

**Claim 3.** \( S \) is directed under the ordering \( \leq \) of projections.

We are going to show that \( Q \) is even a lattice. First, it is clear that if \( q_1, q_2, \ldots, q_n \) in \( Q \) are mutually orthogonal then \( q_1 + q_2 + \ldots + q_n \in Q. \) Then, if \( q_1, q_2 \in Q, \) we have
\[ (q_1 \vee q_2 - q_1) \sim (q_2 - q_1 \wedge q_2) \leq q_2. \]
Hence \( (q_1 \vee q_2 - q_1) \in Q \) by Claim 2, and consequently \( q_1 \vee q_2 = (q_1 \vee q_2 - q_1) + q_1 \in Q. \)

Associate to each \( q \) in \( Q \) a \( t_q \) in \( M \) such that
\[ T\pi_p(q) = \pi_p(t_q). \]
Then \( ||t_q|| = ||\pi_p(t_q)|| \leq ||T|| \) because \( \pi_p \) is an isometry. Since the net \( \{ t_q : q \in Q \} \) is bounded in the \( W^*-\)algebra \( M, \) some subnet \( \{ t_{q_{\lambda}} \} \) converges to some \( t \) in \( M \) with respect to the \( \sigma(M, M_*) \) topology. For every \( xp \) in \( Mp, \) let \( q_x \) be the range projection of \( xp. \) Then \( q_x \in Q \) since \( q_x \lessdot p. \) Consequently, for large enough \( \lambda, \) we have \( q_x \leq q_{\lambda} \) and thus
\[ T(xp) = T(q_{\lambda}xp) = T\pi_p(q_{\lambda})(xp) = t_{q_{\lambda}}xp. \]

It follows that
\[ txp = \lim_{\lambda} t_{q_{\lambda}} xp = T(xp), \quad \forall x \in M. \]
Hence \( \pi_p(t) = T. \) Finally, \( ||t|| = ||\pi_p(t)|| = ||T|| \) since \( \pi_p \) is an isometry. \( \square \)
Theorem 2.3. Let $M$ be a $W^*$-algebra, $p$ a projection in $M$ and $\pi_p$ the left regular representation of $M$ on $Mp$. Then the commutant of $\pi_p(M)$ in $B(Mp)$ is

$$\pi_p(M)' = \{R_{stp} : t \in M\},$$

and the double commutant is

$$\pi_p(M)'' = \pi_p(M)^{\text{SOT}} = \pi_p(M)^{\text{WOT}} = \pi_p(M).$$

Proof. Suppose $T \in \pi_p(M)'$. Let $Tp = tp \in Mp$. Now

$$Txp = T\pi_p(x)p = \pi_p(x)(tp) = xtp, \quad \forall x \in M.$$ 

Since $(1-p)p = 0$, we must have $(1-p)tp = 0$, i.e., $tp = ptp$. Hence $T = R_{stp}$. The opposite inclusion is obvious and thus we have $\pi_p(M)' = \{R_{stp} : t \in M\}$. Since the double commutant of any subset of $B(Mp)$ is closed in both the strong operator topology (SOT) and the weak operator topology (WOT) of $B(Mp)$, the second assertion follows from Lemma 2.2. \qed

3. The left regular representation of a C*-algebra

Let

$$S(A) = \{\varphi \in A^* : \varphi \geq 0, \|\varphi\| = 1\}$$

be the state space and

$$Q(A) = \{\varphi \in A^* : \varphi \geq 0, \|\varphi\| \leq 1\}$$

be the quasi-state space of $A$ equipped with the weak* topology. $Q(A)$ is a weak* compact convex set. A convex subset $F$ of $Q(A)$ is called a face if both $\varphi$ and $\psi$ belong to $F$ whenever $\varphi, \psi \in Q(A)$ and $\lambda \varphi + (1-\lambda)\psi \in F$ for some $0 < \lambda < 1$.

Recall that a projection $p$ in $A^{**}$ is closed if and only if the face

$$F(p) = \{\varphi \in Q(A) : \varphi(1-p) = 0\}$$

of $Q(A)$ supported by $p$ is weak* closed. The relation

$$L = A^{**}(1-p) \cap A$$

establishes a one-to-one correspondence between closed projections in $A^{**}$ and norm closed left ideals of $A$. Also, $L^{**} = A^{**}(1-p)$. Moreover, we have isometrical isomorphisms

$$a + L \mapsto ap \quad \text{and} \quad x + L^{**} \mapsto xp$$

under which

$$A/L \cong Ap \quad \text{and} \quad (A/L)^{**} \cong A^{**}/L^{**} \cong A^{**}p$$

as Banach spaces and also as left $A$-modules, respectively [12, 15, 1, 14].

From now on, $p$ is always the unique closed projection in $A^{**}$ associated to the norm closed left ideal $L = A^{**}(1-p) \cap A$. For simplicity of notation, we write $Ap$ for the left quotient $A/L$ of the C*-algebra $A$ by $L$. Consequently, its Banach double dual $A^{**}p$ is the quotient $A^{**}/L^{**}$. Denote by $\pi_p$ the left regular representation of $A$ on $Ap$ defined by $\pi_p(a)bp = abp$ (or equivalently, $\pi_p(a)(b + L) = ab + L$). As usual, $\pi_p$ can be extended to the left regular
representation of $A^{**}$ into $B(A^{**}p)$, denoted again by $\pi_p$, such that $\pi_p(x) yp = xy p$ (or equivalently, $\pi_p(x)(y + L^{**}) = xy + L^{**}$).

We note that

$$\varphi(x) = \varphi(px) = \varphi(xp) = \varphi(px p), \quad \forall x \in A^{**}, \forall \varphi \in F(p).$$

Let $\varphi \in F(p)$. The GNS construction yields a cyclic representation $(\pi_\varphi, H_\varphi, \omega_\varphi)$ of $A$ such that $\pi_\varphi(A) \omega_\varphi = H_\varphi$ and $\varphi(x) = \langle \pi_\varphi(x) \omega_\varphi, \omega_\varphi \rangle_\varphi$ for all $x$ in $A^{**}$. Here $\pi_\varphi$ also denotes the canonical extension of $\pi_\varphi$ to $A^{**}$, and $\langle \cdot, \cdot \rangle_\varphi$ is the inner product of the Hilbert space $H_\varphi$ (see, e.g., [11]). Set $H_\varphi = \{0\}$ for $\varphi = 0$.

**Notation.** Write $x\omega_\varphi$ for $\pi_\varphi(x) \omega_\varphi$ in $H_\varphi$, $\forall x \in A^{**}, \forall \varphi \in F(p)$.

There is a linear embedding of $A^{**}p$ into the product space $\prod_{\varphi \in F(p)} H_\varphi$ defined by associating to each $x \in A^{**}p$ the vector section $(x\omega_\varphi)_{\varphi \in F(p)}$ in $\prod_{\varphi \in F(p)} H_\varphi$. Note that the fiber Hilbert spaces $H_\varphi$’s are not totally independent. In fact, we have

**Lemma 3.1** ([20, 2.3]). For $\varphi, \psi \in F(p)$ such that $0 \leq \psi \leq \lambda \varphi$ for some $\lambda > 0$, we can define a bounded linear map

$$T_{\psi \varphi} : H_\varphi \to H_\psi$$

by sending $a\omega_\varphi$ to $a\omega_\psi, \forall a \in A$. Moreover, $\|T_{\psi \varphi}\|^2 \leq \lambda$ and

$$T_{\psi \varphi}(x\omega_\varphi) = x\omega_\psi, \quad \forall x \in A^{**}.$$

**Definition 3.2** ([20, 2.4]). A vector section $(x_\varphi)_\varphi$ in $\prod_{\varphi \in F(p)} H_\varphi$ is said to be admissible if

$$T_{\psi \varphi} x_\varphi = x_\psi$$

whenever $\varphi, \psi \in F(p)$ and $0 \leq \psi \leq \lambda \varphi$ for some $\lambda > 0$.

Clearly, each $xp$ in $A^{**}p$ induces an admissible vector section $(x\omega_\varphi)_\varphi$ in $\prod_{\varphi \in F(p)} H_\varphi$. They are exactly all of them.

**Theorem 3.3** ([20, 3.1]). The image of the linear embedding $xp \mapsto (x\omega_\varphi)_\varphi$ of $A^{**}p$ into $\prod_{\varphi \in F(p)} H_\varphi$ coincides with the set of all admissible vector sections in $\prod_{\varphi \in F(p)} H_\varphi$. Moreover, we have

$$\|xp\| = \sup_{\varphi \in F(p)} \|x\omega_\varphi\|_{H_\varphi}.$$  

In particular, admissible vector sections are automatically bounded.

It is natural to ask which properties characterize those admissible vector sections arising from elements of $Ap$. Recall the notion of a continuous field of Hilbert spaces [13, 10]. Note that $\{a\omega_\varphi : a \in A\}$ is norm dense in $H_\varphi$, $\forall \varphi \in F(p)$, and the norm functions $\varphi \mapsto \|a\omega_\varphi\|_\varphi = \varphi(a^*a)^{1/2}$ are continuous on $F(p)$ for $a$ in $A$. Consequently, the image of $Ap$ under the embedding $A^{**}p \mapsto \prod_{\varphi \in F(p)} H_\varphi$ defines a continuous structure of the field of Hilbert spaces $(F(p), \{H_\varphi\})$ with base space $F(p)$ and fiber Hilbert spaces $H_\varphi, \forall \varphi \in F(p)$. In this context,

- A vector section $(x_\varphi)_{\varphi \in F(p)}$ in $\prod_{\varphi \in F(p)} H_\varphi$ is bounded if $\sup_{\varphi \in F(p)} \|x_\varphi\|_{H_\varphi} < \infty$.  


A bounded vector section \((x_\varphi)_{\varphi \in F(p)}\) is weakly continuous if \(\varphi \mapsto \langle x_\varphi, a_\varphi \rangle_\varphi\) is continuous on \(F(p)\) for all \(ap\) in \(Ap\).

A weakly continuous vector section \((x_\varphi)_{\varphi \in F(p)}\) is continuous if \(\varphi \mapsto \langle x_\varphi, x_\varphi \rangle_\varphi\) is also continuous on \(F(p)\).

Let us denote the continuous field of Hilbert spaces thus obtained by \((F(p), \{H_\varphi\}_\varphi, Ap)\).

The following result says that there are no more continuous admissible vector sections in \((F(p), \{H_\varphi\}_\varphi, Ap)\) other than those arising from elements of \(Ap\).

