SUMS OF ORTHOMORPHISMS OF CONTINUOUS FUNCTIONS

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Abstract. A bounded linear operator $T : C_0(X) \to C_0(Y)$ of continuous functions is called an orthomorphism if it is disjointness preserving, i.e.,

$$Tf_0 Tf_1 = 0 \quad \text{whenever} \quad f_0 f_1 = 0.$$ 

We call $T$ an $n$–orthomorphism if it is $n$–disjointness preserving, i.e.,

$$Tf_0 Tf_1 \cdots Tf_n = 0 \quad \text{whenever} \quad f_i f_j = 0, \forall i \neq j.$$ 

It is clear that a sum of $n$ orthomorphisms is an $n$–orthomorphism. But counter examples show that the converse does not hold. In this paper, we study the question of how to write an $n$–orthomorphism as a sum of $n$ orthomorphisms approximately.

1. Introduction

A basic and well studied model in analysis is the space $C(X)$ of continuous functions on a compact Hausdorff space $X$. These spaces are universal Banach spaces in the sense that every Banach space $E$ can be embedded into $C(U_{E^*})$ as a Banach subspace, where $U_{E^*}$ is the weak* compact unit ball of the dual space $E^*$ of $E$. In fact, $C(X)$ carries a very rich structure.

For example, every abelian C*-algebra with an identity is a $C(X)$, and a semi-simple abelian Banach algebra is a subalgebra of some $C(X)$. Here, $X$ is the maximal ideal space of the algebra. On the other hand, every Banach lattice which is an AM-space with a strong unit is also a $C(X)$, and many others can be considered as sublattices of some $C(X)$.

It is now a common knowledge that the full structure of $C(X)$ can be recovered from either the algebraic structure (see, e.g., [13]), or from the vector lattice structure (see, e.g., [2]). Indeed, let $T : C(X) \to C(Y)$ be a bijective linear operator. If $T$ is an algebra isomorphism then there is a homeomorphism $\varphi : Y \to X$ such that $Tf = f \circ \varphi, \forall f \in C(X)$. If $T$ is

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a lattice isomorphism then there is a homeomorphism \( \varphi : Y \rightarrow X \) and a strictly positive \( h \) in \( C(Y) \) such that \( Tf = h \cdot f \circ \varphi, \forall f \in C(X) \). See, e.g., [19, 21, 24] for more expositions.

Let \( T : C(X) \rightarrow C(Y) \) be a linear map. We call \( T \) an algebra homomorphism if it preserves products, i.e.,

\[
T(fg) = Tf Tg, \quad \forall f, g \in C(X).
\]

When the underlying field is \( \mathbb{R} \), we call \( T \) a lattice homomorphism if it preserves the meet, i.e. the infimum, operations, i.e.,

\[
T(f \wedge g) = Tf \wedge Tg, \quad \forall f, g \in C(X).
\]

A bijective algebra (resp. lattice) homomorphism is called an algebra (resp. lattice) isomorphism.

We see that \( T \) is a lattice homomorphism if and only if it preserves zero meets, i.e.,

\[
Tf \wedge Tg = 0 \quad \text{whenever} \quad f \wedge g = 0.
\]

It also amounts to say that \( T \) is positive, i.e. \( Tf \geq 0 \) whenever \( f \geq 0 \), and \( T \) is disjointness preserving, i.e.,

\[
|Tf| \wedge |Tg| = 0 \quad \text{whenever} \quad |f| \wedge |g| = 0.
\]

Being a linear map between continuous functions, \( T \) is disjointness preserving exactly when \( T \) preserves zero products, i.e.,

\[
Tf Tg = 0 \quad \text{whenever} \quad fg = 0.
\]

From these points of view, the algebraic and the lattice structure do have a common point. In other words, the zero products from the algebraic structure coincide with the disjointness from the lattice structure of continuous functions. Indeed, the zero product, or equivalently, the disjointness structure suffices to determine \( C(X) \).

There are many attentions put on disjointness preserving linear operators (also called Lamperti operator in, e.g., [3], or separating map in, e.g., [9]) on Banach algebras and Banach lattices (see, e.g., [1, 3–12, 14–19, 25]). A bounded disjointness preserving linear operator is called an orthomorphism. Note that lattice homomorphisms are exactly positive orthomorphisms. Motivated by the notion of regular operators which are differences of positive operators, and extending the projects in [4, 6], we are interested in the question when a bounded linear operator of continuous functions can be written as a finite sum of orthomorphisms.

If a bounded linear operator \( T = T_1 + T_2 \) is a sum of two orthomorphisms, then \( T \) is a 2–orthomorphism, that is, \( T f T_1 T f_2 = 0 \) for every pairwise
disjoint functions \( f_0, f_1, f_2 \) in \( C(X) \). However, the 2–disjointness preserving property does not guarantee that \( T \) is a finite sum of orthomorphisms. In fact, Example 2.4 below provides us a 2–orthomorphism from \( C([0,1]) \) into \( C([0,1]) \), which cannot be written as a finite sum of orthomorphisms.

