LINEAR ORTHOGONALITY PRESERVERS OF HILBERT MODULES

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Abstract. We verify in this paper that the linearity and orthogonality structures of a (not necessarily local trivial) Hilbert bundle over a locally compact Hausdorff space \( \Omega \) determine its unitary structure. In fact, as Hilbert bundles over \( \Omega \) are exactly Hilbert \( C_0(\Omega) \)-modules, we have a more general set up. A \( C \)-linear map \( \theta \) (not assumed to be bounded) between two Hilbert \( C^* \)-modules is said to be “orthogonality preserving” if \( \langle \theta(x), \theta(y) \rangle = 0 \) whenever \( \langle x, y \rangle = 0 \). We prove that if \( \theta \) is a orthogonality preserving \( C_0(\Omega) \)-module map from a full Hilbert \( C_0(\Omega) \)-module \( E \) into another Hilbert \( C_0(\Omega) \)-module \( F \), then \( \theta \) is bounded and there exists \( \phi \in C_b(\Omega)_+ \) such that

\[
\langle \theta(x), \theta(y) \rangle = \phi \cdot \langle x, y \rangle \quad (x, y \in E).
\]

On the other hand, if \( F \) is a full Hilbert \( C^* \)-module over another commutative \( C^* \)-algebra \( C_0(\Delta) \), we show that an “bi-orthogonality preserving” bijective map \( \theta \) with some “local-type property” will be bounded and satisfy

\[
\langle \theta(x), \theta(y) \rangle = \phi \cdot \langle x, y \rangle \circ \sigma \quad (x, y \in E)
\]

for a map \( \phi \in C_b(\Omega)_+ \) and a homeomorphism \( \sigma : \Delta \to \Omega \). We will also have a look at the non-commutative situation.

1. Introduction

It is a common knowledge that the inner product of a Hilbert space determines both the norm and the orthogonality structure; and conversely, the norm structure determines the inner product structure. On the other hand, it is not hard to check that a non-zero orthogonality preserving linear map between Hilbert spaces is a scalar multiple of an isometry (see, e.g., [5, 4]).

It is natural to explore these equivalences in the setting of Hilbert bundles, i.e. continuous fields of Hilbert spaces [6]. In the terminologies of [7], they are called (F)-Hilbert bundles. It is shown in [7, Theorem 2.6] that (F)-Hilbert bundles over a locally compact Hausdorff space \( \Omega \) are (categorically) equivalent to Hilbert \( C_0(\Omega) \)-modules. Therefore, we will not distinguish them.
Notation 1.1. Throughout this article, Banach bundles and Hilbert bundles mean respectively, (F)-Banach bundles and (F)-Hilbert bundles in the sense of [7].

Let $\Omega$ and $\Delta$ be locally compact Hausdorff spaces, and let $E$ and $F$ be Banach spaces such that $F$ does not contains a copy of the two dimensional space $\ell^\infty_2$. It is shown in [11] (see also [12, 14]) that every linear isometry from $C_0(\Omega, E)$ onto $C_0(\Delta, F)$ carries the form

$$\theta(f)(\nu) = U_\nu(f(\sigma(\nu))) \quad (\nu \in \Delta).$$

Here, $\sigma$ is a homeomorphism from $\Delta$ onto $\Omega$, and $U_\nu$ is a surjective linear isometry from $E$ onto $F$ for every $\nu \in \Delta$. One can easily modify the proof of [11, Theorem 6] to obtain a similar version for surjective isometries between two continuous fields of Hilbert spaces with non-zero fibres over each point. Hence the norm structure (and linearity) determines the unitary structure in this more general situation.

In this paper, we study whether the orthogonality structure of a Hilbert bundle determines the unitary structure. One of the main results in this article (Corollary 2.10) tells us that a bijective $\mathbb{C}$-linear map between two Hilbert bundles that do not have non-zero fibre (over the same base space $\Omega$) preserves the unitary structures up to a strictly positive bounded continuous function if (and only if) it is orthogonality preserving and satisfies some “local”-type properties (which is weaker than $C_0(\Omega)$-linearity). Consequently, one knows the $C_0(\Omega)$-inner product of a “full” Hilbert $C_0(\Omega)$-module up to a strictly positive continuous function if one knows its $C_0(\Omega)$-module structure and its orthogonality structure.

Let us now state clearly what we meant by “local”-type property.

Definition 1.2. Let $\Omega$ be a locally compact Hausdorff space and $A = C_0(\Omega)$. Suppose that $E$ and $F$ are Hilbert $A$-modules. A $\mathbb{C}$-linear map $\theta : E \to F$ is said be local if $\theta(e)a = 0$ whenever $ea = 0$ for any $e \in E$ and $a \in A$.

The idea of local linear maps are found in many researches in analysis. For example, a theorem of Peetre [18] states that local linear maps of the space of smooth functions defined on a manifold modelled on $\mathbb{R}^n$ are exactly linear differential operators (see, e.g., [16]). This is further extended to the case of vector-valued differentiable functions defined on a finite dimensional manifold by Kantrowitz and Neumann [13] and Araujo [2], and in the Banach $C^1[0, 1]$-module setting by Alaminos et. al. [1]. Note that every $A$-module map is local. Conversely, every bounded local map is an $A$-module map ([15, Proposition A.1]). See Remark 2.5 below for more information.

One of our main results (Theorem 2.6) shows that orthogonality preserving local $\mathbb{C}$-linear maps are automatically bounded $C_0(\Omega)$-module maps that preserves the
In this case of Hilbert bundles over the same base space, one can start with a more familiar (but stronger) situation of orthogonality preserving $C_0(\Omega)$-module maps. However, in the case of Hilbert bundles over different base spaces, one cannot define $C_0(\Omega)$-linearity but have to consider some form of “local”-type property. Another main result in this article (Theorem 2.13) states that a bijective $C$-linear map between two Hilbert bundles without zero fibre over two possibly different base spaces preserves the unitary structures, up to a bounded strictly positive continuous function and a homeomorphism between the base spaces, if (and only if) it is bi-orthogonality preserving and satisfies another form of “local”-type property (see Definition 2.11).