**Theorem 3.4** ([20, 3.2]). The image of \(Ap\) under the linear embedding \(xp \mapsto (x_\varphi)_\varphi\) of \(A^{**}p\) into \(\prod_{\varphi \in F(p)} H_\varphi\) coincides with the set of all continuous admissible vector sections in the continuous field of Hilbert spaces \((F(p), \{H_\varphi\}_\varphi, Ap)\). Consequently,

\[
Ap = \{xp \in A^{**}p : \varphi \mapsto \langle x_\varphi, x_\varphi \rangle_\varphi = \varphi(x^*x) \text{ and } \varphi \mapsto \langle x_\varphi, a_\varphi \rangle_\varphi = \varphi(a^*x) \text{ are continuous on } F(p) \text{ for all } a \text{ in } A\}.
\]

Let \(W_p\) be the set of weakly continuous admissible vector sections in \((F(p), \{H_\varphi\}_\varphi, Ap)\).

In other words,

\[
W_p = \{xp \in A^{**}p : \varphi \mapsto \langle x_\varphi, a_\varphi \rangle_\varphi = \varphi(a^*x) \text{ is continuous on } F(p) \text{ for all } a \text{ in } A\}.
\]

The following extension of Kadison function representation is useful for our work. The classical one deals with the case \(p = 1\) (see, e.g., [14]). In the following, \(A_{sa}\) (resp. \(A_{sa}^{**)\)) denotes the set of all self-adjoint elements of \(A\) (resp. \(A^{**}\)).

**Proposition 3.5** ([5, 3.5]). \(pA_{sa}p\) (resp. \(pA_{sa}^{**}p\)) is isometrically linear and order isomorphic to the Banach space of all continuous (resp. bounded) real affine functionals of \(F(p)\) vanishing at zero. In particular, for any \(x\) in \(A^{**}\), we have

\[
pxp \in pAp \quad \text{if and only if} \quad \varphi \mapsto \varphi(pxp) = \varphi(x) \text{ is continuous on } F(p).
\]

**Corollary 3.6** ([20, 4.1]). Let \(xp \in A^{**}p\).

1. \(W_p = \{xp \in A^{**}p : pa^*xp \in pAp, \forall a \in A\}\).
2. \(Ap = \{xp \in A^{**}p : px^*xp \in pAp \text{ and } pa^*xp \in pAp, \forall a \in A\}\).
3. \(Ap = \{xp \in A^{**}p : pw^*xp \in pAp, \forall wp \in W_p\}\).

Motivated by the fact that elements of \(A^{**}p\) are exactly the admissible vector sections in \(\prod_{\varphi \in F(p)} H_\varphi\), we make the following definition.

**Definition 3.7.** Let \(T_\varphi\) be in \(B(H_\varphi)\) for each \(\varphi\) in \(F(p)\). The operator section \((T_\varphi)_{\varphi \in F(p)}\) is said to be admissible if

\[
T_{\psi \varphi}T_\varphi = T_\psi T_{\psi \varphi}
\]

whenever \(\psi, \varphi \in F(p)\) such that \(0 \leq \psi \leq \lambda \varphi\) for some \(\lambda > 0\).

**Lemma 3.8.** Let \((T_\varphi)_{\varphi \in F(p)}\) be an operator section in \(\prod_{\varphi \in F(p)} B(H_\varphi)\). The following are all equivalent to each other.
We apply the closed graph theorem to establish the boundedness of $xp$.

(1) $(T_\varphi)_{\varphi \in F(p)}$ is admissible.

(2) $(T_\varphi)_{\varphi \in F(p)}$ sends continuous admissible vector sections to admissible vector sections; that is, it induces a linear operator $T$ from $Ap$ into $A^{**}p$.

(3) $(T_\varphi)_{\varphi \in F(p)}$ sends admissible vector sections to admissible vector sections; that is, it induces a linear operator $T$ from $A^{**}p$ into $A^{**}p$.

**Proof.** Firstly, we note that the assertions in (2) and (3) follow from Theorems 3.3 and 3.4.

(3) $\implies$ (2) is clear.

(2) $\implies$ (1): Suppose that $(T_\varphi(a\omega_\varphi))_{\varphi \in F(p)}$ is admissible for each $a$ in $A$. Hence there is an $xp$ in $A^{**}p$ such that $x\omega_\varphi = T_\varphi(a\omega_\varphi)$, $\forall \varphi \in F(p)$, by Theorem 3.3. Let $\psi_1, \varphi_1 \in F(p)$ such that $0 \leq \psi_1 \leq \lambda \varphi_1$ for some $\lambda > 0$. Then

$$T_{\psi_1}T_\varphi(a\omega_\varphi) = T_{\psi_1}(x\omega_\varphi) = x\omega_\psi_1 = T_\psi(a\omega_\psi) = T_\psi T_{\psi_1}(a\omega_\varphi).$$

Since $\pi_p(A)\omega_\varphi$ is dense in $H_\varphi$, $T_{\psi_1}T_\varphi = T_\psi T_{\psi_1}$. As a result, $(T_\varphi)_{\varphi \in F(p)}$ is an admissible operator section.

(1) $\implies$ (3): We suppose that $(T_\varphi)_{\varphi \in F(p)}$ is an admissible operator section. We want to show that $y_\varphi = T_\varphi(x\omega_\varphi)$, $\varphi \in F(p)$, defines an admissible vector section for each $x$ in $A^{**}$. Let $\psi, \varphi \in F(p)$ such that $0 \leq \psi \leq \lambda \varphi$ for some $\lambda > 0$. Observe that

$$T_{\psi}(y_\varphi) = T_{\psi}(T_\varphi(x\omega_\varphi)) = T_\psi(T_{\psi_1}(x\omega_\varphi)) = T_\psi(x\omega_\psi) = y_\psi.$$ 

This proves the admissibility of $(y_\varphi)_{\varphi \in F(p)}$. \hfill \square

**Lemma 3.9.** Every admissible operator section $(T_\varphi)_{\varphi \in F(p)}$ induces a unique bounded linear operator $T$ in $B(A^{**}p)$ such that the vector section representing $T(xp)$ is $(T_\varphi(x\omega_\varphi))_{\varphi \in F(p)}$. In this case, we write $T = (T_\varphi)_{\varphi \in F(p)}$.

**Proof.** In view of the proof of Lemma 3.8, we can define $T : A^{**}p \rightarrow A^{**}p$ by

$$T(xp)\omega_\varphi = T_\varphi(x\omega_\varphi), \quad \varphi \in F(p).$$

We apply the closed graph theorem to establish the boundedness of $T$. Assume $x_n p \rightarrow xp$ and $T(x_n p) \rightarrow yp$ in norm. If $yp \neq T(xp)$ then there is a $\varphi$ in $F(p)$ such that $y\omega_\varphi \neq T(xp)\omega_\varphi = T_\varphi(x\omega_\varphi)$. But they are both the limit of $T_\varphi(x_n \omega_\varphi) = T(x_n p)\omega_\varphi$, a contradiction. So $\|T\| < \infty$. \hfill \square

**Definition 3.10.** A bounded linear operator $T$ in $B(A^{**}p)$ is said to be decomposable if for each $\varphi$ in $F(p)$ there is a $T_\varphi$ in $B(H_\varphi)$ such that $(T_\varphi x)\omega_\varphi = T_\varphi(x\omega_\varphi)$ for all $x$ in $A^{**}$. In other words, $T = (T_\varphi)_{\varphi \in F(p)}$ (cf. Lemma 3.9). Note that the operator section $(T_\varphi)_{\varphi \in F(p)}$ must be admissible in this case (Lemma 3.8).

It is clear that all operators $T$ in $\pi_p(A^{**})$ are decomposable. In fact, $T = \pi_p(t)$ for some $t$ in $A^{**}$, and thus we can set $T_\varphi = \pi_\varphi(t)$, $\forall \varphi \in F(p)$. We are going to prove that every decomposable operator in $B(A^{**}p)$ arises in this way.
Lemma 3.11. If \((T_\varphi)_{\varphi \in F(p)}\) is an admissible section of operators in \(\prod_{\varphi \in F(p)} B(H_\varphi)\) then \(T_\varphi\) belongs to the double commutant \(\pi_\varphi(A)^{''}\) of \(\pi_\varphi(A)\) in \(B(H_\varphi)\) for each \(\varphi \in F(p)\).

**Proof.** Let \(\varphi \in F(p)\) and \(q\) a projection in \(\pi_\varphi(A)'' \subseteq B(H_\varphi)\). Define a linear functional \(\psi\) on \(A\) by

\[\psi(a) = \langle a_\omega, q_\omega \rangle_\varphi.\]

It is easy to see that \(\psi \in F(p)\) and \(0 \leq \psi \leq \varphi\). Observe that for \(a, b \in A\),

\[\langle T^*_\psi (a_\omega), b_\omega \rangle_\varphi = \langle a_\psi, T_{\psi \varphi} (b_\omega) \rangle_\psi = \langle a_\psi, b_\omega \rangle_\psi = \psi(b^* a) = \langle b^* a_\omega, q_\omega \rangle_\varphi = \langle a_\varphi, b_\omega \rangle_\varphi = \langle q a_\varphi, b_\omega \rangle_\varphi.\]

We thus have \(q a_\varphi = T^*_\psi \psi (a_\omega)\) for all \(a \in A\). In particular, \(q H_\varphi = \frac{T^*_\psi \psi}{H_\varphi}\). By the admissibility condition, we have \(T_{\psi \varphi} T_\varphi = T_\psi T_{\psi \varphi}\) and thus \(T^*_\varphi T^*_\psi = T^*_\psi T^*_\varphi\). It follows that \(q H_\varphi\) is invariant under \(T^*_\varphi\). Apply the same argument to \(1 - q\), we can conclude that \(q H_\varphi\) is a reducing subspace of \(T^*_\varphi\). Hence \(q T^*_\varphi = T^*_\varphi q\) for every projection \(q\) in the von Neumann algebra \(\pi_\varphi(A)'\). It follows that \(T^*_\varphi \in \pi_\varphi(A)^{''}\) and thus \(T_\varphi \in \pi_\varphi(A)^{''}\) for each \(\varphi \in F(p)\). \(\square\)

**Theorem 3.12.** Let \(A\) be a C*-algebra, \(p\) a closed projection in \(A^{**}\) with central support \(c(p)\) and \(T \in B(A^{**})\). Then \(T \in \pi_p(A^{**})\) if and only if \(T\) is decomposable. In this case, if \(T = (T_\varphi)_{\varphi \in F(p)} = \pi_p(t)\) for some \(t\) in \(A^{**}\) then \(\|T\|_{B(A^{**})} = \sup_{\varphi \in F(p)} \|T_\varphi\| = \|tc(p)\|\).

