SUMS OF ORTHOMORPHISMS OF CONTINUOUS FUNCTIONS

CHING-JOU LIAO, JUNG-HUI LIU AND NGAI-CHING WONG*

ABSTRACT. A bounded linear operator $T : C_0(X) \to C_0(Y)$ of continuous functions is called an orthomorphism if it is disjointness preserving, i.e.,

$$Tf_0\, Tf_1 = 0 \quad \text{whenever} \quad f_0\, f_1 = 0.$$

We call T an n–orthomorphism if it is n–disjointness preserving, i.e.,

$$Tf_0\, Tf_1 \cdots Tf_n = 0 \quad \text{whenever} \quad f_i\, f_j = 0,\ \forall i \neq j.$$

It is clear that a sum of n orthomorphisms is an n–orthomorphism. But counter examples show that the converse does not hold. In this paper, we study the question of how to write an n–orthomorphism as a sum of n orthomorphisms approximately.

1. INTRODUCTION

A basic and well studied model in analysis is the space $C(X)$ of continuous functions on a compact Hausdorff space X. These spaces are universal Banach spaces in the sense that every Banach space E can be embedded into $C(U_{E^*})$ as a Banach subspace, where U_{E^*} is the weak* compact unit ball of the dual space E^* of E. In fact, $C(X)$ carries a very rich structure. For example, every abelian C*-algebra with an identity is a $C(X)$, and a semi-simple abelian Banach algebra is a subalgebra of some $C(X)$. Here, X is the maximal ideal space of the algebra. On the other hand, every Banach lattice which is an AM-space with a strong unit is also a $C(X)$, and many others can be considered as sublattices of some $C(X)$.

It is now a common knowledge that the full structure of $C(X)$ can be recovered from either the algebraic structure (see, e.g., [13]), or from the vector lattice structure (see, e.g., [2]). Indeed, let $T : C(X) \to C(Y)$ be a bijective linear operator. If T is an algebra isomorphism then there is a homeomorphism $\varphi : Y \to X$ such that $Tf = f \circ \varphi, \forall f \in C(X)$. If T is

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*Corresponding author: Ngai-Ching Wong, wong@math.nsysu.edu.tw.

a lattice isomorphism then there is a homeomorphism $\varphi : Y \to X$ and a strictly positive h in $C(Y)$ such that $Tf = h \cdot f \circ \varphi, \forall f \in C(X)$. See, e.g., [19, 21, 24] for more expositions.

Let $T : C(X) \to C(Y)$ be a linear map. We call T an *algebra homomorphism* if it preserves products, i.e.,

$$T(fg) = Tf\,Tg, \quad \forall f, g \in C(X).$$

When the underlying field is $\mathbb{R}$, we call T a *lattice homomorphism* if it preserves the meet, i.e. the infimum, operations, i.e.,

$$T(f \wedge g) = Tf \wedge Tg, \quad \forall f, g \in C(X).$$

A bijective algebra (resp. lattice) homomorphism is called an *algebra* (resp. *lattice*) *isomorphism*.

We see that T is a lattice homomorphism if and only if it preserves zero meets, i.e.,

$$Tf \wedge Tg = 0 \quad \text{whenever} \quad f \wedge g = 0.$$

It also amounts to say that T is *positive*, i.e. $Tf \geq 0$ whenever $f \geq 0$, and T is *disjointness preserving*, i.e.,

$$|Tf| \wedge |Tg| = 0 \quad \text{whenever} \quad |f| \wedge |g| = 0.$$

Being a linear map between continuous functions, T is disjointness preserving exactly when T *preserves zero products*, i.e.,

$$Tf\,Tg = 0 \quad \text{whenever} \quad fg = 0.$$

From these points of view, the algebraic and the lattice structure do have a common point. In other words, the zero products from the algebraic structure coincide with the disjointness from the lattice structure of continuous functions. Indeed, the zero product, or equivalently, the disjointness structure suffices to determine $C(X)$.

There are many attentions put on disjointness preserving linear operators (also called *Lamperti operator* in, e.g., [3], or *separating map* in, e.g., [9]). on Banach algebras and Banach lattices (see, e.g., [1, 3–12, 14–19, 25]). A bounded disjointness preserving linear operator is called an *orthomorphism*. Note that lattice homomorphisms are exactly positive orthomorphisms. Motivated by the notion of regular operators which are differences of positive operators, and extending the projects in [4, 6], we are interested in the question when a bounded linear operator of continuous functions can be written as a finite sum of orthomorphisms.