In Section 3 as well as in the appendix, we will consider the situation of orthogonality preserving local maps between Hilbert $A$-modules where $A$ is a possibly non-commutative $C^*$-algebra. In this case, we only have a conjecture as well as some evidences for the validity of it.

2. Orthogonality preserving maps between Hilbert $C_0(\Omega)$-modules

Notation 2.1. Throughout this section, $\Omega$ and $\Delta$ are two locally compact Hausdorff spaces, and $\Omega_{\infty}$ is the one-point compactification of $\Omega$. Moreover, $E$ and $F$ are respectively, a (right) Hilbert $C_0(\Omega)$-module and a (right) Hilbert $C_0(\Delta)$-module, and $\theta : E \to F$ is a $\mathbb{C}$-linear map (not assumed to be bounded). For any $\omega \in \Omega$, we let $N_\Omega(\omega)$ be the set of all compact neighborhoods of $\omega$ in $\Omega$. If $S \subseteq \Omega$, we denote by $\text{Int}_\Omega(S)$ the interior of $S$ in $\Omega$. Moreover, if $U, V \subseteq \Omega$ such that the closure of $V$ is a compact subset of $\text{Int}_\Omega(U)$, we denote by $\mathcal{U}_\Omega(V, U)$ the collection of all $\lambda \in C_0(\Omega)$ with $0 \leq \lambda \leq 1$, $\lambda \equiv 1$ on $V$ and $\lambda$ vanishes outside $U$.

Note that any Banach $C_0(\Omega)$-module can be regarded as a Banach $C(\Omega_{\infty})$-module and the results in [7] can be applied. In particular, by [7, p. 49], $E$ is the space of $C_0$-sections (i.e., continuous sections that vanish at infinity) of a Hilbert bundle $\Xi^E$ over $\Omega_{\infty}$.

We define $|f|(\omega) := \|f(\omega)\|$ ($f \in E; \omega \in \Omega$). For any closed subset $S \subseteq \Omega_{\infty}$ and $\omega \in \Omega_{\infty}$, set

$$K_S^E := \{f \in E : f(\omega) = 0 \text{ for any } \omega \in S\},$$

$$I_\omega := \bigcup_{V \in \mathcal{N}_\Omega(\omega)} K_V^E \quad \text{ and } \quad K_\omega^E := K_{I_\omega}^E.$$
Note that $K^E_{\infty} = E$ and the fibre of $\Xi^E$ at $\omega \in \Omega_{\infty}$ is given by $\Xi^E_\omega = E/K^E_{\omega}$. Furthermore, $K^E_S$ is a Hilbert $K^C_0(\Omega)$-module and $K^E_S = E \cdot K^C_0(\Omega)$.

On the other hand, we denote

$$\Delta_\theta := \{ \nu \in \Delta : \theta(E) \not\subseteq K^F_{\nu} \}$$

$$= \{ \nu \in \Delta : \theta(e)(\nu) \not= 0 \text{ for some } e \in E \} .$$

Then $\Delta_\theta$ is an open subset of $\Delta$. In the case when $\Delta = \Omega$, we denote by $\mathcal{B}_{C_0(\Omega)}(E; F)$ the set of all bounded $C_0(\Omega)$-module maps from $E$ into $F$. We put $\Omega_E := \Omega_{\text{id}}$, i.e.,

$$\Omega_E = \{ \omega \in \Omega : \Xi^E_\omega \not= (0) \} .$$

Let $\Omega_0 \subseteq \Omega$ be an open set. As in [7, p. 10], we denote by $\Xi^E|_{\Omega_0}$ the restrictions of $\Xi^E$ to $\Omega_0$ and by $E_{\Omega_0}$ the set of $C_0$-sections on $\Xi^E|_{\Omega_0}$. One can identify that

$$C_0(\Omega_0) = K^C_{0|\Omega_0}(\Omega) \text{ and } E_{\Omega_0} = K^E_{\Omega_0}(\Omega).$$

The following technical lemmas follows directly from [15, Lemma 3.1, Lemma 3.3 and Theorem 3.7] (see also [15, Remark 3.4]), which summarize, unify, and generalize techniques sporadically used in the literature, e.g., [10, 3, 9].

**Lemma 2.2.** If $\sigma : \Delta_\theta \rightarrow \Omega_{\infty}$ is a map satisfying $\theta \left( I^E_{\sigma(\nu)} \right) \subseteq K^F_{\nu}$ (for any $\nu \in \Delta_\theta$), then $\sigma$ is continuous.

**Lemma 2.3.** Let $\sigma : \Delta \rightarrow \Omega$ be a map (not assumed to be continuous) such that $\theta \left( I^E_{\sigma(\nu)} \right) \subseteq K^F_{\nu}$ for every $\nu \in \Delta$.

(a) If $\mathcal{U}_\theta := \{ \nu \in \Delta : \sup_{|e| \leq 1} \| \theta(e)(\nu) \| = \infty \}$, then $\sigma(\mathcal{U}_\theta)$ is a finite set.

(b) If $\mathcal{N}_{\theta,\sigma} := \{ \nu \in \Delta : \theta \left( K^E_{\sigma(\nu)} \right) \not\subseteq K^F_{\nu} \}$, then $\mathcal{N}_{\theta,\sigma} \subseteq \mathcal{U}_\theta$ and $\sigma(\mathcal{N}_{\theta,\sigma})$ consists of non-isolated points in $\Omega$.