**Proof.** We check the sufficiency only. Suppose that \(T\) induces an operator section \((T_\varphi)_{\varphi \in F(p)}\) in \(\prod_{\varphi \in F(p)} B(H_\varphi)\). In view of Lemma 2.2, we need only verify that \(T\) commutes with right multiplications \(R_{xyp}\) for all \(x \in A^{**}\); i.e., for every \(y \in A^{**}\), \(T(R_{xyp}yp) = R_{xyp}(Typ)\). In other words,

\[T(yypxp) = (Typ)xp;\]

or equivalently,

\[T(yypxp)a_\varphi = (T(yyp)xp)a_\varphi, \quad \forall \varphi \in F(p).\]

By Lemma 3.11, for each \(\varphi \in F(p)\) we can choose a \(t_\varphi\) in \(A^{**}\) such that

\[\pi_\varphi(t_\varphi) = T_\varphi.\]

The admissibility of \((T_\varphi)_{\varphi \in F(p)}\) says that \(T_{\psi \varphi} T_\varphi = T_{\psi \varphi} T_\varphi\). Consequently,

\[\pi_\psi(t_\psi) T_{\psi \varphi} = T_{\psi \varphi} \pi_\varphi(t_\varphi)\]

whenever \(\varphi, \psi \in F(p)\) such that \(0 \leq \psi \leq \lambda \varphi\) for some \(\lambda > 0\). In this case, we have

\[t_\psi y_\omega = \pi_\psi(t_\psi) T_{\psi \varphi} (y_\omega) = T_{\psi \varphi} \pi_\varphi(t_\varphi) (y_\omega) = T_{\psi \varphi} (t_\varphi y_\omega) = t_\varphi y_\omega\]

for every \(y \in A^{**}\), and thus

\[\pi_\psi(t_\psi) = \pi_\psi(t_\varphi) \text{ in } B(H_\psi).\]

Moreover, we note that

\[p_\varphi = \omega_\varphi \text{ and } T(xp) = (T(xp))p \in A^{**}p, \quad \forall \varphi \in F(p), \forall x \in A^{**}.\]
For each $x$ in $A^{**}$ with $\|x\| \leq 1$ and $\varphi$ in $F(p)$ we define $\psi, \rho$ in $F(p)$ by

$$\psi(t) = \langle a_\varphi, px_\varphi \rangle_{\varphi} \quad \text{and} \quad \rho = \frac{\varphi + \psi}{2}.$$ 

Since $0 \leq \varphi \leq 2\rho$ and $0 \leq \psi \leq 2\rho$, by (1) we have

$$\pi_\varphi(t_\varphi) = \pi_\varphi(t_\rho) \quad \text{and} \quad \pi_\psi(t_\psi) = \pi_\psi(t_\rho).$$

It follows that

$$\pi_\varphi(t_\varphi) = \pi_\varphi(t_\rho) \quad \text{and} \quad \pi_\psi(t_\psi) = \pi_\psi(t_\rho).$$

Therefore, by (4)

$$T(ypxp)x_\varphi = T_\varphi(ypx_\varphi) = \pi_\varphi(t_\varphi)(ypx_\varphi) = \pi_\varphi(t_\rho)(ypx_\varphi) = (t_\rho ypxp)x_\varphi.$$ 

Observe also that for every $y$ in $A^{**}$, by (2) and (3) we have,

$$\langle (Typ)x_\varphi, ypx_\varphi \rangle_{\varphi} = \langle (Typ)y_\varphi, y_\varphi \rangle_{\psi} = \langle T_\psi(y_\varphi), y_\varphi \rangle_{\psi} = \langle \pi_\psi(t_\varphi)y_\varphi, y_\varphi \rangle_{\psi} = \langle \pi_\psi(t_\rho)y_\varphi, y_\varphi \rangle_{\psi} = \langle t_\rho ypxp_\varphi, ypx_\varphi \rangle_{\varphi}.$$ 

Therefore, ((Typ) - t_\rho yyp)x_\varphi \in (A^{**}px_\varphi)^+$. It follows

$$(Typ)x_\varphi = t_\rho ypx_\varphi.$$ 

Consequently, by (4)

$$(Typ)y_\varphi = t_\rho ypx_\varphi = ((Typ)x_\varphi)y_\varphi, \quad \forall \varphi \in F(p),$$

and thus $T(ypxp) = (Typ)x_\varphi$, as asserted.

For the norm equalities, we choose a $t$ in $A^{**}$ by Lemma 2.2 such that $T = \pi_\varphi(t)$ and

$$\|T\|_{B(A^{**})} = \|tc(p)\| = \sup_{\varphi \in F(p)} \|\pi_\varphi(t)\| = \sup_{\varphi \in F(p)} \|T_\varphi\|.$$ 

Let

$$\text{QM}(A, p) = \{x \in A^{**} : pAxAp \subseteq pAp\}$$

the Banach space of relative quasi-multipliers of $A$ associated to $p$. By Corollary 3.6, for any $x$ in $A^{**}$, we have $x \in \text{QM}(A, p)$ if and only if $\pi_p(x) \in B(Ap, W_p)$; i.e., $\pi_p(x)$ sends continuous admissible vector sections to weakly continuous admissible vector sections in $(F(p), \{H_\varphi\}_\varphi, Ap)$.

**Theorem 3.13.** Let $A$ be a C*-algebra and $p$ a closed projection in $A^{**}$ with central support $c(p)$. Assume $T$ in $B(Ap, W_p)$ satisfies the condition that

$$\varphi(a^* a) = 0 \implies \varphi((Tap)^* (Tap)) = 0$$

whenever $\varphi$ is a pure state in $F(p)$ and $a \in A$. Then $T$ can be extended to a decomposable operator in $B(A^{**}p)$, denoted again by $T$, such that $T = \pi_p(t)$ for some $t$ in $\text{QM}(A, p)$ and

$$\|T\|_{B(A^{**}p)} = \|T\|_{B(A^{**})} = \|tc(p)\|.$$ 

**Proof.** We first recall that

$$\|a_\varphi\| = \langle a_\varphi, a_\varphi \rangle_{\varphi} = \varphi(a^* a), \quad \forall a \in A, \forall \varphi \in F(p).$$
Let $X = F(p) \cap P(A)$, where $P(A)$ is the pure state space of $A$. By hypothesis and the Kadison transitivity theorem, for each $\varphi$ in $X$ we can define a linear map $T_{\varphi}$ on $H_{\varphi} = A \omega_{\varphi}$ by

$$T_{\varphi}(a \omega_{\varphi}) = (T(ap)) \omega_{\varphi}.$$ 

Let $\varphi \in X$ and $a \omega_{\varphi} \in H_{\varphi}$ such that $\|a \omega_{\varphi}\| = 1$. Again by the Kadison transitivity theorem, there is a $b$ in $A$ such that $b \omega_{\varphi} = a \omega_{\varphi}$ and $\|b\| = 1$. Hence

$$\|T_{\varphi}(a \omega_{\varphi})\| = \|T_{\varphi}(b \omega_{\varphi})\| = \|(T(bp)) \omega_{\varphi}\| \leq \|T\| \|bp\| \leq \|T\|.$$ 

Therefore, $\|T_{\varphi}\| \leq \|T\|$ for every $\varphi$ in $X$. Consequently, we have $\sup_{\varphi \in X} \|T_{\varphi}\| \leq \|T\|$.

Now assume $\varphi$ belongs to $\overline{X}$, the weak* closure of $X$, and $a, b \in A$. Since $T(ap) \in W_{\varphi}$, the scalar functions $\psi \mapsto \|a \omega_{\varphi}\|_{\psi}$, $\psi \mapsto \|b \omega_{\varphi}\|_{\psi}$ and $\psi \mapsto \langle (T(ap)) \omega_{\varphi}, b \omega_{\varphi} \rangle_{\psi}$ are all continuous on $F(p)$. It follows that

$$| \langle (T(ap)) \omega_{\varphi}, b \omega_{\varphi} \rangle_{\varphi} | \leq (\sup_{\psi \in X} \|T_{\varphi}\|) \|a \omega_{\varphi}\|_{\varphi} \|b \omega_{\varphi}\|_{\varphi} \leq \|T\| \|a \omega_{\varphi}\|_{\varphi} \|b \omega_{\varphi}\|_{\varphi}.$$ 

Hence $T_{\varphi}$ in $B(H_{\varphi})$ exists such that

$$T_{\varphi}(a \omega_{\varphi}) = (T(ap)) \omega_{\varphi}, \quad \forall a \in A, \forall \varphi \in \overline{X}.$$ 

Moreover, $\|T_{\varphi}\| \leq \|T\|$ for every $\varphi$ in $X = (F(p) \cap P(A))$.

Note that $X \cup \{0\}$ is the extreme boundary of the compact convex set $F(p)$. Consequently, continuous affine functionals of $F(p)$ assume extrema at points in $X$. From Proposition 3.5, we know that there is an order-preserving linear isometry from $pA_{sa}p$ into $C_{\mathbb{R}}^R(X)$, the Banach space of continuous real-valued functions defined on the compact Hausdorff space $\overline{X}$. Hence each $\varphi$ in $F(p)$ has a (non-unique) Hahn-Banach positive extension $m_{\varphi}$ in the space $M(\overline{X})$ ($\cong C_{\mathbb{R}}(\overline{X})^*$) of regular finite Borel measures on $\overline{X}$. By handling real and imaginary parts separately, we can write for each $\varphi$ in $F(p)$

$$\varphi(a) = \varphi(pap) = \int_{\overline{X}} \psi(pap) dm_{\varphi}(\psi) = \int_{\overline{X}} \psi(a) dm_{\varphi}(\psi), \quad \forall a \in A.$$ 

For any $a, b$ in $A$, since $T(ap) \in W_{\varphi}$, we have $pb^*(T(ap)) \in pAp$ by Corollary 3.6. Therefore, the continuous affine function $\psi \mapsto \psi(pb^*(Tap)) = \langle (Tap) \omega_{\psi}, b \omega_{\psi} \rangle_{\psi}$ satisfies the barycenter
Let $\psi \in X$. Consequently, by (5) we have

\[
\begin{align*}
| \langle T(ap)\omega_\psi, b\omega_\psi \rangle_{\varphi} | & = \left| \int_X \langle T(ap)\omega_\psi, b\omega_\psi \rangle_{\varphi} \ dm_\varphi(\psi) \right| \\
& = \left| \int_X \langle T_\psi(a\omega_\psi), b\omega_\psi \rangle_{\psi} \ dm_\psi(\psi) \right| \\
& \leq \int_X ||T_\psi|| \|a\omega_\psi\| \|b\omega_\psi\| \ dm_\psi(\psi)
\end{align*}
\]

\[
\leq \left( \sup_{\psi \in X} ||T_\psi|| \right) \left( \int_X \|a\omega_\psi\|^2 \ dm_\psi(\psi) \right)^{\frac{1}{2}} \left( \int_X \|b\omega_\psi\|^2 \ dm_\psi(\psi) \right)^{\frac{1}{2}}
\]

\[
= \left( \sup_{\psi \in X} ||T_\psi|| \right) \left( \int_X \psi(a^*a) \ dm_\psi(\psi) \right)^{\frac{1}{2}} \left( \int_X \psi(b^*b) \ dm_\psi(\psi) \right)^{\frac{1}{2}}
\]

\[
= \left( \sup_{\psi \in X} ||T_\psi|| \right) \varphi(a^*a)^{\frac{1}{2}} \varphi(b^*b)^{\frac{1}{2}}
\]

\[
\leq ||T|| \|a\omega_\varphi\| \|b\omega_\varphi\|.
\]