If a bounded linear operator $T = T_1 + T_2$ is a sum of two orthomorphisms, then T is a 2–orthomorphism, that is, $Tf_0Tf_1Tf_2 = 0$ for every pairwise disjoint functions f_0, f_1, f_2 in $C(X)$. However, the 2–disjointness preserving property does not quarantee that T is a finite sum of orthomorphisms. In fact, Example 2.4 below provides us a 2–orthomorphism from $C([0,1])$ into $C([0,1])$, which cannot be written as a finite sum of orthomorphisms.

However, we can always write an n–orthomorphism T of continuous functions as a sum of at most n orthomorphisms in an approximative way. The approximation here is defined through an approximate order identity $\{g_\lambda\}$, i.e., an increasing net of non-negative functions with $\sup_\lambda g_\lambda h = h$ for every nonnegative h. We call T an *approximate sum of n orthomorphisms* if there is an approximate order identity $\{g_\lambda\}$ such that $g_\lambda T$ is a sum of at most n orthomorphisms for each λ.

In a little more general setting, we consider the C*-algebras and Banach lattices, $C_0(X)$ and $C_0(Y)$, of continuous functions defined respectively on locally compact Hausdorff spaces X and Y vanishing at infinity. In Section 2, along with some preliminary preparation, we provide a counter example to show that a 2–orthomorphism of $C[0,1]$ need not be a sum of finitely many orthomorphisms. In Section 3, we discuss how to write an n–orthomorphism as a sum of n orthomorphisms approximately. In Theorem 3.1, we see that a bounded linear operator $T : C_0(X) \to C_0(Y)$ is an n–orthomorphism if and only if its canonical extension from $C_0(X)$ into the second dual $C_0(Y)^{**}$ of $C_0(Y)$ is a sum of at most n orthomorphisms. In Theorem 3.3, without going through $C_0(Y)^{**}$, among the equivalent conditions, we show that $T : C_0(X) \to C_0(Y)$ is an n–orthomorphism if and only if it is an approximate sum of n orthomorphisms.

Some results of this paper are based on the PhD dissertation of Jung-Hui Liu [20].

2. Preliminaries and a counter example

Proposition 2.1 ([3, 9, 15, 17])**.** *Let X, Y be locally compact Hausdorff spaces. Let $T : C_0(X) \to C_0(Y)$ be a disjointness preserving linear map. Then we can partition $Y = Y_0 \cup Y_c \cup Y_d$ into a closed subset Y_0, an open subset Y_d, and $Y_c = Y \setminus (Y_0 \cup Y_d)$, satisfying the following properties.*

(1) A point $y \in Y_0$ exactly when the linear functional $f \mapsto Tf(y)$ is zero on $C_0(X)$. In other words, $Y_0 = \bigcap_{f\in C_0(X)} (Tf)^{-1}(0)$, and thus,

$$Tf|_{Y_0} = 0.$$

(2) A point $y \in Y_d$ (resp. $y \in Y_c$) exactly when the linear functional $f \mapsto Tf(y)$ is nonzero and discontinuous (resp. continuous) on $C_0(X)$.

(3) *There exist a continuous map* $\varphi : Y_c \to X$ *and a non-vanishing bounded continuous function* h *on* Y_c *such that*

$$Tf|_{Y_c} = h \cdot f \circ \varphi, \quad \forall f \in C_0(X).$$

(4) *When* T *is bijective, we have* $Y = Y_c$*, and thus* T *is automatically bounded in this case.*

(5) *When* T *is bounded, especially when* T *is positive,* $Y_d = \emptyset$ *and* $Y_c = Y \setminus Y_0$ *is open.*

For convenience, we usually write an orthomorphism, i.e., a disjointness preserving bounded linear operator, as $Tf = h \cdot f \circ \varphi$ by setting $h = 0$ on Y_0. Note that $Y_d = \emptyset$ and $\varphi : Y = Y_0 \cup Y_c \to X$ is continuous on the *cozero set* $\operatorname{coz}(h) = \{y \in Y : h(y) \neq 0\} = Y_c$ of h.

Definition 2.2. A bounded linear map $T : C_0(X) \to C_0(Y)$ is called an n*–orthomorphism* if it is n*–disjointness preserving*, i.e.,

$$Tf_0\,Tf_1 \cdots Tf_n = 0 \quad \text{whenever} \quad f_i\,f_j = 0, \forall i \neq j.$$

A sum of n orthomorphisms is clearly an n–orthomorphism. However, an n–orthomorphism is not necessarily a sum of n orthomorphisms. We are grateful to the referee for sharing with us the following example.