However, we can always write an \( n \)–orthomorphism \( T \) of continuous functions as a sum of at most \( n \) orthomorphisms in an approximative way. The approximation here is defined through an approximate order identity \( \{ g_\lambda \} \), i.e., an increasing net of non-negative functions with \( \sup \lambda g_\lambda h = h \) for every nonnegative \( h \). We call \( T \) an *approximate sum of \( n \) orthomorphisms* if there is an approximate order identity \( \{ g_\lambda \} \) such that \( g_\lambda T \) is a sum of at most \( n \) orthomorphisms for each \( \lambda \).

In a little more general setting, we consider the C*-algebras and Banach lattices, \( C_0(X) \) and \( C_0(Y) \), of continuous functions defined respectively on locally compact Hausdorff spaces \( X \) and \( Y \) vanishing at infinity. In Section 2, along with some preliminary preparation, we provide a counter example to show that a 2–orthomorphism of \( C[0,1] \) need not be a sum of finitely many orthomorphisms. In Section 3, we discuss how to write an \( n \)–orthomorphism as a sum of \( n \) orthomorphisms approximately. In Theorem 3.1, we see that a bounded linear operator \( T : C_0(X) \rightarrow C_0(Y) \) is an \( n \)–orthomorphism if and only if its canonical extension from \( C_0(X) \) into the second dual \( C_0(Y)^{**} \) of \( C_0(Y) \) is a sum of at most \( n \) orthomorphisms. In Theorem 3.3, without going through \( C_0(Y)^{**} \), among the equivalent conditions, we show that \( T : C_0(X) \rightarrow C_0(Y) \) is an \( n \)–orthomorphism if and only if it is an approximate sum of \( n \) orthomorphisms.

Some results of this paper are based on the PhD dissertation of Jung-Hui Liu [20].

2. Preliminaries and a counter example

**Proposition 2.1** ([3, 9, 15, 17]). Let \( X, Y \) be locally compact Hausdorff spaces. Let \( T : C_0(X) \rightarrow C_0(Y) \) be a disjointness preserving linear map. Then we can partition \( Y = Y_0 \cup Y_c \cup Y_d \) into a closed subset \( Y_0 \), an open subset \( Y_d \), and \( Y_c = Y \setminus (Y_0 \cup Y_d) \), satisfying the following properties.

1. A point \( y \in Y_0 \) exactly when the linear functional \( f \mapsto Tf(y) \) is zero on \( C_0(X) \). In other words, \( Y_0 = \bigcap_{f \in C_0(X)} (Tf)^{-1}(0) \), and thus,
   \[
   Tf|_{Y_0} = 0.
   \]

2. A point \( y \in Y_d \) (resp. \( y \in Y_c \)) exactly when the linear functional \( f \mapsto Tf(y) \) is nonzero and discontinuous (resp. continuous) on \( C_0(X) \).
(3) There exist a continuous map \( \varphi : Y_c \to X \) and a non-vanishing bounded continuous function \( h \) on \( Y_c \) such that
\[
Tf|_{Y_c} = h \cdot f \circ \varphi, \quad \forall f \in C_0(X).
\]

(4) When \( T \) is bijective, we have \( Y = Y_c \), and thus \( T \) is automatically bounded in this case.

(5) When \( T \) is bounded, especially when \( T \) is positive, \( Y_d = \emptyset \) and \( Y_c = Y \setminus Y_0 \) is open.

For convenience, we usually write an orthomorphism, i.e., a disjointness preserving bounded linear operator, as \( Tf = h \cdot f \circ \varphi \) by setting \( h = 0 \) on \( Y_0 \). Note that \( Y_d = \emptyset \) and \( \varphi : Y = Y_0 \cup Y_c \to X \) is continuous on the cozero set \( \text{coz}(h) = \{ y \in Y : h(y) \neq 0 \} = Y_c \) of \( h \).

**Definition 2.2.** A bounded linear map \( T : C_0(X) \to C_0(Y) \) is called an \( n \)-orthomorphism if it is \( n \)-disjointness preserving, i.e.,
\[
Tf_0 \cdot Tf_1 \cdots \cdot Tf_n = 0 \quad \text{whenever} \quad f_i f_j = 0, \forall i \neq j.
\]

A sum of \( n \) orthomorphisms is clearly an \( n \)-orthomorphism. However, an \( n \)-orthomorphism is not necessarily a sum of \( n \) orthomorphisms. We are grateful to the referee for sharing with us the following example.