(c) If $\sigma$ is injective and sends isolated points in $\Delta$ to isolated points in $\Omega$, then $\mathcal{N}_{\theta,\sigma} = \emptyset$ and there exists a finite set $T$ consisting of isolated points of $\Delta$, a bounded linear map $\theta_0 : K^E_{\sigma(T)} \rightarrow K^F_T$ as well as linear maps $\theta_\nu : \Xi^E_{\sigma(\nu)} \rightarrow \Xi^F_\nu$ ($\nu \in T$) such that $E = K^E_{\sigma(T)} \oplus \bigoplus_{\nu \in T} \Xi^E_{\sigma(\nu)}$,

$$F = K^F_T \oplus \bigoplus_{\nu \in T} \Xi^F_\nu \text{ and } \theta = \theta_0 \oplus \bigoplus_{\nu \in T} \theta_\nu.$$ 

For any $\nu \in \Delta \backslash \mathcal{N}_{\theta,\sigma}$, one can define $\theta_\nu : \Xi^E_{\sigma(\nu)} \rightarrow \Xi^F_\nu$ by

$$\theta_\nu (e + K^E_{\sigma(\nu)}) = \theta(e) + K^F_{\nu} \quad (e \in E),$$

or equivalently, $\theta_\nu(e(\sigma(\nu))) = (\theta(e))(\nu) \quad (e \in E)$.

(2.1)
Lemma 2.4. Let σ and Λσ be the same as in Lemma 2.3. Suppose, in addition, that σ is injective and θ is orthogonality preserving. Then there exists a bounded function ψ : Δ \ \Lambdaσ \rightarrow \mathbb{R}_+ such that

\begin{equation}
(\theta(e) , \theta(g)) (\nu) = \psi(\nu)^2 (e,g)(\sigma(\nu)) \quad (e,g \in E; \nu \in \Delta \ \Lambdaσ).
\end{equation}

Moreover, for each ν ∈ Δσ, there is an isometry νν : Ξσ(ν) → Ξσ(ν) such that

\begin{equation}
\theta(e) (\nu) = \psi(\nu) \iota_\nu (e(\sigma(\nu))) \quad (e \in E; \nu \in \Delta \ \Lambdaσ).
\end{equation}

Proof. Fix any ν ∈ Δσ \ \Lambdaσ. By Lemma 2.3(b), the map θν as in (2.1) is well-defined. Suppose that η1 and η2 are orthogonal elements in Ξσ(ν) with η1 ≠ 0 (it is possible because Δσ \ \mathfrak{M}_{\sigma,\sigma} \subseteq \sigma^{-1}(ΩE)), and g1, g2 ∈ E with g1(σ(ν)) = ηi (i = 1, 2). If V ∈ \mathcal{N}_Ω(\sigma(ν)) such that g1 is non-vanishing on V, then by replacing g2 with

\begin{equation}
\left( g_2 - \frac{\langle g_2, g_1 \rangle}{\|g_1\|^2} g_1 \right) \lambda
\end{equation}

where λ ∈ ΛΩ(\{σ(ν)\}, V), we see that there are orthogonal elements e1, e2 ∈ E with e1(σ(ν)) = ηi (i = 1, 2). Hence, θν is non-zero (because ν ∈ Δσ) and is an orthogonality preserving C-linear map between Hilbert spaces. Consequently, there exist an isometry iν : Ξσ(ν) → Ξσ(ν) and a unique scalar ψ(ν) > 0 such that θν = ψ(ν)iν. For any ν ∈ Δ \ Δσ, we set ψ(ν) = 0. Then clearly (2.2) holds. Next, we show that ψ is a bounded function on Δ \ \Lambdaσ. Suppose that it is not the case. Then there exist distinct points νn ∈ Δσ \ \Lambdaσ such that ψ(νn) > n^3. If e_n ∈ E with \|e_n\| = 1 and \|e_n(\sigma(\nu_n))\| ≥ \frac{n-1}{n} (note that ν_n ∈ \sigma^{-1}(ΩE)), then because of (2.2),

\begin{equation}
|\theta(e_n)(\nu_n)| = \psi(\nu_n)|e_n(\sigma(\nu_n))| > n^2(n-1).
\end{equation}

As {σ(νn)} is a set of distinct points (note that σ is injective) and any subsequence of {σ(νn)} converges to at most one point in Ω, by passing to a subsequence if necessary, we can assume that there are U_n ∈ \mathcal{N}_Ω(σ(ν_n)) such that U_n ∩ U_m = ∅ when m ≠ n. Now, pick any V_n ∈ \mathcal{N}_Ω(σ(ν_n)) with V_n ⊆ \text{Int}_Ω(U_n) and choose a function λ_n ∈ ΛΩ(V_n, U_n) (n ∈ N). Define e := \sum_{k=1}^{\infty} \frac{e_k λ_n^2}{k^2} ∈ E. For any n ∈ N, as \|e - e_nλ_n^2\| ∈ K_{V_n}^E and e_n - e_nλ_n^2 = e_n(1 - λ_n^2) ∈ K_{V_n}^E, we have

\begin{equation}
\|\theta(e)\| ≥ \|\theta(e(\nu_n))\| = \frac{1}{n^2} \|\theta(e_n\lambda_n^2(\nu_n))\| = \frac{1}{n^2} \|\theta(e_n)(\nu_n))\| > n - 1.
\end{equation}

(by the relation between θ and σ) which is a contradiction.

\[\square\]

2.1. Hilbert bundles over the same base space.

Remark 2.5. For any e ∈ E, we denote

\begin{equation}
\text{supp}_Ω e := \{ω ∈ Ω : e(ω) ≠ 0\}.
\end{equation}

It is not hard to check that the following statements are equivalent (which tells us that local map is the same as support shrinking map; see e.g. [9]):
(i) $\theta$ is local (see Definition 1.2);
(ii) $\theta(K^E_V) \subseteq K^F_V$ for any non-empty open set $V$;
(iii) $\text{supp}_\Omega \theta(e) \subseteq \text{supp}_\Omega e$ for every $e \in E$;
(iv) $\text{supp}_\Omega \theta(e) \lambda \subseteq \text{supp}_\Omega e$ for each $e \in E$ and $\lambda \in C_0(\Omega)$.