Example 2.3. Let $\mathbb{T} = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ be the unit circle in the complex plane. Let $T : C(\mathbb{T}) \to C(\mathbb{T})$ be defined by

$$Tf(e^{i\theta}) = f(e^{i\theta/2}) + f(-e^{i\theta/2}), \quad \forall e^{i\theta} \in \mathbb{T}.$$

It is plain that T is a well-defined 2–orthomorphism. However, we cannot write $T = T_1 + T_2$ as a sum of 2 orthomorphisms. Suppose we could, and

$$\begin{aligned} Tf(e^{i\theta}) &= T_1 f(e^{i\theta}) + T_2(e^{i\theta}) \\ &= h_1(e^{i\theta}) f(\varphi_1(e^{i\theta})) + h_2(e^{i\theta}) f(\varphi_2(e^{i\theta})), \quad \forall e^{i\theta} \in \mathbb{T}. \end{aligned}$$

Here, $h_j = T_j\mathbf{1} \in C(\mathbb{T})$ with $\mathbf{1}$ being the constant one function, and $\varphi_j : \mathbb{T} \to \mathbb{T}$ is continuous at y whenever $h_j(y) \neq 0$ for $j = 1, 2$. Dealing with appropriate continuous functions f from $C(\mathbb{T})$, we see that

$$\{\varphi_1(e^{i\theta}), \varphi_2(e^{i\theta})\} = \{e^{i\theta/2}, -e^{i\theta/2}\},$$

and

$$h_1(e^{i\theta}) = h_2(e^{i\theta}) = 1, \quad \forall e^{i\theta} \in \mathbb{T}.$$

Consequently, both φ_1, φ_2 are continuous maps from $\mathbb{T}$ into itself. It follows from a connectedness argument that, with either $j = 1$ or $j = 2$, the map $\varphi_j(e^{i\theta}) = e^{i\theta/2}$ for all θ in $(0, 2\pi)$. However, this prevents φ_j from being continuous at 1. This contradiction shows that T cannot be written as a sum of 2 orthomorphisms.

However, we can write T as a sum of 4 orthomorphisms. To this end, let $1 = g_1 + g_2$ be a continuous partition of $\mathbb{T}$ such that $g_1 = 0$ in a neighborhood of 1, and $g_2 = 0$ in a neighborhood of -1. Then, both $g_1 T$ and $g_2 T$ can be written as sums of 2 orthomorphisms. Thus $T = g_1 T + g_2 T$ is a sum of 4 orthomorphisms. $\square$

In [6, Example 1], there is a positive 2–orthomorphism which cannot be written as a finite sum of lattice homomorphisms. Recall that a lattice homomorphism is a *positive* orthomorphism. In the following we show that the 2–orthomorphism in [6, Example 1] cannot be written as a finite sum of orthomorphisms, either.

Example 2.4. Assume $\varphi_1, \varphi_2 : [0,1] \to [0,1]$ are continuous maps such that $\varphi_1(0) = \varphi_2(0)$ and $\varphi_1(y) < \varphi_2(y)$ for all $0 < y \leq 1$. Let $T : C[0,1] \to C[0,1]$ be defined by

$$Tf(y) = \begin{cases} \dfrac{1+\sin(1/y)}{2} f(\varphi_1(y)) + \dfrac{1-\sin(1/y)}{2} f(\varphi_2(y)), & \text{if } 0 < y \leq 1; \\ f(\varphi_1(0)), & \text{if } y = 0. \end{cases}$$

It is easy to see that T is a 2–orthomorphism. We shall show that T cannot be written as a sum of finitely many orthomorphisms of $C[0,1]$.

Assume on contrary that

$$T = S_1 + S_2 + \cdots + S_n,$$

where each S_i is an orthomorphism, or more precisely,

$$S_i f(y) = r_i(y) f(\psi_i(y)), \quad \forall y \in [0,1], i = 1, 2, \ldots, n.$$

Here, each $r_i = S_i(\mathbf{1}) \in C[0,1]$ with $\mathbf{1}$ being the constant one function, and $\psi_i : [0,1] \to [0,1]$ is continuous at y whenever $r_i(y) \neq 0$.

Let $p_1, p_2, \ldots, p_{2^n}$ be 2^n distinct numbers in $[0,1]$. For each $i = 1, 2, \ldots, 2^n$, let $\{y_{ij}\}_j$ be a sequence in $(0,1]$ such that $\lim_{j\to\infty} y_{ij} = 0$ and

$$\frac{1+\sin(1/y_{ij})}{2} = p_i, \quad \text{for } j = 1, 2, \ldots.$$

Let

$$A_{ij} = \{k : \psi_k(y_{ij}) \neq \varphi_2(y_{ij})\} \subseteq \{1, 2, \ldots, n\}.$$

Choose f_{ij} from $C[0,1]$ such that

$$f_{ij}(\varphi_1(y_{ij})) = f_{ij}(\psi_k(y_{ij})) = 1, \quad \forall k \in A_{ij},$$

and

$$f_{ij}(\varphi_2(y_{ij})) = 0.$$

Consider the value of $Tf(y_{ij})$, we have

$$\sum_{k\in A_{ij}} r_k(y_{ij}) = \frac{1+\sin(1/y_{ij})}{2} = p_i, \quad \text{for } j = 1, 2, \ldots.$$