**Example 2.3.** Let \( T = \{ e^{i\theta} : 0 \leq \theta \leq 2\pi \} \) be the unit circle in the complex plane. Let \( T : C(T) \to C(T) \) be defined by
\[
Tf(e^{i\theta}) = f(e^{i\theta/2}) + f(-e^{i\theta/2}), \quad \forall e^{i\theta} \in T.
\]

It is plain that \( T \) is a well-defined 2–orthomorphism. However, we cannot write \( T = T_1 + T_2 \) as a sum of 2 orthomorphisms. Suppose we could, and
\[
Tf(e^{i\theta}) = T_1 f(e^{i\theta}) + T_2(e^{i\theta})
\]
\[
= h_1(e^{i\theta}) f(\varphi_1(e^{i\theta})) + h_2(e^{i\theta}) f(\varphi_2(e^{i\theta})), \quad \forall e^{i\theta} \in T.
\]

Here, \( h_j = T_j 1 \in C(T) \) with 1 being the constant one function, and \( \varphi_j : T \to T \) is continuous at \( y \) whenever \( h_j(y) \neq 0 \) for \( j = 1, 2 \). Dealing with appropriate continuous functions \( f \) from \( C(T) \), we see that
\[
\{ \varphi_1(e^{i\theta}), \varphi_2(e^{i\theta}) \} = \{ e^{i\theta/2}, -e^{i\theta/2} \},
\]
and
\[
h_1(e^{i\theta}) = h_2(e^{i\theta}) = 1, \quad \forall e^{i\theta} \in T.
\]

Consequently, both \( \varphi_1, \varphi_2 \) are continuous maps from \( T \) into itself. It follows from a connectedness argument that, with either \( j = 1 \) or \( j = 2 \), the map \( \varphi_j(e^{i\theta}) = e^{i\theta/2} \) for all \( \theta \) in \( (0, 2\pi) \). However, this prevents \( \varphi_j \) from being continuous at \( 1 \). This contradiction shows that \( T \) cannot be written as a sum of 2 orthomorphisms.
However, we can write $T$ as a sum of 4 orthomorphisms. To this end, let $1 = g_1 + g_2$ be a continuous partition of $\mathbb{T}$ such that $g_1 = 0$ in a neighborhood of 1, and $g_2 = 0$ in a neighborhood of $-1$. Then, both $g_1T$ and $g_2T$ can be written as sums of 2 orthomorphisms. Thus $T = g_1T + g_2T$ is a sum of 4 orthomorphisms. \qed

In [6, Example 1], there is a positive 2–orthomorphism which cannot be written as a finite sum of lattice homomorphisms. Recall that a lattice homomorphism is a positive orthomorphism. In the following we show that the 2–orthomorphism in [6, Example 1] cannot be written as a finite sum of orthomorphisms, either.

Example 2.4. Assume $\varphi_1, \varphi_2 : [0, 1] \to [0, 1]$ are continuous maps such that $\varphi_1(0) = \varphi_2(0)$ and $\varphi_1(y) < \varphi_2(y)$ for all $0 < y \leq 1$. Let $T : C[0, 1] \to C[0, 1]$ be defined by

$$Tf(y) = \begin{cases} 
\frac{1 + \sin(1/y)}{2}f(\varphi_1(y)) + \frac{1 - \sin(1/y)}{2}f(\varphi_2(y)), & \text{if } 0 < y \leq 1; \\
1 + \sin(1/\varphi_1(0)), & \text{if } y = 0.
\end{cases}$$

It is easy to see that $T$ is a 2–orthomorphism. We shall show that $T$ cannot be written as a sum of finitely many orthomorphisms of $C[0, 1]$.

Assume on contrary that

$$T = S_1 + S_2 + \cdots + S_n,$$

where each $S_i$ is an orthomorphism, or more precisely,

$$S_if(y) = r_i(y)f(\psi_i(y)), \quad \forall y \in [0, 1], i = 1, 2, \ldots, n.$$

Here, each $r_i = S_i(1) \in C[0, 1]$ with 1 being the constant one function, and $\psi_i : [0, 1] \to [0, 1]$ is continuous at $y$ whenever $r_i(y) \neq 0$.

Let $p_1, p_2, \ldots, p_{2^n}$ be $2^n$ distinct numbers in $[0, 1]$. For each $i = 1, 2, \ldots, 2^n$, let $\{y_{ij}\}$ be a sequence in $(0, 1]$ such that $\lim_{j \to \infty} y_{ij} = 0$ and

$$\frac{1 + \sin(1/y_{ij})}{2} = p_i, \quad \text{for } j = 1, 2, \ldots.$$

Let

$$A_{ij} = \{k : \psi_k(y_{ij}) \neq \varphi_2(y_{ij})\} \subseteq \{1, 2, \ldots, n\}.$$

Choose $f_{ij}$ from $C[0, 1]$ such that

$$f_{ij}(\varphi_1(y_{ij})) = f_{ij}(\psi_k(y_{ij})) = 1, \quad \forall k \in A_{ij},$$

and

$$f_{ij}(\varphi_2(y_{ij})) = 0.$$
Consider the value of $Tf(y_{ij})$, we have
\[ \sum_{k \in A_{ij}} r_k(y_{ij}) = \frac{1 + \sin(1/y_{ij})}{2} = p_i, \quad \text{for } j = 1, 2, \ldots. \]