Hence, a bounded linear operator $T_\varphi$ in $B(H_\varphi)$ exists such that $T_\varphi(a\omega_\varphi) = (T(ap))\omega_\varphi$ for every $a$ in $A$. Moreover,

\[ ||T_\varphi|| \leq ||T||, \quad \forall \varphi \in F(p). \]

At this point, we have shown that $T$ can be written as an admissible section of operators $T = (T_\varphi)_{\varphi \in F(p)}$ in $\prod_{\varphi \in F(p)} B(H_\varphi)$ (cf. Lemma 3.8). Extend $T$ to a bounded linear operator on $A^{**}p$ as in Lemma 3.9. Consequently by Theorem 3.12, there is a $t$ in $A^{**}$ such that $T = \pi_p(t)$ and $||T||_{B(A^{**}p)} = \sup_{\varphi \in F(p)} ||T_\varphi||_{B(H_\varphi)} = ||tc(p)||$. Since $T(Ap) \subseteq W_p$, we have $p\varphi(Tap) \in pAp$ by Corollary 3.6. Hence $pAtAp \subseteq pAp$. As a result, $t \in QM(A,p)$. Finally, we note that

\[ ||T||_{B(Ap,W_p)} \leq ||T||_{B(A^{**}p)} = \sup_{\varphi \in F(p)} ||T_\varphi||_{B(H_\varphi)} \leq ||T||_{B(Ap,W_p)}. \]

\[ \square \]

Let

\[ LM(A,p) = \{ x \in A^{**} : xAp \subseteq Ap \}, \]

the Banach algebra of relative left multipliers of $A$ associated to $p$.

**Corollary 3.14.** Let $A$ be a $C^*$-algebra, $p$ a closed projection in $A^{**}$ with central support $c(p)$ and $T \in B(Ap)$. The following are all equivalent.

1. $T \in \pi_p(LM(A,p))$.
2. $T$ is decomposable.
3. $\varphi(a^*a) = 0$ implies $\varphi((Tap)^*(Tap)) = 0$ whenever $\varphi$ is a pure state supported by $p$ and $a$ in $A$.

In this case, if $t \in LM(A,p)$ such that $T = \pi_p(t)$ then $||T||_{B(Ap)} = ||tc(p)||$. 

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4. Commutants and density theorems

**Definition 4.1.** Let $A$ be a C*-algebra and $p$ a closed projection in $A^{**}$. Let

\[
\begin{align*}
LM(A, p) &= \{ x \in A^{**} : xAp \subseteq Ap \}, \\
RM(A, p) &= \{ x \in A^{**} : pAx \subseteq pA \}, \\
M(A, p) &= \{ x \in A^{**} : xAp \subseteq Ap, pAx \subseteq pA \}, \text{ and} \\
QM(A, p) &= \{ x \in A^{**} : pAxAp \subseteq pAp \}
\end{align*}
\]

the sets of relative left multipliers, relative right multipliers, relative multipliers and relative quasi-multipliers associated to $p$. We define the relative left strict topology, relative right strict topology, relative strict topology and relative quasi-strict topology of $A^{**}$ associated to $p$ by the seminorms $x \mapsto \|xap\|$, $x \mapsto \|pax\|$, $x \mapsto \|xap\| + \|pbx\|$ and $x \mapsto \|paxbp\|$ for $a, b \in A$.

**Remarks 4.2.**

1. It is easy to see that $LM(A) \subseteq LM(A, p)$, $RM(A) \subseteq RM(A, p)$, $...$, and all of them are norm closed subspaces of $A^{**}$.

2. $QM(A, p)$ is *-invariant whereas $LM(A, p)^* = RM(A, p)$. Moreover, both $LM(A, p)$ and $RM(A, p)$ are Banach algebras, and $M(A, p) = LM(A, p) \cap RM(A, p)$ is a C*-algebra.

3. The relative strict topologies associated to $p$ are Hausdorff if and only if the central support $c(p)$ of $p$ equals 1.

**Theorem 4.3.** Let $A$ be a C*-algebra and $p$ a closed projection in $A^{**}$. Then $LM(A, p)$ (resp. $RM(A, p)$, $M(A, p)$ and $QM(A, p)$) coincides with the closure of $A$ in $A^{**}$ with respect to the relative left strict (resp. right strict, strict and quasi-strict) topology associated to $p$.

Moreover, the unit ball (resp. its self-adjoint part, positive part) of $A$ is dense in the unit ball (resp. its self-adjoint part, positive part) of $LM(A, p)$, $RM(A, p)$, $M(A, p)$ and $QM(A, p)$ in the corresponding relative strict topologies associated to $p$, respectively.

**Proof.** We prove only the assertion about relative left multipliers since all others follow in a similar manner. In the following, we denote by $B_{sa}$ (resp. $B_+$, $B_1$) the set of all self-adjoint elements (resp. positive elements, elements of norm not greater than 1) in $B$ whenever $B$ is a subset of $A$ or $A^{**}$.

Assume $x \in LM(A, p)$. We want to show that $x$ belongs to the relative left strict closure of $A$. Let $a_1, a_2, ..., a_n \in A$. Consider the convex set $V$ in the direct sum $(Ap)^n = Ap \oplus \ldots \oplus Ap$ given by

\[
V = \{(ba_1p, \ldots, ba_np) : b \in A\}.
\]

(In case $x \in A_1^{**}, x \in A_{sa}^{**} \cap A_1^{**}$ or $x \in A_1^{**} \cap A_1^{**}$, we replace $A$ by $A_1$, $A_{sa} \cap A_1$ or $A_+ \cap A_1$ in the definition of $V$, respectively.) Since $x \in LM(A, p)$, we have $\tilde{x} = (xa_1p, xa_2p, \ldots, xa_np) \in (Ap)^n$. If $\tilde{x} \notin \overline{V}^{\|\cdot\|}$ then there is an $f$ in $(Ap)^n$ such that

\[
\text{Re } \tilde{f}(\tilde{x}) < -1 \leq \text{Re } \tilde{f}(\tilde{b}), \quad \forall \tilde{b} \in V,
\]

(7)
where \( \tilde{b} = (ba_1p, ba_2p, \ldots, ba_np) \). Since \((Ap)^* \cong A^{**}F(p)\) (see, e.g., [12]), we can write \( \tilde{f} = f_1 + f_2 + \ldots + f_n \) such that \( f_k = y_k^* \varphi_k \) for some \( y_k \) in \( A^{**} \) and \( \varphi_k \) in \( F(p) \), \( k = 1, 2, \ldots, n \). Hence
\[
\tilde{f}(\tilde{x}) = \sum_{k=1}^{n} f_k(xakp) = \sum_{k=1}^{n} \varphi_k(y_k^* xa_k) = \sum_{k=1}^{n} \langle xa_k \omega_{\varphi_k}, y_k \omega_{\varphi_k} \rangle_{\varphi_k}
\]
and
\[
\tilde{f}(\tilde{b}) = \sum_{k=1}^{n} f_k(bakp) = \sum_{k=1}^{n} \varphi_k(y_k^* bak) = \sum_{k=1}^{n} \langle bak \omega_{\varphi_k}, y_k \omega_{\varphi_k} \rangle_{\varphi_k}.
\]
Let \( \{b_\lambda\}_\lambda \) be a net in \( A \) such that \( b_\lambda \) converges to \( x \) \( \sigma \)-weakly. (In case \( x \in A_1^{**}, x \in A_{sa}^{**} \cap A_1^{**} \) or \( x \in A_+^{**} \cap A_1^{**} \), the Kaplansky density theorem (see, e.g., [11]) enables us to choose \( b_\lambda \)'s from \( A_1, A_{sa} \cap A_1 \) or \( A_+ \cap A_1 \), respectively.) In particular,
\[
\langle b_\lambda a_k \omega_{\varphi_k}, y_k \omega_{\varphi_k} \rangle_{\varphi_k} \longrightarrow \langle xa_k \omega_{\varphi_k}, y_k \omega_{\varphi_k} \rangle_{\varphi_k} \quad \text{for} \quad k = 1, 2, \ldots, n.
\]
Therefore, \( \tilde{f}(\tilde{b}) \longrightarrow \tilde{f}(\tilde{x}) \) where \( \tilde{b}_\lambda = (b_\lambda a_1p, b_\lambda a_2p, \ldots, b_\lambda a_np) \in V \). This contradicts (7) and thus \( \tilde{x} \in \mathbf{V}_0 \). This shows that for any positive \( \varepsilon \) and \( a_1, a_2, \ldots, a_n \) in \( A \) there is a \( b \) in \( A \) such that
\[
\| (x - b) a_k p \| < \varepsilon \quad \text{for} \quad k = 1, 2, \ldots, n.
\]
In other words, \( x \) belongs to the relative left strict closure of \( A \). (In case \( x \) comes from \( A_1^{**}, A_{sa}^{**} \cap A_1^{**} \) or \( A_+^{**} \cap A_1^{**} \), we can choose \( b \) from \( A_1, A_{sa} \cap A_1 \) or \( A_+ \cap A_1 \), respectively.) Our assertion follows since the opposite inclusion is obvious.

**Theorem 4.4.** The closure of \( \pi_p(A) \) in \( B(Ap) \) with respect to the strong operator topology (SOT) as well as the weak operator topology (WOT) coincides with \( \pi_p(LM(A,p)) \). Moreover, the unit ball of \( \pi_p(A) \) is SOT as well as WOT dense in the unit ball of \( \pi_p(LM(A,p)) \).

**Proof.** It is well-known that a linear functional on \( B(E), E \) a Banach space, is continuous with respect to SOT if and only if it is continuous with respect to WOT. Since \( \pi_p(A) \) is convex, its closures in \( B(Ap) \) with respect to these topologies coincide. We are going to show that they are identical to \( \pi_p(LM(A,p)) \).

Let \( \{a_\lambda\}_\lambda \) be a net in \( A \) such that \( \pi_p(a_\lambda) \) converges to some bounded linear operator \( T \) in SOT. By Corollary 3.14, to see \( T \in \pi_p(LM(A,p)) \) we just need to check whether the condition \( \varphi(a^*a) = 0 \) implies \( \varphi((Tap)^*(Tap)) = 0 \) whenever \( \varphi \) is a pure state in \( F(p) \) and \( a \in A \). In this case, \( ap_\varphi = 0 \) where \( p_\varphi \) is the support projection of the pure state \( \varphi \). Now
\[
(Tap)p_\varphi = (\lim \pi_p(a_\lambda)ap)p_\varphi = \lim a_\lambda ap_\varphi = 0.
\]
Hence \( \varphi((Tap)^*(Tap)) = 0 \), as asserted. Thus
\[
\pi_p(A) \overset{\text{SOT}}{\subseteq} \pi_p(LM(A,p)).
\]
The opposite inclusion and other assertions follow from Theorem 4.3 since the strong operator topology of \( B(Ap) \) restricted to \( \pi_p(LM(A,p)) \) coincides with the one induced by the relative left strict topology of \( A^{**} \) associated to \( p \).