Although the nonempty set A_{ij} can be different for each $j = 1, 2, \ldots$, there are only $2^n - 1$ of such choices as $A_{ij} \subseteq \{1, 2, \ldots, n\}$. Therefore, we can assume there is a nonempty subset A_i of $\{1, 2, \ldots, n\}$ such that

$$\sum_{k\in A_i} r_k(y_{ij}) = p_i, \quad \text{for infinitely many } j = 1, 2, \ldots.$$

By continuity, we have

$$\sum_{k\in A_i} r_k(0) = p_i, \quad \text{for } i = 1, 2, \ldots, 2^n.$$

Since there are exactly $2^n - 1$ distinct nonempty subsets of $\{1, 2, \ldots, n\}$, we will have some $A_{i_1} = A_{i_2}$ with $i_1 \neq i_2$. Then a contradiction arrives:

$$p_{i_1} = \sum_{k\in A_{i_1}} r_k(0) = \sum_{k\in A_{i_2}} r_k(0) = p_{i_2}.$$

$\square$

Let Y be a locally compact Hausdorff space and $C_0(Y, \mathbb{C})$ be the C*-algebra of all continuous complex-valued functions on Y vanishing at infinity. The dual space of $C_0(Y, \mathbb{C})$ is the Banach space $M(Y, \mathbb{C})$ of all complex-valued regular Borel measures on Y with finite variation norm. By Zorn's Lemma and the Radon-Nikodym theorem, $M(Y, \mathbb{C})$ can be described as an l^1–direct sum

$$M(Y, \mathbb{C}) = \oplus_1\{L^1(\mu, \mathbb{C}) : \mu \in \mathcal{C}\} \oplus_1 l^1(Y, \mathbb{C}),$$

where $\mathcal{C}$ is a maximal family of mutually singular continuous positive measures in $M(Y, \mathbb{C})$ of norm one. Accordingly, the double dual space of $C_0(Y, \mathbb{C})$ is given by an ℓ^∞–direct sum

$$C_0(Y, \mathbb{C})^{**} = \oplus_\infty\{L^\infty(\mu, \mathbb{C}) : \mu \in \mathcal{C}\} \oplus_\infty l^\infty(Y, \mathbb{C}). \tag{2.1}$$

The canonical embedding J sends $C_0(Y, \mathbb{C})$ into $C_0(Y, \mathbb{C})^{**}$. More precisely, in the setting of (2.1), for any f in $C_0(X)$ we have

$$Jf = \oplus\{f_\mu : \mu \in \mathcal{C}\} \oplus f_a.$$

Here, the atomic part f_a in $l^\infty(Y, \mathbb{C})$ agrees with f pointwisely, and each of the continuous part, f_μ in $L^\infty(\mu, \mathbb{C})$, agrees μ–almost everywhere with f on Y.

Being a commutative W*-algebra, $C_0(Y, \mathbb{C})^{**} \cong C(\widetilde{Y}, \mathbb{C})$. The spectrum $\widetilde{Y}$ of $C_0(Y, \mathbb{C})^{**}$ consists of pure states of $C_0(Y, \mathbb{C})^{**}$, and Y can be considered as a subset of $\widetilde{Y}$ consisting of normal pure states, i.e., those being weak* continuous.

By restricting to the real forms of the algebras, we can also assume the above hold when the underlying field is the real, $\mathbb{R}$. In particular, we will use the identification $C_0(Y)^{**} \cong C(\widetilde{Y})$ for both the real and complex cases. Moreover, the realization $C_0(Y)^{**} \cong \oplus_\infty\{L^\infty(\mu) : \mu \in \mathcal{C}\} \oplus_\infty l^\infty(Y)$ also helps us to visualize our arguments more constructively.

Convention. In the following, we will deal with the real case, and corresponding statements for the complex case follow from the real case with simple modifications. We also assume that Y consists of infinitely many points, for else the assertions being trivial.

Remark that $\widetilde{Y}$ is a compact and extremely disconnected space (see, e.g., [23]), that is, the closure of any open set in $\widetilde{Y}$ is again open in $\widetilde{Y}$. It follows that $C(\widetilde{Y})$ is Dedekind complete; namely, every nonempty set in $C(\widetilde{Y})$ bounded form above has a least upper bound ([22]).

3. Writing an n–orthomorphism as a sum of n orthomorphisms

In the following, we assume that X and Y are locally compact Hausdorff spaces, and let $J : C_0(Y) \to C(\widetilde{Y}) \cong C_0(Y)^{**}$ be the canonical embedding.