Although the nonempty set $A_{ij}$ can be different for each $j = 1, 2, \ldots$, there are only $2^n - 1$ of such choices as $A_{ij} \subseteq \{1, 2, \ldots, n\}$. Therefore, we can assume there is a nonempty subset $A_i$ of $\{1, 2, \ldots, n\}$ such that
\[ \sum_{k \in A_i} r_k(y_{ij}) = p_i, \quad \text{for infinitely many } j = 1, 2, \ldots. \]

By continuity, we have
\[ \sum_{k \in A_i} r_k(0) = p_i, \quad \text{for } i = 1, 2, \ldots, 2^n. \]

Since there are exactly $2^n - 1$ distinct nonempty subsets of $\{1, 2, \ldots, n\}$, we will have some $A_{i_1} = A_{i_2}$ with $i_1 \neq i_2$. Then a contradiction arrives:
\[ p_{i_1} = \sum_{k \in A_{i_1}} r_k(0) = \sum_{k \in A_{i_2}} r_k(0) = p_{i_2}. \]

Let $Y$ be a locally compact Hausdorff space and $C_0(Y, \mathbb{C})$ be the $C^*$-algebra of all continuous complex-valued functions on $Y$ vanishing at infinity. The dual space of $C_0(Y, \mathbb{C})$ is the Banach space $M(Y, \mathbb{C})$ of all complex-valued regular Borel measures on $Y$ with finite variation norm. By Zorn’s Lemma and the Radon-Nikodym theorem, $M(Y, \mathbb{C})$ can be described as an $l^1$–direct sum
\[ M(Y, \mathbb{C}) = \oplus_1 \{L^1(\mu, \mathbb{C}) : \mu \in C\} \oplus_1 l^1(Y, \mathbb{C}), \]

where $C$ is a maximal family of mutually singular continuous positive measures in $M(Y, \mathbb{C})$ of norm one. Accordingly, the double dual space of $C_0(Y, \mathbb{C})$ is given by an $\ell^\infty$–direct sum
\[ C_0(Y, \mathbb{C})^{**} = \oplus_\infty \{L^\infty(\mu, \mathbb{C}) : \mu \in C\} \oplus_\infty \ell^\infty(Y, \mathbb{C}). \]

The canonical embedding $J$ sends $C_0(Y, \mathbb{C})$ into $C_0(Y, \mathbb{C})^{**}$. More precisely, in the setting of (2.1), for any $f$ in $C_0(X)$ we have
\[ Jf = \oplus \{f_\mu : \mu \in C\} \oplus f_a. \]

Here, the atomic part $f_a$ in $l^\infty(Y, \mathbb{C})$ agrees with $f$ pointwisely, and each of the continuous part, $f_\mu$ in $L^\infty(\mu, \mathbb{C})$, agrees $\mu$–almost everywhere with $f$ on $Y$.

Being a commutative $W^*$-algebra, $C_0(Y, \mathbb{C})^{**} \cong C(\tilde{Y}, \mathbb{C})$. The spectrum $\tilde{Y}$ of $C_0(Y, \mathbb{C})^{**}$ consists of pure states of $C_0(Y, \mathbb{C})^{**}$, and $Y$ can be considered
as a subset of $\tilde{Y}$ consisting of normal pure states, i.e., those being weak* continuous.

By restricting to the real forms of the algebras, we can also assume the above hold when the underlying field is the real, $\mathbb{R}$. In particular, we will use the identification $C_0(Y)^{**} \cong C(\tilde{Y})$ for both the real and complex cases. Moreover, the realization $C_0(Y)^{**} \cong \oplus_{\infty} \{ L^\infty(\mu) : \mu \in C \} \oplus_{\infty} l^\infty(Y)$ also helps us to visualize our arguments more constructively.

**Convention.** In the following, we will deal with the real case, and corresponding statements for the complex case follow from the real case with simple modifications. We also assume that $Y$ consists of infinitely many points, for else the assertions being trivial.

Remark that $\tilde{Y}$ is a compact and extremely disconnected space (see, e.g., [23]), that is, the closure of any open set in $\tilde{Y}$ is again open in $\tilde{Y}$. It follows that $C(\tilde{Y})$ is Dedekind complete; namely, every nonempty set in $C(\tilde{Y})$ bounded form above has a least upper bound ([22]).

3. Writing an $n$–orthomorphism as a sum of $n$ orthomorphisms

In the following, we assume that $X$ and $Y$ are locally compact Hausdorff spaces, and let $J : C_0(Y) \to C(\tilde{Y}) \cong C_0(Y)^{**}$ be the canonical embedding.