**Theorem 2.6.** Let $\Omega$ be a locally compact Hausdorff space, and let $E$ and $F$ be two Hilbert $C_0(\Omega)$-modules. Suppose that $\theta : E \to F$ is an orthogonality preserving local $\mathbb{C}$-linear map. The following assertions hold.

(a) $\theta \in \mathcal{B}_{C_0(\Omega)}(E, F)$.

(b) There is a bounded non-negative function $\varphi$ on $\Omega$ which is continuous on $\Omega_E$ such that

$$\langle \theta(e), \theta(g) \rangle = \varphi \cdot \langle e, g \rangle \quad (e, g \in E).$$

(c) There exist a strictly positive element $\psi_0 \in C_b(\Omega_\theta)_+$ and $I \in \mathcal{B}_{C_0(\Omega_\sigma)}(E_{\Omega_\sigma}, F_{\Omega_\sigma})$ such that $I_\omega$ is an isometry ($\omega \in \Omega_\theta$) and

$$\theta(e)(\omega) = \psi_0(\omega)I(e)(\omega) \quad (e \in E; \omega \in \Omega_\theta).$$

**Proof.** Note that the conclusions of Lemmas 2.3 and 2.4 hold for $\Omega = \Delta$ and $\sigma = \text{id}_\Omega$.

(a) By Lemma 2.3(c), we see that $\theta$ is a $C_0(\Omega)$-module map. Furthermore, as $\theta_\nu$ is an orthogonality preserving (and hence bounded) linear map between Hilbert spaces for any $\nu \in T$ (where $T$ is as in Lemma 2.3(c), with $\sigma = \text{id}_\Omega$), we know from Lemma 2.3(c) that $\theta$ is bounded (note that $T$ is finite).

(b) By part (a), $\mathcal{U}_\theta = \emptyset$. Thus, Lemma 2.4 tells us that there exists a bounded non-negative function $\psi$ on $\Omega$ with $\langle \theta(e), \theta(f) \rangle = |\psi|^2 \cdot \langle e, f \rangle$. Let $\omega \in \Omega_E$ and pick any $e \in E$ such that there is $U_\omega \subseteq \mathcal{N}_\Omega(\omega)$ with $e(\nu) \neq 0 (\nu \in U_\omega)$. Then $\psi(\omega) = \frac{|\theta(e)|^2(\omega)}{|e(\omega)|}$ ($\omega \in U_\omega$). Hence $\psi$ is continuous at $\omega$, and $\varphi(\omega) = \psi(\omega)^2$ is the required function.

(c) Note that $\Omega_\theta \subseteq \Omega_E$ because of part (a). Since $\varphi(\omega) > 0 (\omega \in \Omega_\theta)$, we know from part (b) that $\psi = \varphi^{1/2}$ gives a strictly positive element $\psi_0$ in $C_b(\Omega_\theta)_+$. The equivalence in [7, (2.2)] (consider $E$ and $F$ as Hilbert $C(\Omega_\infty)$-bundles) tells us that the restriction of $\theta$ induces a bounded Banach bundle map, again denoted by $\theta$, from $\Xi^E|_{\Omega_\theta}$ into $\Xi^F|_{\Omega_\theta}$. For each $\eta \in \Xi^E|_{\Omega_\theta}$, we define $I(\eta) := \psi_0(\pi(\eta))^{-1}\theta(\eta)$ (where $\pi : \Xi^E \to \Omega$ is the canonical projection). Then $I : \Xi^E|_{\Omega_\theta} \to \Xi^F|_{\Omega_\theta}$ is a Banach bundle map (as $\eta \mapsto \psi_0(\pi(\eta))^{-1}$ is continuous) which is an isometry on each fibre (hence $I$ is bounded) such that $\theta|_{\Omega_\theta} = \psi(\pi(\eta))I(\eta)$. This map $I$ induces a map, again denoted by $I$, in $\mathcal{B}_{C_0(\Omega_\sigma)}(E_{\Omega_\sigma}, F_{\Omega_\sigma})$ that satisfies the requirement of part (c).

$\square$
It is natural to ask if one can find \( \varphi \in C_b(\Omega) \) such that the conclusion of Theorem 2.6(b) holds. Unfortunately, the following example tells us that it is not the case in general.

**Example 2.7.** Let \( \Omega = \mathbb{R}^\infty \). Consider \( E = C_0(\mathbb{R}) = F \) as Hilbert \( C(\Omega) \)-modules and \( \theta(f)(t) = f(t) \cos t \) \((f \in E; t \in \mathbb{R})\). Then \( \Omega \setminus \Omega_E = \{\infty\} \) and \( \varphi(t) = \cos t \) for any \( t \in \mathbb{R} = \Omega_E \). Thus, one cannot extend \( \varphi \) to a continuous function on \( \Omega \).

Now, we can obtain the following commutative analogue of [19, 2.3]. This, together with Corollary 2.10, asserts that the orthogonality structure of a Hilbert bundle determines essentially its unitary structure, as we claimed in the Introduction. Note also that a large portion of Lemma 2.3 were used to deal with the possibility of \( \theta(K^E_\sigma(\nu)) \notin K^F_\nu \) (such situation does not exist for \( C_0(\Omega) \)-module map), and this corollary actually have a much easier proof.

**Corollary 2.8.** Let \( \Omega \) be a locally compact Hausdorff space, and \( E \) and \( F \) be two Hilbert \( C_0(\Omega) \)-modules. Suppose that \( \theta : E \to F \) is a \( C_0(\Omega) \)-module map which preserves orthogonality. Then \( \theta \) is bounded and there exists a bounded non-negative function \( \varphi \) on \( \Omega \) that is continuous on \( \Omega_E \) and satisfies \( \langle \theta(e), \theta(f) \rangle = \varphi \cdot \langle e, f \rangle \) \((e, f \in E)\).