It is plain that if $T : C_0(X) \to C_0(Y)$ is an n–orthomorphism, then its canonical extension $JT : C_0(X) \to C(\widetilde{Y})$ is also an n–orthomorphism. In fact,

$$(JTf_1)(JTf_2)\cdots(JTf_{n+1}) = J(Tf_1 \cdot Tf_2 \cdots Tf_{n+1}) = J(0) = 0$$

if $f_i f_j = 0$ for all $i \neq j$.

Although Examples 2.3 and 2.4 tell us that we might not be able to write T as a sum of at most n orthomorphisms, we can always do so for JT. The following result is a consequence of [4, Theorems 5 and 6]. The original results in [4] deal with n–orthomorphisms from a Reisz space into a Dedekind complete Riesz space. Note that the reason of passing through to JT in the following results is to utilize the Dedekind completeness of $C(Y)^{**} \cong C(\widetilde{Y})$. If $C_0(Y)$ is itself Dedekind complete, all statements below are valid with T directly.

Theorem 3.1. *Let X, Y be locally compact Hausdorff spaces, let $J : C_0(Y) \to C(\widetilde{Y})$ $(\cong C_0(Y)^{**})$ be the canonical embedding, and let $T : C_0(X) \to C_0(Y)$ be an n–orthomorphism. Then there are n orthomorphisms $T_1, T_2, \ldots, T_n$ from $C_0(X)$ into $C(\widetilde{Y})$ such that*

$$JT = \Sigma_{i=1}^n T_i$$

Moreover, if T is positive then all T_i can be chosen to be positive.

As a demonstration, consider the 2–orthomorphism T of $C[0,1]$ in Example 2.4, we can set $\tilde{h}_i = \oplus_\mu h_{i,\mu} \oplus h_i$ in $C[0,1]^{**} = \oplus_\infty\{L^\infty(\mu) : \mu \in \mathcal{C}\} \oplus_\infty \ell^\infty([0,1])$ with

$$h_1(y) = \begin{cases} \dfrac{1+\sin(1/y)}{2}, & y \in (0,1]; \\ \dfrac{1}{2}, & y = 0, \end{cases} \qquad h_2(y) = \begin{cases} \dfrac{1-\sin(1/y)}{2}, & y \in (0,1]; \\ \dfrac{1}{2}, & y = 0, \end{cases}$$

and $h_{i,\mu}$ agrees μ–almost everywhere with h_i on $[0,1]$ for all μ in $\mathcal{C}$ and $i = 1,2$. Then

$$\begin{aligned} JTf &= \tilde{h}_1(Jf) \circ \varphi_1 + \tilde{h}_2(Jf) \circ \varphi_2 \\ &= (\oplus_{\mu\in\mathcal{C}}\, h_{1,\mu} f_\mu \circ \varphi_1) \oplus\ h_1 f_a \circ \varphi_1\ +\ (\oplus_{\mu\in\mathcal{C}}\, h_{2,\mu} f_\mu \circ \varphi_2) \oplus\ h_2 f_a \circ \varphi_2 \end{aligned}$$

is a sum of 2 orthomorphisms.

We shall show that any n–orthomorphism can be written as a sum of (at most) n orthomorphisms approximately. In Example 2.4, although the 2–orthomorphism T cannot be written as a finite sum of orthomorphisms, T might be expressed as such a finite sum if we avoid the point $y = 0$. More explicitly, if $g \in C[0,1]$ with $g(0) = 0$, then the operator gT can be written as a sum of 2 orthomorphisms. This suggests us the following definition.

Recall that an increasing net $\{g_\lambda\}$ of non-negative functions in $C_0(Y)$ is called an *approximate order identity* if $\sup_\lambda g_\lambda h = h$ for every non-negative h in $C_0(Y)$. The supremum here is taken in the sense of the lattice order on $C_0(Y)$, as opposed to pointwise supremum. Indeed, such an increasing net $\{g_\lambda\}$ satisfies exactly the conditions that $0 \leq g_\lambda \leq 1$ for all λ and $\sup_\lambda g_\lambda(y) = 1$ for all y in a dense subset of Y.

Definition 3.2. A bounded linear operator $T : C_0(X) \to C_0(Y)$ is called an *approximate sum of n orthomorphisms* if there exists an approximate order identity $\{g_\lambda\}$ in $C_0(Y)$ such that for all λ we have

$$g_\lambda T = \Sigma_{i=1}^n T_i^\lambda,$$

where T_i^λ (can be zero) is an orthomorphism for $i = 1, 2, \cdots n$.

