METRIC SEMIGROUPS THAT DETERMINE LOCALLY COMPACT GROUPS

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Abstract. Let G be a locally compact group. Let A be any one of the (complex) Banach algebras: $L^1(G)$, $M(G)$, $\mathrm{WAP}(G)$ and $\mathrm{LUC}(G)$, consisting of integrable functions, regular Borel complex measures, weakly almost periodic functions, and bounded left uniformly continuous functions, respectively, on G. We show that the metric semigroup $A_+^{\mathbf{1}} := \{f \in A : f \geq 0 \text{ and } \|f\| = 1\}$ (the convex structure is not considered) is a complete invariant for G.

1. Introduction

In this paper, we find several new and simple complete invariants for locally compact groups.

Let G and H be locally compact groups. Wendel showed in [24] (respectively, Johnson showed in [10]) that G and H are isomorphic if and only if there exists an isometric algebra isomorphism $\Phi : L^1(G) \to L^1(H)$ (respectively, $\Phi : M(G) \to M(H)$). Optimistically, as $\Phi(sf) = s\Phi(f)$, information in the one dimensional subspace $\{sf : s \in \mathbb{C}\}$ is somehow encoded in the element $\{f\}$. This leads to a quest of a "smaller invariant". As a candidate, however, the unit sphere of $L^1(G)$ is not closed under the convolution product and hence cannot be served as an invariant for G.

On the other hand, Kawada showed in [11] that G and H are isomorphic whenever there is an algebra isomorphism $\Psi : L^1(G) \to L^1(H)$ satisfying: $\Psi(f) \geq 0$ if and only if $f \geq 0$. Observe that $L^1(G)_+^{\mathbf{1}}$, the positive part of the unit sphere of $L^1(G)$, is closed under the convolution product. This suggests us to consider $L^1(G)_+^{\mathbf{1}}$ as a candidate of a complete invariant of G.

In this article, we will show that the metric and the semigroup structures of $L^1(G)_+^{\mathbf{1}}$, or those of $M(G)_+^{\mathbf{1}}$, (note that the convexity is not needed) determines G. This result supplements the above mentioned results of Wendel [24], Johnson [10] and Kawada [11].

Furthermore, Ghahramani, Lau and Losert ([8]), as well as Lau and McKennon ([13]), showed that either one of the dual Banach algebras $\mathrm{LUC}(G)^*$ and $\mathrm{WAP}(G)^*$ determines G, too. We will also show that the positive parts of the unit spheres of $\mathrm{LUC}(G)^*$ and $\mathrm{WAP}(G)^*$ are complete invariants for G.

For a subset $S \subseteq E$ of an ordered Banach space E, we set

$$S_+^{\mathbf{1}} := \big\{f \in S : \|f\| = 1; f \geq 0\big\}.$$

Our main results (namely, Theorems 5 and 6) can be subsumed and simplified in the following statement.

Theorem 1. *Two locally compact groups G and H are isomorphic as topological groups if and only if any one of the following holds*

(1) $L^1(G)_+^{\mathbf{1}} \cong L^1(H)_+^{\mathbf{1}}$ *as metric semigroups;*
(2) $M(G)_+^{\mathbf{1}} \cong M(H)_+^{\mathbf{1}}$ *as metric semigroups.*
(3) $(\mathrm{WAP}(G)^*)_+^{\mathbf{1}} \cong (\mathrm{WAP}(H)^*)_+^{\mathbf{1}}$ *as metric semigroups;*

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(4) $(\mathrm{LUC}(G)^*)_+^{\mathbf{1}} \cong (\mathrm{LUC}(H)^*)_+^{\mathbf{1}}$ *as metric semigroups.*

We will obtain the above assertions by verifying that those metric preserving semigroup isomorphisms actually extend to isometric algebra isomorphisms between the whole Banach algebras, and then the corresponding established results in [10, 11, 24] apply. This task is nontrivial. Although it has been shown in [19] that metric preserving bijection from the unit sphere of $L^1(G;\mathbb{R})$ (the space of real valued integrable functions) onto that of $L^1(H;\mathbb{R})$ extends to a real linear isometry from $L^1(G;\mathbb{R})$ onto $L^1(H;\mathbb{R})$, neither this statement nor the argument in [19] can be used in our cases. In fact, on top of elementary arguments, our proofs also depend on a theorem of Dye from [6] and its applications given in [16], which are results concerning W^*-algebras.