**Remark 2.9.** Recall that a Hilbert \( C_0(\Omega) \)-module \( E \) is said to be full if the \( \mathbb{C} \)-linear span of \( \{\langle e, f \rangle : e, f \in E\} \) is dense in \( C_0(\Omega) \). Note that if \( E \) is full, then \( \Omega_E = \Omega \) and so, the map \( \varphi \) in Theorem 2.6(b) (and Corollary 2.8) is an element of \( C_0(\Omega) \).

**Corollary 2.10.** Let \( \Omega \) be a locally compact Hausdorff space, and let \( E \) and \( F \) be two Hilbert \( C_0(\Omega) \)-modules. Suppose that \( F \) is full and \( \theta : E \to F \) is an orthogonality preserving surjective local \( \mathbb{C} \)-linear map. Then \( \theta \in \mathcal{B}_{C_0(\Omega)}(E, F) \). Moreover, there exist a strictly positive element \( \psi \in C_0(\Omega)_+ \) and an unitary \( J \in \mathcal{B}_{C_0(\Omega)}(E, F) \) such that \( \theta = \psi \cdot J \).

**Proof.** If there exists \( \omega \in \Omega \setminus \Omega_\theta \), then \( F = \theta(E) \subseteq K^F_\omega \) which contradicts the fullness of \( F \). This means that \( \Omega_\theta = \Omega \). By the surjectivity of \( \theta \), the bounded Banach bundle map \( I \) in Theorem 2.6 is a unitary on each fibre. Therefore, the element \( J \in \mathcal{B}_{C_0(\Omega)}(E, F) \) corresponding to \( I \) as given by [7, (2.2)] is an unitary. \( \square \)

### 2.2. Hilbert bundles over different base spaces.

**Definition 2.11.** \( \theta \) is said to be quasi-local if it is bijective and for any \( e \in E \) and \( \lambda \in C_0(\Delta) \), we have

\[
\text{supp}_\Omega \theta^{-1}(\theta(e)\lambda) \subseteq \text{supp}_\Omega e.
\]

(2.3)
Note that if $\Delta = \Omega$, and if $\theta$ is both local and bijective (hence $\theta^{-1}$ is also local), then $\theta$ is quasi-local by Remark 2.5.

**Lemma 2.12.** Suppose that $\theta$ is bijective and quasi-local and that both $\theta$ and $\theta^{-1}$ are orthogonality preserving. Then $|\theta(e)||\theta(g)| = 0$ whenever $e, g \in E$ with $\text{supp}_\Omega e \cap \text{supp}_\Omega g = \emptyset$.

**Proof.** Suppose on the contrary that there exist $e_1, e_2 \in E$ and $\nu \in \Delta$ such that $\text{supp}_\Omega e_1 \cap \text{supp}_\Omega e_2 = \emptyset$ but $\|\theta(e_1)(\nu)\|\|\theta(e_2)(\nu)\| \neq 0$. As $\theta$ is orthogonality preserving, we may assume that $\theta(e_1)(\nu)$ and $\theta(e_2)(\nu)$ are two orthogonal unit vectors in $\Xi^E$. Let $U, W \in N_\Delta(\nu)$ with $W \subseteq \text{Int}_\Delta(U)$ and $\|\theta(e_i)(\mu)\| > 1/2$ for any $\mu \in U$. Pick any $\lambda \in U(\nu; U)$. Define $h_i \in F \setminus \{0\}$ $(i = 1, 2)$ by

$$h_i(\mu) := \begin{cases} \frac{\theta(e_i)(\mu)}{\|\theta(e_i)(\mu)\|} & \mu \in \text{Int}_\Delta(U) \\ 0 & \mu \notin \text{Int}_\Delta(U) \end{cases}$$

and set $e'_i := \theta^{-1}(h_i)$. The orthogonality of $h_1$ and $h_2$ (note that $e_1$ and $e_2$ are orthogonal), together with that of $h_1 + h_2$ and $h_1 - h_2$ (as $|h_1| = \lambda = |h_2|$), ensure the orthogonality of $e'_1$ and $e'_2$, as well as that of $e'_1 + e'_2$ and $e'_1 - e'_2$. It follows that $|e'_1| = |e'_2| \neq 0$ which contradicts the fact that $|e'_1||e'_2| = 0$ (as $\theta$ is quasi-local). \(\square\)

**Theorem 2.13.** Let $\Omega$ and $\Delta$ be locally compact Hausdorff spaces. Suppose that $E$ is a full Hilbert $C_0(\Omega)$-module and $F$ is a full Hilbert $C_0(\Delta)$-module. If $\theta : E \to F$ is a bijective $\mathbb{C}$-linear map such that both $\theta$ and $\theta^{-1}$ are quasi-local and orthogonality preserving, then $\theta$ is bounded and

$$\theta(e)(\nu) = \psi(\nu) J_\nu(e(\sigma(\nu))) \quad (e \in E; \nu \in \Delta),$$

where $\sigma : \Delta \to \Omega$ is a homeomorphism, $\psi$ is a strictly positive element in $C_b(\Delta)_+$, and $J_\nu$ is a unitary operator from $\Xi^E_{\sigma(\nu)}$ onto $\Xi^F_{\nu}$ such that for each fixed $f \in E$, the map $\nu \mapsto J_\nu(f(\sigma(\nu)))$ is continuous.