It is plain that if $T : C_0(X) \to C_0(Y)$ is an $n$–orthomorphism, then its canonical extension $JT : C_0(X) \to C(\tilde{Y})$ is also an $n$–orthomorphism. In fact,

$$(JTf_1)(JTf_2)\cdots(JTf_{n+1}) = J(Tf_1 \cdot Tf_2 \cdots Tf_{n+1}) = J(0) = 0$$

if $f_i f_j = 0$ for all $i \neq j$.

Although Examples 2.3 and 2.4 tell us that we might not be able to write $T$ as a sum of at most $n$ orthomorphisms, we can always do so for $JT$. The following result is a consequence of [4, Theorems 5 and 6]. The original results in [4] deal with $n$–orthomorphisms from a Reisz space into a Dedekind complete Riesz space. Note that the reason of passing through $JT$ in the following results is to utilize the Dedekind completeness of $C(Y)^{**} \cong C(\tilde{Y})$. If $C_0(Y)$ is itself Dedekind complete, all statements below are valid with $T$ directly.

**Theorem 3.1.** Let $X, Y$ be locally compact Hausdorff spaces, let $J : C_0(Y) \to C(\tilde{Y}) (\cong C_0(Y)^{**})$ be the canonical embedding, and let $T : C_0(X) \to C_0(Y)$ be an $n$–orthomorphism. Then there are $n$ orthomorphisms $T_1, T_2, \ldots, T_n$ from $C_0(X)$ into $C(\tilde{Y})$ such that

$$JT = \Sigma_{i=1}^{n} T_i$$
Moreover, if $T$ is positive then all $T_i$ can be chosen to be positive.

As a demonstration, consider the 2–orthomorphism $T$ of $C[0,1]$ in Example 2.4, we can set $\tilde{h}_i = \oplus_\mu h_{i,\mu} \oplus h_i$ in $C[0,1]^{**} = \oplus_\infty \{L^\infty(\mu) : \mu \in C\} \oplus_\infty \ell^\infty([0,1])$ with

$$h_1(y) = \begin{cases} 
\frac{1 + \sin(1/y)}{2}, & y \in (0, 1]; \\
\frac{1}{2}, & y = 0,
\end{cases} \quad h_2(y) = \begin{cases} 
\frac{1 - \sin(1/y)}{2}, & y \in (0, 1]; \\
\frac{1}{2}, & y = 0,
\end{cases}$$

and $h_{i,\mu}$ agrees $\mu$–almost everywhere with $h_i$ on $[0,1]$ for all $\mu$ in $C$ and $i = 1, 2$. Then

$$JTf = \tilde{h}_1(Jf) \circ \varphi_1 + \tilde{h}_2(Jf) \circ \varphi_2$$

$$= (\oplus_{\mu \in C} h_{1,\mu} f_\mu \circ \varphi_1) \oplus h_1 f_a \circ \varphi_1 + (\oplus_{\mu \in C} h_{2,\mu} f_\mu \circ \varphi_2) \oplus h_2 f_a \circ \varphi_2$$

is a sum of 2 orthomorphisms.

We shall show that any $n$–orthomorphism can be written as a sum of (at most) $n$ orthomorphisms approximately. In Example 2.4, although the 2–orthomorphism $T$ cannot be written as a finite sum of orthomorphisms, $T$ might be expressed as such a finite sum if we avoid the point $y = 0$. More explicitly, if $g \in C[0,1]$ with $g(0) = 0$, then the operator $gT$ can be written as a sum of 2 orthomorphisms. This suggests us the following definition.

Recall that an increasing net $\{g_\lambda\}$ of non-negative functions in $C_0(Y)$ is called an approximate order identity if $\sup_\lambda g_\lambda h = h$ for every non-negative $h$ in $C_0(Y)$. The supremum here is taken in the sense of the lattice order on $C_0(Y)$, as opposed to pointwise supremum. Indeed, such an increasing net $\{g_\lambda\}$ satisfies exactly the conditions that $0 \leq g_\lambda \leq 1$ for all $\lambda$ and $\sup_\lambda g_\lambda(y) = 1$ for all $y$ in a dense subset of $Y$.

**Definition 3.2.** A bounded linear operator $T : C_0(X) \rightarrow C_0(Y)$ is called an approximate sum of $n$ orthomorphisms if there exists an approximate order identity $\{g_\lambda\}$ in $C_0(Y)$ such that for all $\lambda$ we have

$$g_\lambda T = \sum_{i=1}^n T_i^\lambda,$$

where $T_i^\lambda$ (can be zero) is an orthomorphism for $i = 1, 2, \cdots n$.

In Example 2.4, for each $n = 1, 2, \ldots$, let

$$g_n(y) = \begin{cases} 
0, & 0 \leq y \leq \frac{1}{2n}; \\
2ny - 1, & \frac{1}{2n} \leq y \leq \frac{1}{n}; \\
1, & \frac{1}{n} \leq y \leq 1.
\end{cases}$$
Then \( \{g_n\} \) is an approximate order identity of \( C[0,1] \), and
\[
g_nTf = h_{1n}f \circ \varphi_1 + h_{2n}f \circ \varphi_2
\]
is a sum of 2 orthomorphisms from \( C[0,1] \) into \( C[0,1] \). Here, \( h_{in} = g_nh_i \) in \( C[0,1] \) agrees with \( h_i \) on \([1/n, 1] \) for \( i = 1, 2 \) and \( n = 1, 2, \ldots \).