In Example 2.4, for each $n = 1, 2, \ldots$, let

$$g_n(y) = \begin{cases} 0, & 0 \leq y \leq \frac{1}{2n}; \\ 2ny - 1, & \frac{1}{2n} \leq y \leq \frac{1}{n}; \\ 1, & \frac{1}{n} \leq y \leq 1. \end{cases}$$

Then $\{g_n\}$ is an approximate order identity of $C[0,1]$, and

$$g_n Tf = h_{1n} f \circ \varphi_1 + h_{2n} f \circ \varphi_2$$

is a sum of 2 orthomorphisms from $C[0,1]$ into $C[0,1]$. Here, $h_{in} = g_n h_i$ in $C[0,1]$ agrees with h_i on $[1/n, 1]$ for $i = 1, 2$ and $n = 1, 2, \ldots$.

The following result extends and enriches [6, Theorem 2] to the case of (not necessarily positive) linear operators between continuous functions on locally compact spaces.

Theorem 3.3. *Let X, Y be locally compact Hausdorff spaces, let $T : C_0(X) \to C_0(Y)$ be a bounded linear operator, and let n be a fixed positive integer. The following are equivalent.*

(1) *T is an approximate sum of n orthomorphisms.*
(2) *T is an n–orthomorphism.*
(3) *There are orthomorphisms $T_i : C_0(X) \to C(\widetilde{Y})$, such that*

$$JT = T_1 + T_2 + \cdots + T_n.$$

Here, $J : C_0(Y) \to C_0(Y)^{**}\ (\cong C(\widetilde{Y}))$ is the canonical embedding

(4) *For each y in Y, there are scalars $a_1, a_2, \ldots a_n$ and points $x_1, x_2, \ldots, x_n$ in X satisfying*

$$Tf(y) = \sum_{i=1}^{n} a_i f(x_i), \quad \forall f \in C_0(X).$$

(5) *There is a scalar valued function k on $X \times Y$ such that for each y in Y we have $k(x, y) = 0$ except for at most n of x in X, and*

$$Tf(y) = \int_Y k(x, y) f(x) d\sigma, \quad \forall f \in C_0(X).$$

Here, σ is the counting measure.

(6) *There are (maybe empty) disjoint subsets $Y_0, H_1, H_2, \ldots, H_n$ of Y such that their union H is denes in Y. Each H_m $(m = 1, 2, \ldots, n)$ is open, and on which there exist non-vanishing bounded scalar functions $a_1, a_2 \ldots a_m$, and continuous maps $x_i : H_m \to X$ with $x_i(y) \neq x_j(y)$ for all y in H_m and $i \neq j$, satisfying that*

$$Tf(y) = \sum_{i=1}^{m} a_i(y) f(x_i(y)), \quad \forall f \in C_0(X), \forall y \in H_m.$$

Moreover, Y_0 is closed in Y, and

$$Tf(y) = 0, \quad \forall f \in C_0(X), \forall y \in Y_0.$$

Proof. (1) implies (2): Assume that $f_0, f_1, \ldots, f_n \in C_0(X)$ and $f_i f_j \neq 0$ for $i \neq j$. Suppose that $\{g_\lambda\}$ is an approximate order identity of $C_0(Y)$ such that $g_\lambda \cdot T = \sum_{i=1}^n T_i^\lambda$, where each T_i^λ is an orthomorphism. In particular, $g_\lambda T$ is n–disjointness preserving. Thus

$$g_\lambda^n(Tf_0Tf_1\cdots Tf_n) = 0, \quad \forall\lambda.$$

As $g_\lambda(Tf_m)^\pm \uparrow (Tf_m)^\pm$ for each $m = 0, 1, 2, \ldots, n$, we see that

$$Tf_0Tf_1\cdots Tf_n = 0.$$

So T is an n–orthomorphism.

(2) implies (3): This is Theorem 3.1.

(3) implies (4): Since each $T_i : C_0(X) \to C(\widetilde{Y})$ is an orthomorphism, by Proposition 2.1 there exist continuous functions $h_i : \widetilde{Y} \to \mathbb{R}$ and maps $\varphi_i : \widetilde{Y} \to X$ such that $T_i f = h_i \cdot f \circ \varphi_i$. Consequently,

$$JTf(y) = \sum_{i=1}^n h_i(y) f(\varphi_i(y)), \quad \forall f \in C_0(X), \forall y \in \widetilde{Y}.$$

For each y in Y, setting $a_i = h_i(y)$ and $x_i = \varphi_i(y)$ we have

$$Tf(y) = \sum_{i=1}^n a_i f(x_i), \quad \forall f \in C_0(X).$$

(4) $\Leftrightarrow$ (5): Let $Tf(y) = \sum_{i=1}^n a_i f(x_i)$, where a_i and x_i depend on y. Define

$$k(x,y) = \begin{cases} a_i, & x = x_i \text{ for } i = 1, 2, \ldots, n; \\ 0, & \text{otherwise.} \end{cases}$$

We thus have (4) $\implies$ (5). It is also plain for the reverse implication.