**Proof.** We consider $E$ as a Hilbert $C(\Omega_\infty)$-module. For each $\nu \in \Delta$, let

$$S_\nu := \{ \omega \in \Omega_\infty : \theta(K^E_{\Omega_\infty \setminus W}) \not\subset K^F_\nu \text{ for every } W \in N_{\Omega_\infty}(\omega) \}.$$

We first show that $S_\nu$ is a singleton set. Indeed, assume that $S_\nu = \emptyset$. Then for any $\omega \in \Omega_\infty$, there is $W_\omega \in N_{\Omega_\infty}(\omega)$ such that $\theta(K^E_{\Omega_\infty \setminus W_\omega}) \subseteq K^F_\nu$. Consider $\omega_1, \ldots, \omega_n \in \Omega_\infty$ with

$$\bigcup_{k=1}^n \text{Int}_{\Omega_\infty}(W_{\omega_k}) = \Omega_\infty$$

and consider $\{ \varphi_k \}_{k=1}^n$ to be a partition of unity subordinate to $\{ \text{Int}_{\Omega_\infty}(W_{\omega_k}) \}_{k=1}^n$. Then for any $e \in E$, we have $e \varphi_k \in K^E_{\Omega_\infty \setminus W_{\omega_k}}$ and so $\theta(e) \in K^F_\nu$. This shows that $F = K^F_\nu$ (as $\theta$ is surjective) which contradicts Remark 2.9. Now, assume that there are distinct elements $\omega_1, \omega_2 \in S_\nu$. Let $V_1 \in N_{\Omega_\infty}(\omega_1)$ and $V_2 \in N_{\Omega_\infty}(\omega_2)$ with
Consider any \( V \in \mathbb{N}_{\Omega_\infty}(\sigma(\nu)) \) and \( e \in K_{\nu}^E \). Pick any \( U \in \mathbb{N}_{\Omega_\infty}(\sigma(\nu)) \) with \( U \subseteq \text{Int}_{\Omega_\infty}(V) \). By the definition of \( \sigma \), there exists \( g \in K_{\Omega_\infty \setminus U}^E \) such that \( \theta(g)(\nu) \neq 0 \). Hence, there is \( W \in \mathbb{N}_\Delta(\nu) \) such that \( \theta(g)(\mu) \neq 0 \) (\( \mu \in W \)) and Lemma 2.12 implies that \( \theta(e) \in K_W^E \) as claimed. If there exists \( \nu \in \Delta \setminus \Delta_\theta \), then for any \( f \in F \), we have \( f(\nu) = 0 \) (because \( \theta \) is surjective) which contradicts the fullness of \( F \). Thus, \( \Delta_\theta = \Delta \) and \( \sigma : \Delta \to \Omega_\infty \) is continuous (by Lemma 2.2). As \( \theta^{-1} \) is also quasi-local and orthogonality preserving, a similar argument as the above gives a continuous map \( \tau : \Omega \to \Delta_\infty \) satisfying \( \theta^{-1} \left( I_{\nu}^E(\omega) \right) \subseteq I_{\omega}^E (\omega \in \Omega) \). Now, the argument of [15, Theorem 5.3] tells us that \( \sigma \) is a homeomorphism from \( \Delta \) to \( \Omega \) such that

\[
\theta(e \cdot \varphi) = \theta(e) \cdot \varphi \circ \sigma \quad (e \in E; \varphi \in C_0(\Omega)),
\]
and by Lemma 2.3(c), there exists a finite set \( T \) consisting of isolated points of \( \Delta \) such that \( \theta \) restricts to a bounded map from \( K_{\sigma(T)}^E \) to \( K_\nu^E \). Since any \( \nu \in T \) is an isolated point, \( \theta \) induces an orthogonality preserving (hence bounded) map \( \theta_\nu \) from the Hilbert space \( \Xi_{\sigma(\nu)}^E \) onto the Hilbert space \( \Xi_\nu^E \). This shows that \( \theta \) is bounded (because of Lemma 2.3(c) and the fact that \( T \) is finite). By Lemma 2.4, there is a surjective isometry \( J_\nu : \Xi_{\sigma(\nu)}^E \to \Xi_\nu^E \) such that

\[
\theta(e)(\nu) = \psi(\nu) J_\nu(e(\sigma(\nu))) \quad (e \in E; \nu \in \Delta).
\]
Now the fullness of \( E \) implies that \( \psi(\nu) > 0 \) (for every \( \nu \in \Delta \)) and clearly \( \nu \mapsto \frac{\theta(e)(\nu)}{\psi(\nu)} \) is continuous.

Note that the assumption of \( \theta^{-1} \) being orthogonality preserving is necessary in Theorem 2.13 as can be seen from the following example.

**Example 2.14.** Let \( \Omega \) be a (non-empty) locally compact Hausdorff space, and \( \Omega_2 \) be the topological disjoint sum of two copies of \( \Omega \) with \( j_1, j_2 : \Omega \to \Omega_2 \) being respectively the embeddings into the first and the second copies of \( \Omega \) in \( \Omega_2 \). Let \( H \) be a (non-zero) Hilbert space, and let \( H_2 \) be the Hilbert space direct sum of two copies of \( H \). Then the map \( \theta : C_0(\Omega_2, H) \to C_0(\Omega, H_2) \) defined by

\[
\theta(f)(\omega) = (f(j_1(\omega)), f(j_2(\omega)))
\]
is a bijective \( \mathbb{C} \)-linear map preserving orthogonality satisfying Condition (2.3). However, \( \theta \) is not of the expected form. Note that \( \theta^{-1} \) does not preserves orthogonality.
3. A conjecture concerning the non-commutative case and some of its evidences

In this section, we consider the possibility of a non-commutative generalization of Theorem 2.6(a) & (b) (see also Remark 2.9) that takes the form of the following conjecture:

**Conjecture 3.1.** Let $A$ be a $C^*$-algebra, and $E$ and $F$ be Hilbert $A$-modules. Suppose that $E$ is full and $\theta : E \to F$ is an orthogonality preserving local $\mathbb{C}$-linear map. Then $\theta$ is bounded and there exists a central multiplier $u \in M(A)$ such that 

$$
\langle \theta(e), \theta(f) \rangle = u \langle e, f \rangle \quad (e, f \in E).
$$

Note that this conjecture fails badly if the local property is dropped.