The following result extends and enriches [6, Theorem 2] to the case of (not necessarily positive) linear operators between continuous functions on locally compact spaces.

**Theorem 3.3.** Let \( X, Y \) be locally compact Hausdorff spaces, let \( T : C_0(X) \to C_0(Y) \) be a bounded linear operator, and let \( n \) be a fixed positive integer. The following are equivalent.

1. \( T \) is an approximate sum of \( n \) orthomorphisms.
2. \( T \) is an \( n \)-orthomorphism.
3. There are orthomorphisms \( T_i : C_0(X) \to C(\tilde{Y}) \), such that
\[
JT = T_1 + T_2 + \cdots + T_n.
\]

Here, \( J : C_0(Y) \to C_0(Y)^{**} (\cong C(\tilde{Y})) \) is the canonical embedding.

4. For each \( y \) in \( Y \), there are scalars \( a_1, a_2, \ldots, a_n \) and points \( x_1, x_2, \ldots, x_n \) in \( X \) satisfying
\[
Tf(y) = \sum_{i=1}^{n} a_i f(x_i), \quad \forall f \in C_0(X).
\]

5. There is a scalar valued function \( k \) on \( X \times Y \) such that for each \( y \) in \( Y \) we have \( k(x,y) = 0 \) except for at most \( n \) of \( x \) in \( X \), and
\[
Tf(y) = \int_{Y} k(x,y) f(x) d\sigma, \quad \forall f \in C_0(X).
\]

Here, \( \sigma \) is the counting measure.

6. There are (maybe empty) disjoint subsets \( Y_0, H_1, H_2, \ldots, H_n \) of \( Y \) such that their union \( H \) is dense in \( Y \). Each \( H_m \) \((m = 1, 2, \ldots, n)\) is open, and on which there exist non-vanishing bounded scalar functions \( a_1, a_2 \ldots a_m \), and continuous maps \( x_i : H_m \to X \) with \( x_i(y) \neq x_j(y) \) for all \( y \) in \( H_m \) and \( i \neq j \), satisfying that
\[
Tf(y) = \sum_{i=1}^{m} a_i(y) f(x_i(y)), \quad \forall f \in C_0(X), \forall y \in H_m.
\]

Moreover, \( Y_0 \) is closed in \( Y \), and
\[
Tf(y) = 0, \quad \forall f \in C_0(X), \forall y \in Y_0.
\]

**Proof.** (1) implies (2): Assume that \( f_0, f_1, \ldots, f_n \in C_0(X) \) and \( f_i f_j \neq 0 \) for \( i \neq j \). Suppose that \( \{g_\lambda\} \) is an approximate order identity of \( C_0(Y) \) such
that \( g_\lambda \cdot T = \sum_{i=1}^n T_i^\lambda \), where each \( T_i^\lambda \) is an orthomorphism. In particular, \( g_\lambda T \) is \( n \)-disjointness preserving. Thus

\[
g_\lambda^n(T f_0 T f_1 \cdots T f_n) = 0, \quad \forall \lambda.
\]

As \( g_\lambda(T f_m)^\pm \uparrow (T f_m)^\pm \) for each \( m = 0, 1, 2, \ldots, n \), we see that

\[
T f_0 T f_1 \cdots T f_n = 0.
\]

So \( T \) is an \( n \)-orthomorphism.

(2) implies (3): This is Theorem 3.1.

(3) implies (4): Since each \( T_i : C_0(X) \to C(\tilde{Y}) \) is an orthomorphism, by Proposition 2.1 there exist continuous functions \( h_i : \tilde{Y} \to \mathbb{R} \) and maps \( \varphi_i : \tilde{Y} \to X \) such that \( T_i f = h_i \cdot f \circ \varphi_i \). Consequently,

\[
JTf(y) = \sum_{i=1}^n h_i(y) f(\varphi_i(y)), \quad \forall f \in C_0(X), \forall y \in \tilde{Y}.
\]

For each \( y \) in \( Y \), setting \( a_i = h_i(y) \) and \( x_i = \varphi_i(y) \) we have

\[
Tf(y) = \sum_{i=1}^n a_i f(x_i), \quad \forall f \in C_0(X).
\]

(4) \iff (5): Let \( T f(y) = \sum_{i=1}^n a_i f(x_i) \), where \( a_i \) and \( x_i \) depend on \( y \). Define

\[
k(x, y) = \begin{cases} a_i, & x = x_i \text{ for } i = 1, 2, \ldots, n; \\ 0, & \text{otherwise.} \end{cases}
\]

We thus have (4) \implies (5). It is also plain for the reverse implication.