(4) implies (6): Clearly, the set $Y_0 = \bigcap_{f\in C_0(X)} (Tf)^{-1}(0)$ is closed in Y, and on which every Tf vanishes. Let Y_n be the subset of the open set $Y \setminus Y_0$ consisting of all points y_0 in Y such that there are n distinct points $x_1(y_0)$, $x_2(y_0)$, $\ldots$, $x_n(y_0)$ in X and n non-zero real numbers $a_1(y_0)$, $a_2(y_0)$, $\ldots$, $a_n(y_0)$ satisfying

$$Tf(y_0) = \sum_{i=1}^n a_i(y_0) f(x_i(y_0)), \quad \forall f \in C_0(X).$$

Assume $y_0 \in Y_n$. Let U_i be an open neighborhood of $x_i(y_0)$ in X such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Choose by Uryshon's Lemma $g_1, g_2, \ldots, g_n$ from $C_0(X)$ such that $0 \leq g_i \leq 1$, $g_i(x_i(y_0)) = 1$, and $g_i = 0$ outside U_i, for $i = 1, 2, \ldots, n$. As $Tg_i(y_0) = a_i(y_0) \neq 0$, and the continuity of Tg_i, we have

$$Tg_1(y)Tg_2(y)\cdots Tg_n(y) \neq 0$$

for all y in an open neighborhood V of y_0 in Y. By (4), for all y in V there are (maybe not all distinct) points $x_1(y), x_2(y), \ldots, x_n(y)$ in X and (maybe zero) real numbers $a_1(y)$, $a_2(y)$, $\ldots$, $a_n(y)$ such that

$$Tf(y) = \sum_{i=1}^{n} a_i(y) f(x_i(y)), \quad \forall f \in C_0(X). \tag{3.1}$$

However, if there are less than n distinct points in $\{x_1(y), x_2(y), \ldots, x_n(y)\}$, or any one of $a_1(y), a_2(y), \ldots, a_n(y)$ is zero, then there will be some $Tg_i(y) = 0$, as $g_1, g_2, \ldots, g_n$ are pairwise disjoint. This forces $V \subseteq Y_n$, and thus Y_n is an open subset of $Y \setminus Y_0$. Moreover, we can arrange $x_i(y)$'s so that each $x_i(y)$ belongs to exactly U_i for all y in V for $i = 1, 2, \ldots, n$. It is then routine to see that all a_i are continuous on V and all x_i are continuous from V into X.

Let V' be another open subset of Y_n such that on V' a similar sum as in (3.1) can be obtained. If V' is disjoint from V, then in a trivial manner we can extend the continuous functions a_i and x_i from V to $V \cup V'$, for $i = 1, 2, \ldots, n$.

Denote by the tuple $(\{a_i, x_i\}_{i=1}^n, V)$ a nonempty open subset V of the open set Y_n, on which

$$Tf(y) = \sum_{i=1}^{n} a_i(y) f(x_i(y)), \quad \forall f \in C_0(X).$$

Here, all a_i are continuous and nonvanishing scalar functions on V and all x_i are continuous from V into X with distinct values everywhere. Order the non-emtpy family of tuples $(\{a_i, x_i\}_{i=1}^n, V)$ by extension. In other words, $(\{a_i, x_i\}_{i=1}^n, V) \leq (\{a_i', x_i'\}_{i=1}^n, V')$ whenever $V \subseteq V'$ and all a_i' agree with a_i and x_i' agree with x_i on V. Using Zorn's Lemma, we have a maximal element $(\{a_i'', x_i''\}_{i=1}^n, H_n)$. It follows from the above arguments that H_n is an open dense subset of Y_n, and (3.1) holds on H_n.

If H_n is dense in $Y \setminus Y_0$, then the assertion is obtained by setting $H_{n-1} = \cdots = H_1 = \emptyset$. If it is not, consider the nonempty open subset $Y' = Y \setminus \overline{H_n}$ of Y. The induced operator $T' : C_0(X) \to C_0(Y')$ defined by restriction clearly satisfies (4), but with n replaced with $n-1$. Let Y_{n-1} be the open set of points y_0 in Y' such that there are $n-1$ distinct points $x_1(y_0), x_2(y_0), \ldots, x_{n-1}(y_0)$ in X and $n-1$ non-zero real numbers $a_1(y_0)$, $a_2(y_0)$, $\ldots$, $a_{n-1}(y_0)$ satisfying

$$Tf(y_0) = \sum_{i=1}^{n-1} a_i(y_0) f(x_i(y_0)), \quad \forall f \in C_0(X). \tag{3.2}$$

In a similar manner, we obtain an open dense subset H_{n-1} of Y_{n-1}, which is open in Y', and thus also in Y, such that (3.2) holds on H_{n-1}. If $H_n \cup H_{n-1}$ is dense in $Y \setminus Y_0$, the assertion is obtained; otherwise, we continue to find H_{n-2} from $Y \setminus \overline{H_n \cup H_{n-1}}$, $\ldots$. Eventually, we will have n disjoint open sets, H_n, H_{n-1}, …, H_1, some of them can be empty, such that the union $H_1 \cup \cdots \cup H_n$ is an open dense subset of $Y \setminus Y_0$, and on each H_i the asserted sum representation as in (3.1) is established.