□
Remark 4.5. In [18], Tomita defined the notion of $Q^*$–topology. In fact, it is the double strong operator topology (DSOT) of $\pi_p(M(A,p))$ which is defined by seminorms

$$\pi_p(x) \mapsto \|xap\| + \|x^*ap\|, \quad \forall a \in A.$$ 

Since $RM(A_p)^* = LM(A_p)$ and $M(A,p) = LM(A_p) \cap RM(A,p)$, Theorems 4.3 and 4.4 imply $\pi_p(A)_{DSOT} = \pi_p(M(A,p))$. Moreover, the unit ball of $\pi_p(A)$ (resp. its self-adjoint part, positive part) is DSOT dense in the unit ball (resp. its self-adjoint part, positive part) of $\pi_p(M(A,p))$. Another way to look at $\pi_p(M(A,p))$ is to observe that it coincides with the family of all adjointable admissible operator sections $\{T_\varphi\}_\varphi$ in $\prod_{\varphi \in F(p)} B(H_\varphi)$. We say that $\{T_\varphi\}_\varphi$ is adjointable if the operator section $\{T_\varphi^*\}_\varphi$ is admissible (see Corollary 3.14). Tomita expected that in some situations the double commutant $\pi_p(A)^{\prime\prime}$ of $\pi_p(A)$ in $B(Ap)$ is the $C^*$-algebra $\pi_p(M(A,p))$. However, as indicated by the Theorem 4.8 below, we shall see that the Banach algebra $\pi_p(LM(A,p))$ is a more appropriate object to look for.

Recall that a projection $r$ in $A^{\ast\ast}$ is closed if the face $F(r) = \{\varphi \in Q(A) : \varphi(1 - r) = 0\}$ of $Q(A)$ supported by $r$ is weak* closed, and $r$ is compact if $F(r) \cap S(A)$ is weak* closed [2]. An element $h$ of $pA^{\ast\ast}_sa$ is called $q$–continuous on $p$ [4] if the spectral projection $E_F(h)$ (computed in $pA^{\ast\ast}p$) is closed for every closed subset $F$ of $\mathbb{R}$. Also, $h$ is called strongly $q$–continuous on $p$ [5] if, in addition, $E_F(h)$ is compact whenever $F$ is closed and $0 \notin F$.

Lemma 4.6 ([5, 3.43]). Let $h \in pA^{\ast\ast}_sa$.

1. $h$ is strongly $q$–continuous on $p$ if and only if $h = pa = ap$ for some $a$ in $A_{sa}$.
2. In case $A$ is $\sigma$–unital, $h$ is $q$–continuous on $p$ if and only if $h = px = xp$ for some $x$ in $M(A)_{sa}$.

In general, $h$ in $pA^{\ast\ast}p$ is said to be $q$–continuous or strongly $q$–continuous if both Re$h$ and Im$h$ are. Denote by $QC(p)$ (resp. $SQC(p)$) the set of all $q$–continuous elements (resp. strongly $q$–continuous elements) on $p$. $SQC(p)$ is always a $C^*$-algebra, and so is $QC(p)$ if $A$ is $\sigma$–unital. We say that $p$ has MQC (“many $q$–continuous elements”) or MSQC (“many strongly $q$–continuous elements”) if $QC(p)$ or $SQC(p)$ is $\sigma$–weakly dense in $pA^{\ast\ast}p$, respectively [8].

Lemma 4.7 ([8, 3.1 and 3.3]). The following statements are all equivalent.

1. $p$ has MSQC.
2. $pAp = SQC(p)$.
3. $pAp$ is an algebra.
4. $pAp$ is a Jordan algebra.
5. $F(p)$ is isomorphic to the quasi-state space of a $C^*$-algebra.
6. $p \in M(A,p)$, i.e., $pAp \subseteq pA \cap Ap$.
7. $p \in QM(A,p)$, i.e., $pApA \subseteq pAp$.

In this case,


When the closed projection \( p \) has MSQC, it shares many good properties with the projection 1. Moreover, every central closed projection in \( A^{**} \) has MSQC.

The first part of the following theorem says that all bounded \( A \)-module maps in \( B(Ap) \) are right multiplications provided that \( A \) is \( \sigma \)-unital.

**Theorem 4.8.** Let \( A \) be a C*-algebra, \( p \) a closed projection in \( A^{**} \) and \( \pi_p \) the left regular representation of \( A \) on \( Ap \). Denote by \( \pi_p(A)' \) the commutant and by \( \pi_p(A)'' \) the double commutant of \( \pi_p(A) \) in \( B(Ap) \). Denote by \( \mathcal{Y} \) the set \( \{ x \in \text{RM}(A) : xp = pxp \} \). If \( A \) is \( \sigma \)-unital then

\[
\pi_p(A)' = \{ R_{pxp} : x \in \mathcal{Y} \}.
\]

If \( A \) is \( \sigma \)-unital and \( p \) has MQC then we also have

\[
\pi_p(A)'' = \pi_p(LM(A,p)).
\]

Here \( R_{pxp}(ap) := apxp = axp, \forall a \in A, \forall x \in \mathcal{Y} \).

**Proof.** It is clear that all right multiplications of the form \( R_{pxp} \) with \( x \) in \( \mathcal{Y} \) commute with elements of \( \pi_p(A) \). Conversely, assume \( T \in \pi_p(A)' \subseteq B(Ap) \). If \( \{ u_\lambda \}_\lambda \) is a (bounded) approximate unit of \( A \), the bounded net \( \{ T(u_\lambda p) \}_\lambda \) in \( Ap \) has a weak* cluster point \( xp \) in \( A^{**}p \). For each \( a \) in \( A \), \( axp \) is a weak* cluster point of \( \{ aT(u_\lambda p) \}_\lambda = \{ T(u_\lambda p) \}_\lambda \). But \( T(au_\lambda p) \to T(ap) \) in norm. It follows that \( T(ap) = axp \in Ap \). Therefore, \( \text{Axp} = T(Ap) \subseteq Ap \). By [5, 3.9], we have \( xp \in \text{RM}(A)p \) if \( A \) is \( \sigma \)-unital. Moreover, if \( a, b \in A \) and \( ap = bp \) then \( T(ap) = T(bp) \). This is equivalent to that \( axp = bxp \). Consequently, \( Lxp = \{ 0 \} \) where \( L = A^{**}(1 - p) \cap A \), the norm closed left ideal of \( A \) related to the closed projection \( p \). It follows that \( L^{**}xp = \{ 0 \} \); i.e., \( A^{**}(1 - p)xp = \{ 0 \} \). This forces \( (1 - p)xp = 0 \). Therefore \( xp = pxp \). Hence \( T(ap) = axp = apxp = R_{pxp}(ap) \).

By Theorem 4.4, \( \pi_p(LM(A,p)) \subseteq \pi_p(A)'' \). Let \( T \in \pi_p(A)'' \subseteq B(Ap) \), \( a \in A \) and \( \varphi \) a pure state in \( F(p) \). Assume that \( \varphi(a^*a) = 0 \), or equivalently \( ap_{\varphi} = 0 \), where \( p_{\varphi} \) is the support projection of \( \varphi \) in \( A^{**} \). Since \( p \) is assumed to have MQC and \( A \) is \( \sigma \)-unital, there is a net \( \{ m_{\lambda}p \}_\lambda \) with \( m_{\lambda} \) in \( M(A) \) such that

\[
(8) \quad m_{\lambda}p = pm_{\lambda} \quad \text{and} \quad m_{\lambda}p \to p_{\varphi} \quad \sigma \text{-weakly}
\]

by Lemma 4.6. Hence, \( am_{\lambda}p \to ap_{\varphi} = 0 \) \( \sigma \)-weakly. In particular, \( am_{\lambda} \to 0 \) with respect to \( \sigma(Ap, (Ap)^*) \) since \( (Ap)^* \cong (A/L)^* \cong L^0 \) can be considered as a subspace of \( A^* \), and the \( \sigma \)-weak topology of \( A^{**} \) coincides with \( \sigma(A^{**}, A^*) \). Here \( L^0 \) is the polar of the left ideal \( L = A^{**}(1 - p) \cap A \) in \( A^* \). As a bounded Banach space operator, \( T \) is \( \sigma(Ap, (Ap)^*) - \sigma(Ap, (Ap)^*) \) continuous. Therefore, \( T(am_{\lambda}p) \to 0 \) in the \( \sigma(Ap, (Ap)^*) \) topology of \( Ap \) and thus also \( \sigma \)-weakly. On the other hand, the right multiplication \( R_{pm_{\lambda}p} \) belongs to \( \pi_p(A)' \). As a result, by (8) we have

\[
T(am_{\lambda}p) = T(apm_{\lambda}p) = TR_{pm_{\lambda}p}(ap) = R_{pm_{\lambda}p}T(ap)
\]

\[
= (Tap)pm_{\lambda}p \to (Tap)p_{\varphi} \quad \sigma \text{-weakly}.
\]
We now define $\varphi((\text{Tap})^*(\text{Tap})) = 0$. Now, Corollary 3.14 implies $T \in \pi_p(\text{LM}(A, p))$.

Although it follows from Theorem 4.4 that we always have $\pi_p(\text{LM}(A, p)) \subseteq \pi_p(A)^\prime\prime$, the following example indicates that the inclusion can be strict in case $p$ does not have MQC.

**Example 4.9.** (This example is based on one given in [8, 3.4]) Let $A = C[0, 1] \otimes K$ where $K$ is the C*-algebra of all compact operators on a separable infinite dimensional Hilbert space $H$. Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis of $H$ and $P_n$ the projection on span $\{e_1, e_2, \ldots, e_n\}$. A closed projection in $A$ is given by a projection-valued function $P : [0, 1] \rightarrow B(H)$ such that

$$h \text{ is any weak cluster point of } P(y) \text{ as } y \rightarrow x, \text{ then } h \leq P(x) \text{ [5, Section 5.G].}$$

$P$ describes the atomic part of a closed projection $p$ in $A^{**} \cong C[0, 1]^{**} \otimes B(H)$, and $P$ determines $p$ since a closed projection is determined by its atomic part. In our case $p$ will equal its atomic part.

We now define $P$.

For each $n = 0, 1, 2, \ldots$ we construct recursively a countable subset $S_n$ of $[0, 1]$ and a unit vector $v(x)$ for each $x$ in $S_n$.