**Example 3.2.** Let $H$ be an infinite dimensional Hilbert space and $A = \mathcal{K}(H)$ be the $C^*$-algebra of all compact operators on $H$. Suppose that $\overline{H}$ is a vector space that is conjugate-linear isomorphic to $H$. When equipped with the operations: $\langle \eta_1, \eta_2 \rangle := \eta_1 \otimes \eta_2$ and $\overline{T} \eta_1 := \overline{T^* \eta_1}$ ($\overline{\eta_1, \eta_2} \in \overline{H}; T \in A$), we see that $\overline{H}$ is a Hilbert $A$-module. Suppose that $\theta$ is any unbounded bijective $\mathbb{C}$-linear map from $\overline{H}$ to $\overline{H}$. Since $\langle x, y \rangle = 0$ if and only if $x = 0$ or $y = 0$, we see that both $\theta$ and $\theta^{-1}$ preserves orthogonality.

In the following, we will give several evidences for the validity of the above conjecture.

**Proposition 3.3.** If $A$ is a $C^*$-algebra, $F$ is a Hilbert $A$-module and $\theta : A \to F$ is an orthogonality preserving $A$-module map, then the conclusion of Conjecture 3.1 holds.

**Proof.** Note that by a similar argument as in [17, 3.12.2], $\theta$ is bounded. Consider $F^{**}$ as a Hilbert $A^{**}$-module. Let $\{e_i\}$ be an approximate unit of $A$ and $x$ be a $\sigma(F^{**}, F^*)$-cluster point of $\{\theta(e_i)\}$. Then 

$$
\theta(a) = \lim \theta(e_i a) = \lim \theta(e_i) a = xa \quad (a \in A).
$$

The element $u := \langle x, x \rangle \in A^{**}$ is a quasi-multiplier of $A$. Since $\theta$ preserves orthogonality, $b^* u c = 0$ whenever $b, c \in A$ with $b^* c = 0$. Modifying slightly the argument of [20, Lemma 2.2], we see that for any $b, c \in M(A)_{sa}$ with $bc = 0$, we have $buc = 0$. Let $a \in M(A)_+$ and let $\Phi : C(\sigma(a))^{**} \to A^{**}$ be the restriction of the canonical normal *-homomorphism from $M(A)^{**}$ to $A^{**}$. Suppose that $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ and $p = \Phi(\chi_{\sigma(a) \cap (\alpha, \beta)})$. Consider two bounded sequences $\{g_n\}$ and $\{h_n\}$ in $C(\sigma(a))_+$ such that $g_n h_n = 0$ ($n \in \mathbb{N}$), as well as $g_n \to \chi_{\sigma(a) \cap (\alpha, \beta)}$ and $h_n \to \chi_{\sigma(a) \cap (\alpha, \beta)}$ point-wisely. Using the Dominated Convergence Theorem, we see that $\Phi(g_n) \to p$ and
\[ \Phi(h_n) \to 1 - p \text{ weakly. As } \Phi(g_n)u\Phi(h_n) = 0 \text{ (} n \in \mathbb{N} \text{), we have } pu(1 - p) = 0 \text{ which means that } p \text{ commutes with } u. \text{ By approximating the identity function in } C(\sigma(a)) \text{ by linear combinations of elements of the form } \chi_{\sigma(a)^c} \text{, we see that } u \text{ commutes with } a. \text{ Consequently, } u \text{ is in the centre of } M(A). \]

**Proposition 3.4.** Let \( A \) be a W*-algebra and \( E \) and \( F \) be Hilbert \( A \)-modules. Suppose that there is \( x \in E \) with \( \langle x, x \rangle \) being invertible, and \( \theta : E \to F \) is an orthogonality preserving \( A \)-module map. Then the conclusion of Conjecture 3.1 holds.

**Proof.** By replacing \( x \) with \( \langle x, x \rangle^{-1/2} x \), we assume \( \langle x, x \rangle = 1 \). Let \( a := \langle \theta(x), \theta(x) \rangle \). For any symmetry \( u \in A \), as \( x + xu \) and \( x - xu \) are orthogonal to each other, so are \( \theta(x) + \theta(x)u \) and \( \theta(x) - \theta(x)u \). Hence \( a = \langle \theta(x), \theta(x) \rangle = \langle \theta(xu), \theta(xu) \rangle = uau \), and \( a \in Z(A)_+ \) (by the Russo-Dye theorem). Pick any \( z \in E \) with \( \langle x, z \rangle = 0 \). Then \( z + x \langle z, z \rangle^{1/2} \) is orthogonal to \( z - x \langle z, z \rangle^{1/2} \). It follows from the orthogonality preserving property that

\[
\langle \theta(z), \theta(z) \rangle = \langle z, z \rangle^{1/2} \langle \theta(x), \theta(x) \rangle \langle z, z \rangle^{1/2} = a \langle z, z \rangle.
\]

For any \( y \in E \), the element \( z = y - x \langle y, x \rangle \) is orthogonal to \( x \). Hence,

\[
\langle \theta(y), \theta(y) \rangle = \langle x, y \rangle \langle \theta(x), \theta(x) \rangle \langle y, x \rangle + \langle \theta(z), \theta(z) \rangle = a \langle y, y \rangle.
\]

The result now follows from polarization. \( \square \)

The following is a slight extension of [19, 2.3] (the \( A \)-linearity is replaced by the local property).