(4) implies (6): Clearly, the set \( Y_0 = \bigcap_{f \in C_0(X)} (T f)^{-1}(0) \) is closed in \( Y \), and on which every \( T f \) vanishes. Let \( Y_n \) be the subset of the open set \( Y \setminus Y_0 \) consisting of all points \( y_0 \) in \( Y \) such that there are \( n \) distinct points \( x_1(y_0), x_2(y_0), \ldots, x_n(y_0) \) in \( X \) and \( n \) non-zero real numbers \( a_1(y_0), a_2(y_0), \ldots, a_n(y_0) \) satisfying

\[
T f(y_0) = \sum_{i=1}^n a_i(y_0) f(x_i(y_0)), \quad \forall f \in C_0(X).
\]

Assume \( y_0 \in Y_n \). Let \( U_i \) be an open neighborhood of \( x_i(y_0) \) in \( X \) such that \( U_i \cap U_j = \emptyset \) for \( i \neq j \). Choose by Uryshon’s Lemma \( g_1, g_2, \ldots, g_n \) from \( C_0(X) \) such that \( 0 \leq g_i \leq 1, \ g_i(x_i(y_0)) = 1, \) and \( g_i = 0 \) outside \( U_i \), for \( i = 1, 2, \ldots, n \). As \( T g_i(y_0) = a_i(y_0) \neq 0 \), and the continuity of \( T g_i \), we have

\[
T g_1(y) T g_2(y) \cdots T g_n(y) \neq 0
\]

for all \( y \) in an open neighborhood \( V \) of \( y_0 \) in \( Y \). By (4), for all \( y \) in \( V \) there are (maybe not all distinct) points \( x_1(y), x_2(y), \ldots, x_n(y) \) in \( X \) and (maybe
If there are less than \( n \) distinct points in \( \{x_1(y), x_2(y), \ldots, x_n(y)\} \), or any one of \( a_1(y), a_2(y), \ldots, a_n(y) \) is zero, then there will be some \( Tg_i(y) = 0 \), as \( g_1, g_2, \ldots, g_n \) are pairwise disjoint. This forces \( V \subseteq Y_n \), and thus \( Y_n \) is an open subset of \( Y \setminus Y_0 \). Moreover, we can arrange \( x_i(y) \)'s so that each \( x_i(y) \) belongs to exactly \( U_i \) for all \( y \) in \( V \) for \( i = 1, 2, \ldots, n \). It is then routine to see that all \( a_i \) are continuous on \( V \) and all \( x_i \) are continuous from \( V \) into \( X \).

Let \( V' \) be another open subset of \( Y_n \) such that on \( V' \) a similar sum as in (3.1) can be obtained. If \( V' \) is disjoint from \( V \), then in a trivial manner we can extend the continuous functions \( a_i \) and \( x_i \) from \( V \) to \( V \cup V' \), for \( i = 1, 2, \ldots, n \).

Denote by the tuple \( \{a_i, x_i\}_{i=1}^n \) a nonempty open subset \( V \) of the open set \( Y_n \), on which

\[
Tf(y) = \sum_{i=1}^n a_i(y)f(x_i(y)), \quad \forall f \in C_0(X). \tag{3.1}
\]

Here, all \( a_i \) are continuous and nonvanishing scalar functions on \( V \) and all \( x_i \) are continuous from \( V \) into \( X \) with distinct values everywhere. Order the non-empty family of tuples \( \{a_i, x_i\}_{i=1}^n \) by extension. In other words, \( \{a_i, x_i\}_{i=1}^n \) \( \leq \) \( \{a_i', x_i'\}_{i=1}^n \) whenever \( V \subseteq V' \) and all \( a'_i \) agree with \( a_i \) and \( x_i \) agree with \( x_i \) on \( V \). Using Zorn’s Lemma, we have a maximal element \( \{a_i', x_i'\}_{i=1}^n, H_n \). It follows from the above arguments that \( H_n \) is an open dense subset of \( Y_n \), and (3.1) holds on \( H_n \).

If \( H_n \) is dense in \( Y \setminus Y_0 \), then the assertion is obtained by setting \( H_{n-1} = \cdots = H_1 = \emptyset \). If it is not, consider the nonempty open subset \( Y' = Y \setminus \overline{H_n} \) of \( Y \). The induced operator \( T' : C_0(X) \rightarrow C_0(Y') \) defined by restriction clearly satisfies (4), but with \( n \) replaced with \( n - 1 \). Let \( Y_{n-1} \) be the open set of points \( y_0 \) in \( Y' \) such that there are \( n - 1 \) distinct points \( x_1(y_0), x_2(y_0), \ldots, x_{n-1}(y_0) \) in \( X \) and \( n - 1 \) non-zero real numbers \( a_1(y_0), a_2(y_0), \ldots, a_{n-1}(y_0) \) satisfying

\[
Tf(y_0) = \sum_{i=1}^{n-1} a_i(y_0)f(x_i(y_0)), \quad \forall f \in C_0(X). \tag{3.2}
\]