**Step 0:** Take $S_0 = \{\frac{1}{2}\}$ and $v(\frac{1}{2}) = e_1$.

**Step 1:** Take $S_1 = \{x_1, x_2, \ldots\}$ where the $x_i$’s are distinct, $x_i \neq \frac{1}{2}$, and $x_i \rightarrow \frac{1}{2}$ as $i \rightarrow \infty$. Let $v(x_i) = 2^{-\frac{1}{2}}e_1 + 2^{-\frac{1}{2}}e_{i+1}$ for $i = 1, 2, \ldots$.

**Step $n$ ($n > 1$):** Write $S_{n-1} = \{x_1, x_2, \ldots\}$. Choose distinct $y_{ij}$’s from $[0, 1]$ but outside $\bigcup_{k=0}^{n-1}S_k$ such that $|y_{ij} - x_i| \leq 2^{-(i+j)}$. Let $S_n = \{y_{ij} : i, j = 1, 2, \ldots\}$ and $v(y_{ij}) = n^{-\frac{1}{2}}v(x_i) + (1 - n^{-1})^{\frac{1}{2}}w_{ij}$, where $w_{ij}$ is a unit vector such that $\langle w_{ij}, v(x_i) \rangle_H = 0$ and $P_{i+j+n}w_{ij} = 0$.

Let $S = \bigcup_{n=0}^\infty S_n$. Define a projection-valued function $P$ on $[0, 1]$ by setting $P(x)$ to be the projection on span $\{v(x)\}$ if $x \in S$, and $P(x) = 0$ otherwise. It is shown in [8] that $P$ describes a closed projection $p$ in $A^{**}$ which is atomic and abelian. Moreover, if $h$ in $pA^{**}p$ satisfies that $h \in pAp$ and $h^2 \in pAp$ then $h = 0$. (In [8], this fact is used to show that $SQC(p) = \{0\}$.)

Now consider the C*-algebra $B = C[-1, 1] \otimes K$. Define a projection-valued function $Q$ on $[-1, 1]$ by putting $Q(t) = P(1 \otimes t), \forall t \in [-1, 1]$. It is clear that $Q$ determines an atomic, abelian and closed projection $q$ in $B^{**}$ such that $k = 0$ whenever $k \in qB^{**}q$ satisfying that $k \in qBq$ and $k^2 \in qBq$.

Let $\tilde{A}$ be the C*-algebra obtained by adjoining an identity to $A$ and $\tilde{p} = p + p_\infty$ where $p_\infty = 0 \oplus 1$ in $\tilde{A}^{**} \cong A^{**} \otimes C$. Thus $\tilde{p} = p \oplus 1$. In [8], it is shown that $\tilde{p}$ is closed, and hence compact, in $\tilde{A}^{**}$ and that $QC(\tilde{p}) = \mathbb{C}\tilde{q}$. Similarly, a compact projection $\tilde{q}$ in $\tilde{B}^{**}$ can be obtained such that $QC(\tilde{q}) = \mathbb{C}\tilde{q}$ and thus $\tilde{q}$, like $\tilde{p}$, does not have MQC.
We now consider the left regular representation $\pi_{\tilde{q}} : \tilde{B} \to B(\tilde{B}\tilde{q})$. Since $\tilde{B}$ is unital, $\text{RM}(\tilde{B}) = \tilde{B}$ and thus

$$\pi_{\tilde{q}}(\tilde{B})' = \{ R_{\tilde{x}} : \tilde{x} = \tilde{r}\tilde{q} = \tilde{q}\tilde{r}\tilde{q} \text{ for some } \tilde{r} \text{ in } \tilde{B} \}$$

by Theorem 4.8. Suppose $\tilde{x} = \tilde{r}\tilde{q} = \tilde{q}\tilde{r}\tilde{q}$ for some $\tilde{r}$ in $\tilde{B}$. Here $\tilde{r} = r + \lambda = (r + \lambda) \oplus \lambda$ for some $\lambda$ in $\mathbb{C}$ and $r$ in $B$. It follows $rq = qrq \in qBq$. Now $(qrq)^2 = qrqrq = qr^2q \in qBq$ implies $qrq = 0$. Therefore,

$$\tilde{x} = \tilde{q}\tilde{r}\tilde{q} = \lambda q.$$

Consequently, $\pi_{\tilde{q}}(\tilde{B})' = CR_\tilde{q}$ and thus $\pi_{\tilde{q}}(\tilde{B})'' = B(\tilde{B}\tilde{q})$, since the right multiplication $R_\tilde{q}$ induced by $\tilde{q}$ is the identity in $B(\tilde{B}\tilde{q})$.

It is easy to see that $B(\tilde{B}\tilde{q}) \neq \pi_{\tilde{q}}(\text{LM}(\tilde{B}, \tilde{q}))$. For example, we define an isometry $T$ in $B(\tilde{B}\tilde{q})$ by

$$T((\lambda + a)\tilde{q}) := (\lambda + \overline{a})\tilde{q}, \quad \lambda \in \mathbb{C}, \quad a \in B,$$

where

$$\pi(t) := a(-t), \quad t \in [-1, 1].$$

To see that $T$ is not implemented as a left multiplication $\pi_{\tilde{q}}(\tilde{h})$ for any $\tilde{h}$ in $\text{LM}(\tilde{B}, \tilde{q})$, we just need to show that $T$ is not decomposable by Corollary 3.14. Let $t \in (S \cup (-S)) - \{0\}$, and $\varphi_t$ the corresponding pure state in $F(\tilde{q})$. Since there is $b$ in $B$ such that $\varphi_t(b^*b) = 0$ but $\varphi_{-t}(b^*b) \neq 0$, it is clear that $T$ is not decomposable. \hfill \square

5. THE C*-ALGEBRA ASSOCIATED TO A CLOSED PROJECTION

Recall that for a C*-algebra $A$ and a closed projection $p$ in $A^{**}$, the Banach space $Ap$ (resp. $W_p$) consists of all continuous (resp. weakly continuous) admissible vector sections in $A^{**}p$ (see Theorem 3.4). It follows from Corollary 3.6 that

$$\pi_p(x)Ap \subseteq Ap \iff \pi_p(x^*)W_p \subseteq W_p, \quad \forall x \in A^{**}.$$

We collect these facts in the following.

$$\text{LM}(A, p) = \{ x \in A^{**} : \pi_p(x)Ap \subseteq Ap \},$$

$$\text{RM}(A, p) = \{ x \in A^{**} : \pi_p(x)W_p \subseteq W_p \},$$

$$M(A, p) = \{ x \in A^{**} : \pi_p(x)Ap \subseteq Ap, \pi_p(x)W_p \subseteq W_p \},$$

and

$$\text{QM}(A, p) = \{ x \in A^{**} : \pi_p(x)Ap \subseteq W_p \}.$$
a C*-subalgebra $B$ of $A^**c(p)$ such that

(a) $\text{LM}(A, p)c(p) = \text{LM}(B)$,
(b) $\text{RM}(A, p)c(p) = \text{RM}(B)$,
(c) $M(A, p)c(p) = M(B)$,
(d) $\text{QM}(A, p)c(p) = \text{QM}(B)$.

Consider $A = \text{Alg}(A, p) = \{x \in A^{**}: \pi_p(x)\mathcal{W}_p \subseteq Ap\}$.

We think of $Ac(p)$ as a natural candidate for $B$. It is easy to see that $A$ is an ideal of the C*-algebra $M(A, p)$. Moreover, $\text{LM}(A, p)A \subseteq A$, $\text{ARM}(A, p) \subseteq A$, $M(A, p)A + AM(A, p) \subseteq A$ and $A \text{QM}(A, p)A \subseteq A$.

**Example 5.1.** If $p$ is central, or equivalently if the ideal $L = A^**(1 - p) \cap A$ is two-sided, then $Ap \cong A/L$ as C*-algebras. Consequently, we have $Ac(p) = Ap$ and (a), (b), (c) and (d) hold for $B = Ac(p)$.

It follows from definitions and Corollary 3.6 that we have

**Lemma 5.2.** Let $x \in A^{**}$.

1. $x \in \text{Alg}(A, p)$ if and only if $pv^*xup \in pAp$, $\forall up, vp \in \mathcal{W}_p$.
2. $x \in \text{LM}(A, p)$ if and only if $pv^*xap \in pAp$, $\forall ap \in Ap, \forall up \in \mathcal{W}_p$.
3. $x \in \text{RM}(A, p)$ if and only if $pb^*xup \in pAp$, $\forall up \in \mathcal{W}_p, \forall bp \in Ap$.
4. $x \in \text{QM}(A, p)$ if and only if $pv^*xap$, $pb^*xup \in pAp$, $\forall ap, bp \in Ap, \forall up, vp \in \mathcal{W}_p$.
5. $x \in \text{QM}(A, p)$ if and only if $pb^*xap \in pAp$, $\forall ap, bp \in Ap$.

**Theorem 5.3.** The following conditions are all equivalent and each of them implies (a), (b), (c) and (d) for $B = Ac(p)$.

2. $\pi_p(A)\mathcal{W}_p$ is norm dense in $Ap$.
3. $A$ is non-degenerately represented on $H_{\text{univ}}$, i.e., $\overline{\pi_\varphi(A)H_\varphi} = H_\varphi, \forall \varphi \in Q(A)$, where $H_{\text{univ}} = \bigoplus_2 \{H_\varphi : \varphi \in Q(A)\}$ is the underlying Hilbert space of the universal representation of $A$.
4. $A$ is $\sigma$-weakly dense in $A^{**}$.
5. $\pi_\varphi(A) \neq \{0\}$ for all pure states $\varphi$ in $F(p)$.

**Proof.** (1) $\implies$ (2) is trivial.

(2) $\implies$ (3): Since $A$ contains $A^{**}(1 - c(p))$, we may assume $\varphi$ is supported by $c(p)$. Now, since $\pi_p(A)\mathcal{W}_p$ is norm dense in $Ap$, we see that $\pi_\varphi(A)(\mathcal{W}_pH_\varphi)$ is dense in $\pi_\varphi(Ap)H_\varphi = ApH_\varphi$, which is dense in $A^{**}pH_\varphi$. Let $q = v^*pv$ be a projection for some partial isometry $v$ in $A^{**}$. We see that $qH_\varphi = v^*pvH_\varphi \subseteq A^{**}pH_\varphi$. Hence $\pi_\varphi(A)H_\varphi$ is also dense in $H_\varphi$, and thus (3) follows.
(3) \(\implies\) (4): This follows from the fact that \(AA \subseteq A\).

(4) \(\implies\) (5) is obvious.