(6) implies (1): Set up the index $\alpha = (K, K')$, in which K, K' are two nonempty compact subsets of $H_1 \cup \cdots \cup H_n$ such that K is contained in the interior of K'. Choose h_α from $C_0(Y)$ such that $0 \leq h_\alpha \leq 1$, $h_\alpha|_K = 1$ and $h_\alpha|_{Y \setminus K'} = 0$. Order $\alpha_1 = (K_1, K_1') \leq \alpha_2 = (K_2, K_2')$ if $K_1' \subseteq K_2$. Then $\sup_\alpha h_\alpha f = f$ whenever f is a nonnegative function in $C_0(Y)$ vanishing outside the open set $H_1 \cup \cdots \cup H_n$. In a similar manner, let $\{k_\beta\}$ be an increasing net of nonnegative functions in $C_0(Y)$ such that $\sup_\beta k_\beta f = f$ whenever f is a nonnegative function in $C_0(Y)$ vanishing outside the open set $Y \setminus \overline{H_1 \cup \cdots \cup H_n}$, which is contained in Y_0. Order the indices $\lambda = (\alpha, \beta) \leq \lambda' = (\alpha', \beta')$ whenever $\alpha \leq \alpha'$ and $\beta \leq \beta'$. Let $g_\lambda = h_\alpha + k_\beta$ for each $\lambda = (\alpha, \beta)$. Clearly, $\{g_\lambda\}$ is an approximate order identity of $C_0(Y)$, and $g_\lambda T = h_\alpha T$ is a sum of at most n orthomorphisms. □

We remark that in proving the implication "(4) ⇒ (6)", one might not be able to choose the set H_n to be the whole of Y_n. As in Example 2.3, $Y_2 = \mathbb{T}$ while any choice of H_2 misses at least one point from $\mathbb{T}$.

The equivalence "(2) ⇔ (6)" in Theorem 3.3 can be rephrased in the following result.

Corollary 3.4. *A bounded linear operator $T : C_0(X) \to C_0(Y)$ is an n–orthomorphism if and only if restricting the range to some dense subset H of Y, we can write T as a sum of at most n orthomorphisms. In this case, there are bounded continuous scalar functions $h_1, \ldots, h_n$ on H and maps $\varphi_1, \ldots, \varphi_n : H \to X$ such that*

$$Tf|_H = \sum_{i=1}^{n} h_i f \circ \varphi_i, \quad \forall f \in C_0(X).$$

Moreover, the symbol map φ_i is continuous wherever the weight function h_i is nonvanishing for $i = 1, 2, \ldots, n$.

In Example 2.4, on the dense subset $H = H_2 = (0, 1]$ of $[0, 1]$, we can write T as a sum of two orthomorphisms.

Finally, let us repeat that all results in this paper are valid in both the real and the complex cases. For example, $f_1 + if_2, g_1 + ig_2$ in $C_0(X, \mathbb{C})$ are disjoint if and only if their real parts and imaginary parts are disjoint, namely, $f_j g_k = 0$ for $j, k = 1, 2$. It follows that a complex linear operator $T_\mathbb{C} : C_0(X, \mathbb{C}) \to C_0(Y, \mathbb{C})$ is n–disjointness preserving if and only if its real form $T_\mathbb{R} : C_0(X, \mathbb{R}) \to C_0(Y, \mathbb{R})$ is n–disjointness preserving. Here,

$T_{\mathbb{C}}(f_1 + if_2) = T_{\mathbb{R}}f_1 + iT_{\mathbb{R}}f_2$ for f_1, f_2 in $C_0(X, \mathbb{R})$. The same is true for $T_{\mathbb{C}}$ and $T_{\mathbb{R}}$ being (or approximately being) finite sums of weighted composition operators, or satisfying other equivalent properties stated in Theorem 3.3.

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(Liao) Department of Mathematics, Hong Kong Baptist University, Hong Kong.

E-mail address: cjliao@hkbu.edu.hk

(Liu) Department of Mathematics, National Kaohsiung Normal University, Kaohsiung, 82444, Taiwan.

E-mail address: liujhtw88@yahoo.com.tw

(Wong) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan.

E-mail address: wong@math.nsysu.edu.tw