**Proposition 3.5.** Let \( A \) be a standard \( C^* \)-algebra on a Hilbert space \( H \), and \( E \) and \( F \) be Hilbert \( A \)-modules. A \( \mathbb{C} \)-linear map \( \theta : E \to F \) is local and orthogonality preserving if and only if there is a \( \lambda \geq 0 \) such that

\[
\langle \theta(x), \theta(y) \rangle = \lambda \langle x, y \rangle \quad (x, y \in E).
\]

**Proof.** One only needs to check the necessity. Let \( x \in E \) and \( p \in \mathcal{K}(H) \) be a projection. If \( \{u_i\} \) is an approximate unit for \( A \), then \( (1 - p)u_i(1 - p) \uparrow 1 - p \) and \( pu_i p \uparrow p \). Hence \( \theta(xp)(1 - p)u_i(1 - p) = 0 \) and \( \theta(x(1 - p))pu_i p = 0 \) which will implies that \( \theta(xp)(1 - p) = 0 \) and \( \theta(x(1 - p))p = 0 \). Thus, \( \theta(x)p = (\theta(xp) + \theta(x(1 - p)))p = \theta(xp) \), and \( \theta(xu) = \theta(x)u \) for all finite rank operator \( u \in \mathcal{K}(H) \). Beware of this, we can use the same arguments in the proof of [19, 2.3] to obtain the assertion. \( \square \)
APPENDIX A. A STRONGER VERSION OF ORTHOGONALITY PRESERVERS

In this Appendix, we will show that the conclusion of Conjecture 3.1 holds for a larger class of \( C^* \)-algebras which includes both the standard ones and the commutative ones if \( \theta \) is a \( A \)-module map satisfying a stronger version of orthogonality preserving property.

\( \theta \) is called uniform-weak orthogonality preserving if for any \( \rho \in A^* \), there is a \( \delta > 0 \) such that

\[
(A.1) \quad |\rho(\langle \theta(x), \theta(y) \rangle)| < 1 \quad \text{for any } x, y \in E \text{ with } \|\langle x, y \rangle\| < \delta.
\]

Note that uniform-weak orthogonality preserving is stronger than orthogonality preserving but weaker than the conclusion of Conjecture 3.1.

Let \( \Omega \) be the Gelfand spectrum of \( ZM(A) \) (the centre of the multiplier algebra of \( A \)). Then \( A \) is the space of continuous sections on a \( C^* \)-algebraic bundle \( \Xi^A \) over \( \Omega \) (with the fibre over \( \omega \in \Omega \) being \( \Xi^A_\omega := A/AK_{\omega}^C(\Omega) \); see e.g. [8]). Let \( \Omega_A := \{ \omega \in \Omega : \Xi^A_\omega \neq (0) \} \). Then \( \Omega_A \) is an open subset of \( \Omega \) and \( A \) is the space of continuous section on the restriction of \( \Xi^A \) on \( \Omega_A \), again denoted by \( \Xi^A \). \( A \) is said to be quasi-standard if for any \( \omega \in \Omega_A \), there exists a Hilbert space \( H_\omega \) such that \( \mathcal{K}(H_\omega) \subseteq \Xi^A_\omega \subseteq \mathcal{B}(H_\omega) \). By the Dauns-Hofmann Theorem, \( C^* \)-algebras having continuous trace are quasi-standard.

**Proposition A.1.** Suppose that \( A \) is a quasi-standard \( C^* \)-algebra, \( E \) and \( F \) are Hilbert \( A \)-modules, \( E \) is full, and \( \theta : E \to F \) is a uniform-weak orthogonality preserving \( A \)-module map. Then the conclusion of Conjecture 3.1 holds.

**Proof.** Let \( \Xi^E \) be a Banach bundle over \( \Omega_A \) such that \( E \) is the space of continuous sections on \( \Xi^E \) ([7, 2.5]). Fix any \( \omega \in \Omega_A \). It is easy to check that \( \Xi^E_\omega = E/EK^C_{\omega}(\Omega_A) \) is a Hilbert \( \Xi^A_\omega \)-module. Suppose that \( \xi, \eta \in \Xi^E_\omega \setminus \{0\} \) with \( \langle \xi, \eta \rangle_{\Xi^A_\omega} = 0 \). Let \( x, y \in E \) such that \( x(\omega) = \xi \) and \( y(\omega) = \eta \). For any \( \rho \in (\Xi^A_\omega)^* \subseteq A^* \), there exists \( \delta > 0 \) such that \((A.1)\) holds. Moreover, there exists \( V \in \mathcal{N}_{\Omega_A}(\omega) \) such that \( \|\langle x, y \rangle(\nu)\| < \delta \) for each \( \nu \in V \). Pick \( U \in \mathcal{N}_{\Omega_A}(x) \) with \( U \subseteq \text{Int}_{\Omega_A}(V) \) and \( \lambda_\delta \in \mathcal{U}(U; V) \). If \( x_\delta := x\lambda_\delta \) and \( y_\delta := y\lambda_\delta \), then \( x_\delta(\omega) = \xi \), \( y_\delta(\omega) = \eta \), and \( \|\langle x_\delta, y_\delta \rangle\| < \delta \). Thus, \( |\rho(\langle \theta(\omega)(\xi), \theta(\omega)(\eta) \rangle)| = |\rho(\langle \theta(x_\delta), \theta(y_\delta) \rangle)| < 1 \). Hence, \( \langle \theta(\omega)(\xi), \theta(\omega)(\eta) \rangle = 0 \), and \( \theta(\omega) \) is orthogonality preserving. By [19, 2.3], there is \( \phi(\omega) \in \mathbb{R}_+ \) with \( \langle \theta(x), \theta(y) \rangle(\omega) = \phi(\omega)\langle x, y \rangle(\omega) \) \((x, y \in E) \). As \( \Xi^E_\omega \) is a full Hilbert \( \Xi^A_\omega \)-module, there is \( x \in E \) with \( \langle x, x \rangle(\omega) \neq 0 \) (as \( \Xi^A_\omega \neq (0) \)). For any \( \rho \in A^* \) with \( \rho(\langle x, x \rangle(\omega)) \neq 0 \), we have \( \phi(\nu) = \frac{\rho(\langle \theta(x), \theta(y) \rangle(\nu))}{\rho(\langle x, x \rangle(\nu))} \) when \( \nu \in \Omega_A \) is close to \( \omega \). This shows that \( \phi \) is continuous on \( \Omega_A \). By a similar argument for the boundedness of \( \psi \) in Lemma 2.4(a), we see that \( \phi \) is bounded. Now, \( \phi \in C_b(\Omega_A)_+ \subseteq ZM(A)_+ \). \( \square \)
References


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