(5) \(\implies\) (1): Suppose the norm closure \(\overline{\pi_p(A)p} \neq Ap\). Choose a nonzero \(\varphi\) in \((Ap)^*\) such that \(\varphi(\pi_p(A)p) = \{0\}\). Let \(\{v_\lambda\}_\lambda\) be a positive increasing approximate identity in the C*-subalgebra \(A\) of \(A^{**}\), and note that \(v_\lambda \not\preceq q\) for some projection \(q\) in \(A^{**}\). For every \(a\) in \(A\), \(pa^*v_\lambda ap \not\preceq pa^*qap\). Note that \(pa^*v_\lambda ap \in pAp\). It follows from the continuity of \(pa^*v_\lambda ap\) that \(pa^*qap\) is lower semi-continuous on \(F(p)\). Since \(AA \subseteq A\), we see that \(\overline{\pi_\psi(A)H_\psi}\) is an invariant subspace for \(\pi_\psi(A)\) for every \(\psi\) in \(F(p)\). For each pure state \(\psi\) in \(F(p)\), the hypothesis \(\pi_\psi(A) \neq \{0\}\) implies \(\overline{\pi_\psi(A)H_\psi} = H_\psi\) and hence \(\pi_\psi(q) = 1\). Therefore, the non-positive lower semicontinuous affine function

\[\psi \mapsto \psi(pa^*(q - 1)ap), \quad \psi \in F(p),\]

vanishes on the extreme boundary \((F(p) \cap P(A)) \cup \{0\}\) of the weak* compact convex set \(F(p)\), where \(P(A)\) is the pure state space of \(A\). Thus \(pa^*(q - 1)ap = 0\). We then have \(qap = ap\) for every \(a\) in \(A\). Consequently,

\[\varphi(ap) = \varphi(qap) = \lim \varphi(v_\lambda ap) = 0, \quad \forall a \in A.\]

This contradiction establishes the implication.

From now on, we assume these equivalent conditions are satisfied and we are going to verify (a) to (d). We prove only that \(\text{LM}(B) \subseteq \text{LM}(A,p)c(p)\) since the opposite inclusions are obvious and the other assertions will follow similarly. Note that we can consider \(\text{LM}(B)\) as a subset of \(A^{**}c(p)\) (cf. [3, 4.3]).

Let \(x\) be a nonzero element of \(\text{LM}(B)\) and \(\varepsilon > 0\). For each \(a\) in \(A\), it follows from (2) that there exist \(a_1, a_2, \ldots, a_n\) in \(A\) and \(w_1p, w_2p, \ldots, w_np\) in \(W_p \subseteq A^{**}p\) such that

\[\|ap - \sum_{k=1}^n a_kw_kp\| < \varepsilon.\]

Hence

\[\|xap - \sum_{k=1}^n xa_kw_kp\| < \varepsilon.\]

Since \(x \in \text{LM}(B) \subseteq A^{**}c(p), xa_k = x(a_kw_k(p)) \in x(Ac(p)) = xB \subseteq B\). Note that elements of \(\pi_p(B)\) send \(W_p\) into \(Ap\). Consequently, \(\pi_p(xa_k)w_kp \in Ap\) for \(k = 1, 2, \ldots, n\). It follows that \(xap \in \overline{Ap} = Ap\). That is, \(x \in \text{LM}(A,p)\). Since \(x = xc(p)\), we have \(x \in \text{LM}(A,p)c(p)\), too.

**Corollary 5.4.** If \(p\) has MSQC then (a) to (d) will be satisfied for \(B = Ac(p)\). Moreover, we have \(Ap + pA \subseteq A\) in this case.

**Proof.** By Theorem 5.3, it suffices to show that \(\pi_p(A)p = Ap\) (since \(p \in W_p\)). One inclusion is easy:

\[\pi_p(A)p \subseteq \pi_p(A)W_p \subseteq Ap.\]
For the opposite inclusion, as well as the assertion $Ap + pA \subseteq A$, it will sufficient to show that $Ap \subseteq A$. To this end, let $up, vp \in W_p$ and $a \in A$. Observe that

$$pu^*(apvp) = (pa^*up)^*vp \in (pAp)^*vp = pApvp \subseteq pAvp,$$

since $pAp \subseteq pA$ as $p$ has MSQC, $\subseteq pAp$. Hence $ap \in A$ by Lemma 5.2. □

We remark that the inclusion in Corollary 5.4 does not hold if $p$ fails to have MSQC (see Example 5.7). Even when $p$ does have MSQC, the inclusion can be strict (see Example 5.8). The rest of this section is devoted to a few assorted results and examples about what $A$ contains.

**Proposition 5.5.** Let $B = pA^{**}p \cap QM(A, p)$. Then $A$ contains the norm closure of the linear space spanned by $ABA$.

**Proof.** Since $A$ is a C*-algebra, we only need to prove that if $a, c \in A$, $b \in B$ then $abc \in A$. It is equivalent to show that $pu^*abcvp \in pAp$ for every $up, vp$ in $W_p$ by Lemma 5.2. In fact,

$$pu^*abcvp = pu^*apbpcvp, \text{ since } b \in pA^{**}p,$$

$$\in pApbpAp, \text{ since } up, vp \in W_p,$$

$$= pAbAp, \text{ since } b \in pA^{**}p,$$

$$\subseteq pAp, \text{ since } b \in QM(A, p).$$

□

**Corollary 5.6.** Let $C = SQC(p) \cap M(A, p)$. Then $A$ contains $C$ as a C*-subalgebra.

**Proof.** Note that $C$ is a C*-algebra. In particular, $C = C^3$. The assertion now follows from Proposition 5.5 since $C \subseteq pA^{**}p \cap QM(A, p)$ and $C^3 \subseteq ACA$ (see Lemma 4.6). □

To convince the readers that $B$ and $C$ in Proposition 5.5 and Corollary 5.6 can be nonzero, we present the following example. In particular, the closed span of $B$ is the whole of $A$ and $C$ is only a proper subalgebra of $A$ in this example.

**Example 5.7.** In this example, $A$ is a separable scattered C*-algebra and $p$ is a closed projection in $A^{**}$ with central support $c(p) = 1$. But $p$ does not have MSQC. We shall see that (a) to (d) are all satisfied here. In fact, $A = A$, $LM(A, p) = LM(A)$, $RM(A, p) = RM(A)$, $M(A, p) = M(A)$ and $QM(A, p) = QM(A)$. Moreover, $B$ and $C$ are both nonzero. Furthermore, $ABA$ is norm dense in $A$ but $Ap \not\subseteq A$ (cf. Corollary 5.4).
Let \( A \) be the C*-subalgebra of \( c \otimes M_2 \) consisting of all sequences of \( 2 \times 2 \) matrices \( x = (x_n)_{n \geq 1} \) such that

\[
 x_n = \begin{pmatrix} a_n & b_n \\
 c_n & d_n \end{pmatrix} \rightarrow \begin{pmatrix} a & 0 \\
 0 & 0 \end{pmatrix}.
\]

\( A^{**} \) can be represented as the C*-algebra of all uniformly bounded sequences of \( 2 \times 2 \) matrices plus a copy of \( \mathbb{C} \). More precisely, every element of \( A^{**} \) is of the form \( x = (x_n)_{n \geq 1} \) where

\[
 x_n = \begin{pmatrix} a_n & b_n \\
 c_n & d_n \end{pmatrix}, \quad n = 1, 2, \ldots, \text{ and } x_\infty = a \in \mathbb{C}.
\]

The norm of \( A^{**} \) (and \( A \)) is given by \( \|x\| := \sup_{1 \leq n \leq \infty} \|x_n\| < \infty \). Put

\[
 p_n = \frac{1}{2} \begin{pmatrix} 1 & 1 \\
 1 & 1 \end{pmatrix}, \quad n = 1, 2, \ldots, \text{ and } p_\infty = 1 \in \mathbb{C}.
\]

Then \( p = (p_n)_{n=1}^\infty \) is a closed projection in \( A^{**} \) and \( c(p) = 1 \). Let \( x = (x_n)_{n=1}^\infty \in A^{**} \), with notation as above. We have:

1. \( x \in Ap \Leftrightarrow x_n = \frac{1}{2} \begin{pmatrix} u_n & u_n \\
 v_n & v_n \end{pmatrix}, \quad u_n \rightarrow a, \text{ and } v_n \rightarrow 0. \)
2. \( x \in W_p \Leftrightarrow x_n = \frac{1}{2} \begin{pmatrix} u_n & u_n \\
 v_n & v_n \end{pmatrix} \) and \( u_n \rightarrow a. \)
3. \( x \in pA^{**}p \Leftrightarrow x_n = \frac{1}{2} \begin{pmatrix} s_n & s_n \\
 s_n & s_n \end{pmatrix} \) for some scalars \( s_n. \)
4. \( x \in pAp \Leftrightarrow x_n = \frac{1}{2} \begin{pmatrix} s_n & s_n \\
 s_n & s_n \end{pmatrix} \) for some scalars \( s_n \rightarrow a. \)
5. \( x \in SQC(p) \Leftrightarrow x_n = \frac{1}{4} \begin{pmatrix} s_n & s_n \\
 s_n & s_n \end{pmatrix} \) for some scalars \( s_n \rightarrow a = 0. \)
6. \( x \in LM(A) = LM(A, p) \Leftrightarrow a_n \rightarrow a \) and \( c_n \rightarrow 0. \)
7. \( x \in RM(A) = RM(A, p) \Leftrightarrow a_n \rightarrow a \) and \( b_n \rightarrow 0. \)
8. \( x \in M(A) = M(A, p) \Leftrightarrow a_n \rightarrow a \) and \( b_n, c_n \rightarrow 0. \)
9. \( x \in QM(A) = QM(A, p) \Leftrightarrow a_n \rightarrow a. \)
10. \( x \in A = A \Leftrightarrow a_n \rightarrow a \) and \( b_n, c_n, d_n \rightarrow 0. \)

Since \( pAp \neq SQC(p) \), \( p \) does not have MSQC by Lemma 4.7. It is clear that both \( B = QM(A, p) \cap pA^{**}p \) and \( C = SQC(p) \cap M(A, p) = SQC(p) \) are nonzero. In addition, the closed span \( ABA = A = A. \)

\[ \square \]

**Example 5.8.** In this example we shall see that \( LM(A, p) \neq LM(A) \), etc., and \( A \) is neither a subset nor a superset of \( A \) even when \( p \) has MSQC and its central support \( c(p) = 1. \) However, (a) to (d) are all satisfied.