In a similar manner, we obtain an open dense subset \( H_{n-1} \) of \( Y_{n-1} \), which is open in \( Y' \), and thus also in \( Y \), such that (3.2) holds on \( H_{n-1} \). If \( H_n \cup H_{n-1} \) is dense in \( Y \setminus Y_0 \), the assertion is obtained; otherwise, we continue to find \( H_{n-2} \) from \( Y \setminus \overline{H_n} \cup \overline{H_{n-1}} \), \ldots. Eventually, we will have \( n \) disjoint open
sets, $H_n$, $H_{n-1}$, \ldots, $H_1$, some of them can be empty, such that the union $H_1 \cup \cdots \cup H_n$ is an open dense subset of $Y \setminus Y_0$, and on each $H_i$ the asserted sum representation as in (3.1) is established.

(6) implies (1): Set up the index $\alpha = (K, K')$, in which $K, K'$ are two nonempty compact subsets of $H_1 \cup \cdots \cup H_n$ such that $K$ is contained in the interior of $K'$. Choose $h_\alpha$ from $C_0(Y)$ such that $0 \leq h_\alpha \leq 1$, $h_\alpha|_{K'} = 1$ and $h_\alpha|_{Y \setminus K'} = 0$. Order $\alpha_1 = (K_1, K_1') \leq \alpha_2 = (K_2, K_2')$ if $K_1' \subseteq K_2$. Then $\sup_{\alpha} h_\alpha f = f$ whenever $f$ is a nonnegative function in $C_0(Y)$ vanishing outside the open set $H_1 \cup \cdots \cup H_n$. In a similar manner, let $\{k_\beta\}$ be an increasing net of nonnegative functions in $C_0(Y)$ such that $\sup_{\beta} k_\beta f = f$ whenever $f$ is a nonnegative function in $C_0(Y)$ vanishing outside the open set $Y \setminus H_1 \cup \cdots \cup H_n$, which is contained in $Y_0$. Order the indices $\lambda = (\alpha, \beta) \leq \lambda' = (\alpha', \beta')$ whenever $\alpha \leq \alpha'$ and $\beta \leq \beta'$. Let $g_\lambda = h_{\alpha'} + k_\beta$ for each $\lambda = (\alpha, \beta)$. Clearly, $\{g_\lambda\}$ is an approximate order identity of $C_0(Y)$, and $g_\lambda T = h_\alpha T$ is a sum of at most $n$ orthomorphisms. \hfill $\Box$

We remark that in proving the implication “(4) $\Rightarrow$ (6)” one might not be able to choose the set $H_n$ to be the whole of $Y_n$. As in Example 2.3, $Y_2 = \mathbb{T}$ while any choice of $H_2$ misses at least one point from $\mathbb{T}$.

The equivalence “(2) $\Leftrightarrow$ (6)” in Theorem 3.3 can be rephrased in the following result.

**Corollary 3.4.** A bounded linear operator $T : C_0(X) \to C_0(Y)$ is an $n$–orthomorphism if and only if restricting the range to some dense subset $H$ of $Y$, we can write $T$ as a sum of at most $n$ orthomorphisms. In this case, there are bounded continuous scalar functions $h_1, \ldots, h_n$ on $H$ and maps $\varphi_1, \ldots, \varphi_n : H \to X$ such that

$$Tf|_H = \sum_{i=1}^n h_i f \circ \varphi_i, \quad \forall f \in C_0(X).$$

Moreover, the symbol map $\varphi_i$ is continuous wherever the weight function $h_i$ is nonvanishing for $i = 1, 2, \ldots, n$.

In Example 2.4, on the dense subset $H = H_2 = (0, 1]$ of $[0, 1]$, we can write $T$ as a sum of two orthomorphisms.

Finally, let us repeat that all results in this paper are valid in both the real and the complex cases. For example, $f_1 + i f_2, g_1 + i g_2$ in $C_0(X, \mathbb{C})$ are disjoint if and only if their real parts and imaginary parts are disjoint, namely, $f_j g_k = 0$ for $j, k = 1, 2$. It follows that a complex linear operator $T_{\mathbb{C}} : C_0(X, \mathbb{C}) \to C_0(Y, \mathbb{C})$ is $n$–disjointness preserving if and only if its real form $T_{\mathbb{R}} : C_0(X, \mathbb{R}) \to C_0(Y, \mathbb{R})$ is $n$–disjointness preserving. Here,
$T_C(f_1 + if_2) = T_R f_1 + iT_R f_2$ for $f_1, f_2$ in $C_0(X, \mathbb{R})$. The same is true for $T_C$ and $T_R$ being (or approximately being) finite sums of weighted composition operators, or satisfying other equivalent properties stated in Theorem 3.3.

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