Let \( A \) be the C*-subalgebra of \( c \otimes M_2 \) given by

\[
 A = \left\{ \begin{pmatrix} a_n & b_n \\
 c_n & d_n \end{pmatrix} ; \begin{pmatrix} a_n & b_n \\
 c_n & d_n \end{pmatrix} \rightarrow \begin{pmatrix} a & 0 \\
 0 & d \end{pmatrix} \right\}.
\]

Let \( p = (p_n) \in A^{**} \) with

\[
p_n = \begin{pmatrix} 1 & 0 \\
 0 & 0 \end{pmatrix}, \quad n = 1, 2, \ldots, \text{ and } p_\infty = \begin{pmatrix} 1 & 0 \\
 0 & 1 \end{pmatrix}.
\]
Then $p$ is a closed projection in $A^{**}$. Let $x = (x_n) \in A^{**}$ with
\[ x_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \quad n = 1, 2, \ldots, \text{ and } x_\infty = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}. \]
We have

1. $x \in Ap \iff x_n = \begin{pmatrix} a_n & 0 \\ c_n & 0 \end{pmatrix}$ such that $a_n \to a$ and $c_n \to 0$.
2. $x \in W_p \iff x_n = \begin{pmatrix} a_n & 0 \\ c_n & 0 \end{pmatrix}$ such that $a_n \to a$.
3. $x \in pAp \iff x_n = \begin{pmatrix} a_n & 0 \\ 0 & 0 \end{pmatrix}$ and $a_n \to a$.
4. $x \in LM(A, p) \iff a_n \to a$ and $c_n \to 0$.
5. $x \in RM(A, p) \iff a_n \to a$ and $b_n \to 0$.
6. $x \in M(A, p) \iff a_n \to a$ and $b_n, c_n \to 0$.
7. $x \in QM(A, p) \iff a_n \to a$.
8. $x \in A \iff a_n \to a$ and $b_n, c_n, d_n \to 0$.

We first note that $c(p) = 1$. Since $pAp$ is an algebra, $p$ has MSQC by Lemma 4.7. Thus, (a) to (d) are satisfied for $B = A$. On the other hand, obviously we have $A \not\subseteq A$. We want to point out also that $A$ is not contained in $A$, either. For example, the element $x = (x_n)$ of $A \subseteq A^{**}$ given by $x_n = 0$, $n = 1, 2, \ldots$, and $x_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ does not belong to $A$. It is clear that $LM(A, p) \neq LM(A) = A$, etc., since $A$ is unital.

**Example 5.9.** Consider the C*-algebra $A = c \otimes K$ and
\[ A^{**} = \{ (h_n) : h_n \in B(H), 1 \leq n \leq \infty, \|h\| = \sup \|h_n\| < \infty \}. \]
Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis of the Hilbert space $H$. Let
\[ v_n = \frac{1}{\sqrt{2}} e_1 + \frac{1}{\sqrt{2}} e_{n+1}, \quad n < \infty \quad \text{and} \quad v_\infty = e_1, \]
and
\[ p_n = v_n \otimes v_n, \quad n = 1, 2, \ldots, \infty. \]
Then $p = (p_n)$ is a closed projection in $A^{**}$ without MSQC (cf. [8]) and the central support $c(p)$ of $p$ is 1. We have

1. $Ap = \{ (x_n p_n) \in A^{**} p : x_n v_n \xrightarrow{\|\|} \frac{1}{\sqrt{2}} x_\infty e_1 \}$.
2. $W_p = \{ (x_n p_n) \in A^{**} p : x_n v_n \xrightarrow{\text{weakly}} \frac{1}{\sqrt{2}} x_\infty e_1 \}$.
3. $pAp = \{ (p_n b_n p_n) : (b_n v_n, v_n) \xrightarrow{\text{weakly}} \frac{1}{2} (b_\infty e_1, e_1) \}$.
4. $LM(A) = LM(A, p) = \{ (t_n) \in A^{**} : t_n \xrightarrow{\text{SOT}} t_\infty \}$.
5. $RM(A) = RM(A, p) = \{ (t_n) \in A^{**} : t_n \xrightarrow{\text{SOT}} t_\infty \}$.
6. $M(A) = M(A, p) = \{ (t_n) \in A^{**} : t_n \xrightarrow{\text{DSOT}} t_\infty \}$.
7. $QM(A) = QM(A, p) = \{ (t_n) \in A^{**} : t_n \xrightarrow{\text{WOT}} t_\infty \}$.
8. $A = \{ (t_n) \in A^{**} : t_n \xrightarrow{\|\|} t_\infty, \quad t_\infty \in K \}.$
By Theorem 5.3 and the fact that \( A \subseteq A \), the equations \( LM(A, p) = LM(A) \), etc. are satisfied in this case. This can also be verified by direct calculation. \( \square \)

**Remark 5.10.** In [6], it is shown that for two separable C*-algebras \( A_1 \) and \( A_2 \), the multiplier algebras \( M(A_1) \) and \( M(A_2) \) are isomorphic if and only if \( A_1 \) and \( A_2 \) are isomorphic. In fact, \( A_1 \) (resp. \( A_2 \)) is the largest separable closed, two-sided ideal of \( M(A_1) \) (resp. \( M(A_2) \)). However, in the inseparable case, this may not be true. A perhaps less artificial than usual example to this fact is provided by Example 5.9, since \( M(A) = M(A) \), \( A \) is separable and \( A \) is not separable.

### 6. Atomic parts of relative multipliers

In the following, \( z = z_{at} \) denotes the maximal atomic projection in \( A^{**} \); in other words, \( z \) is the smallest central projection in \( A^{**} \) supporting all pure states of \( A \).

**Lemma 6.1.** Let \( x_p \) and \( y_p \) be in \( W_p \). If \( zx_p = zy_p \) then \( xp = yp \). Moreover, we have \( \| xp \| = \| zxp \| \). In other words, weakly continuous vector sections are determined by their atomic parts.

**Proof.** For each \( a \) in \( A \), the continuous affine function \( \varphi \mapsto \varphi(a^*(x - y)) \) on \( F(p) \) vanishes at all pure states in \( F(p) \). Consequently, it is identically zero on \( F(p) \). As a result, \( pA(x - y)p = \{0\} \) and thus, \( xp = yp \). For the norm equality, we note that the bounded affine function \( \varphi \mapsto \varphi(x^*x) \) is lower semi-continuous on the weak* compact convex set \( F(p) \) [9, Lemma 2.1]. It follows from the Krein-Milman theorem that

\[
\| xp \|^2 \leq \sup \{ \varphi(x^*x) : \varphi \text{ is a pure state in } F(p) \} = \| zxp \|^2 \leq \| xp \|^2.
\]

\( \square \)

The following theorem says that if the operator section \( \pi_p(x) \) preserves the continuity of the atomic part of every vector section in \( A^{**} \) then \( x \) itself must have an appropriate atomic part.

**Theorem 6.2.** Let \( x \) be an element of \( A^{**} \).

1. \( zxAp \subseteq zAp \) if and only if \( zx \in zLM(A, p) \).
2. \( zxW_p \subseteq zW_p \) if and only if \( zx \in zRM(A, p) \).
3. \( zxAp \subseteq zAp \) and \( zxW_p \subseteq zW_p \) if and only if \( zx \in zM(A, p) \).
4. \( zxAp \subseteq zW_p \) if and only if \( zx \in zQM(A, p) \).
5. \( zxW_p \subseteq zAp \) if and only if \( zx \in zAlg(A, p) \).

**Proof.** The sufficiency is obvious and thus we verify the necessity only. Suppose first that \( zxAp \subseteq zW_p \). By Lemma 6.1, we can define a linear map \( T \) from \( Ap \) into \( W_p \). More precisely, we set \( Tap = up \) if \( zxap = zup \). Moreover, \( \| T \| \leq \| x \| \) since \( \| zyp \| = \| yp \| \) for all \( yp \) in \( W_p \). Suppose that \( \varphi \) is a pure state in \( F(p) \) and \( a \) is in \( A \) such that \( \varphi(a^*a) = 0 \).
Then \( \varphi((Tap)^*(Tap)) = \varphi(u^*u) = \varphi((zup)^*(zup)) = \varphi((xap)^*(xap)) = \varphi(pa^*x^*xap) \leq \|x\|^2\varphi(a^*a) = 0 \). By Theorem 3.13, there is a relative quasi-multiplier \( q \) in \( QM(A, p) \) such that \( Tap = qap \) for all \( a \) in \( A \). Therefore \( zxap = zTap = zqap \) for all \( a \) in \( A \). Consequently, \( z(x - q)Ap = \{0\} \), and thus, \( zxc(p) = zq \in zQ(A, p) \).

Consider next the case \( zxAp \subseteq zAp \). A similar argument yields a bounded linear map \( T \) from \( Ap \) into \( Ap \) (by restricting the co-domain of \( T \)). We thus have an \( l \) in \( A^{**}c(p) \) such that \( lap = Tap \in Ap \) for all \( a \) in \( A \). Consequently, \( l \in \text{LM}(A, p) \), and thus \( zxc(p) = zl \in zLM(A, p) \).

For the case \( zxW_p \subseteq zW_p \), we note that \( zx^*Ap \subseteq zAp \). To see this, we observe that \( zpy^*x^*ap = (pa^*xxyp)^* \in zpAp \) for all \( yp \) in \( W_p \), and quote [9, Theorem 1.7] which says \( zup \in zAp \) if and only if \( zpAp \subseteq zpAp \) and \( zpu^*up \subseteq zpAp \). Hence there is a relative left multiplier \( l \) in \( A^{**} \) such that \( zx^* = zl \). By setting \( r = l^* \), we have \( zx = zr \in zR(A, p) \).

The case \( zxW_p \subseteq zAp \) is similar.

Finally, we suppose that \( zxAp \subseteq zAp \) and \( zxW_p \subseteq zW_p \). By above observation, there is an \( l \) in \( \text{LM}(A, p) \) and an \( r \) in \( \text{RM}(A, p) \) such that \( zx = zl = zr \). Now, \( pa_1(l - r)a_2p \) belongs to \( pAp \) and vanishes at each pure state in \( F(p) \) for all \( a_1, a_2 \) in \( A \). It follows that \( pA(l - r)Ap = \{0\} \). Therefore, \( le(p) = rc(p) \), and thus \( zx \in M(A, p) \).

The following is the special case when \( p = 1 \).

**Corollary 6.3.** Let \( x \) be an element of \( A^{**} \).

1. If \( zxA \subseteq zA \) then \( zx = zl \) for some left multiplier \( l \) of \( A \) in \( A^{**} \).
2. If \( zx\text{RM}(A) \subseteq z\text{RM}(A) \) then \( zx = zr \) for some right multiplier \( r \) of \( A \) in \( A^{**} \).
3. If \( zxA \subseteq zA \) and \( zx\text{RM}(A) \subseteq z\text{RM}(A) \) then \( zx = zm \) for some multiplier \( m \) of \( A \) in \( A^{**} \).
4. If \( zxA \subseteq z\text{RM}(A) \) then \( zx = zq \) for some quasi-multiplier \( q \) of \( A \) in \( A^{**} \).
5. If \( zx\text{RM}(A) \subseteq zA \) then \( zx = za \) for some \( a \) in \( A \).

**References**


Department of Mathematics, Purdue University, West Lafayette, Indiana 47907, U. S. A.
E-mail address: lgb@math.purdue.edu

Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan, R.O.C.
E-mail address: wong@math.nsysu.edu.tw