2. The proof of the main theorem

Theorem 1 is a consequence of the following result, which should be of independent interest (in particular, it tells us that the metric structure on the normal state space of a W^*-algebra encodes its convex structure, when the algebra is abelian).

Proposition 2. *Let M and N be W^*-algebras with one of them being abelian. If $\Phi : (M_*)_+^{\mathbf{1}} \to (N_*)_+^{\mathbf{1}}$ is a bijection satisfying $\|\Phi(f) - \Phi(g)\| = \|f - g\|$ $(f, g \in M)$, then there is a *-isomorphism $\Theta : M \to N$ satisfying $\Theta_*^{-1}|_{(M_*)_+^{\mathbf{1}}} = \Phi$.*

Let us first do some preparation for the proof of Proposition 2. In the following, for any subset Δ of a set X, we denote by $\chi_\Delta : X \to \{0, 1\}$ the characteristic function of Δ. Moreover, for any function $g : X \to \mathbb{C}$, we set $\operatorname{supp} g$ to be the *support* of g; namely,

$$\operatorname{supp} g := \{x \in X : g(x) \neq 0\}.$$

Lemma 3. *Suppose that (X, Ω, μ) is a measure space and $n \in \mathbb{N}$.*

(a) Let $E \in \Omega$ and $c > 0$ such that $0 < c\mu(E) \leq 1$. Suppose that $f \in L^1(\mu)_+$ with $\operatorname{supp} f \subseteq E$ such that $\int_X f\, d\mu = c\mu(E)$. Then

$$f = c\chi_E$$

if and only if for any $\Delta \in \Omega$ with $\Delta \subseteq E$ and $\mu(\Delta) > 0$, there exists $g_\Delta \in L^1(\mu)_+^{\mathbf{1}}$ satisfying $\operatorname{supp} g_\Delta \subseteq \Delta$ and

$$\|f - g_\Delta\|_{L^1(\mu)} = 1 + c\mu(E) - 2c\mu(\Delta).$$

(b) Let $E_1, ..., E_n \in \Omega$ and $c_1, ..., c_n > 0$ satisfying $\sum_{k=1}^n c_k\mu(E_k) = 1$ as well as $E_i \cap E_j = \emptyset$ and $\mu(E_i) > 0$ for any $1 \leq i \neq j \leq n$. Consider $f \in L^1(\mu)_+^{\mathbf{1}}$ with $\int_{E_l} f\, d\mu = c_l\mu(E_l)$ $(l = 1, ..., n)$. Then

$$f = \sum_{k=1}^n c_k\chi_{E_k}$$

if and only if for any $l \in \{1, ..., n\}$ and $\Delta \in \Omega$ with $\Delta \subseteq E_l$ and $\mu(\Delta) > 0$, there exists $h_{\Delta,l} \in L^1(\mu)_+^{\mathbf{1}}$ satisfying $\operatorname{supp} h_{\Delta,l} \subseteq \Delta$ and

$$\|f - h_{\Delta,l}\|_{L^1(\mu)} = 2 - 2c_l\mu(\Delta).$$

Proof: (a) $\Rightarrow$). This implication is clear if we take $g_\Delta := \frac{1}{\mu(\Delta)}\chi_\Delta$.

$\Leftarrow$). For any $r > 0$, we set

$$\Delta_r := \{x \in E : f(x) \leq r\}$$

Assume on the contrary that $f \neq c\chi_E$. Then one can find $d \in (0, c)$ with $\mu(\Delta_d) > 0$ (otherwise, $f(x) \geq c$ for μ-almost every $x \in E$, which, together with $\int_X f\, d\mu = c\mu(E)$, will imply $f = c\chi_E$). Hence, we can find $e \in (0,d]$ satisfying $\int_{\Delta_d} f\,d\mu = e\mu(\Delta_d)$. Suppose $g_{\Delta_d} \in L^1(\mu)_+^{\mathbf{1}}$ is as in the statement. Since $\operatorname{supp} g_{\Delta_d} \subseteq \Delta_d$, we know that

$$\begin{aligned}\|f - g_{\Delta_d}\|_{L^1(\mu)} &= \|f \cdot \chi_{E\setminus\Delta_d}\|_{L^1(\mu)} + \|g_{\Delta_d} - f\cdot\chi_{\Delta_d}\|_{L^1(\mu)}\\ &\geq \int_{E\setminus\Delta_d} f\,d\mu + \big(1 - e\mu(\Delta_d)\big) = 1 + c\mu(E) - 2e\mu(\Delta_d).\end{aligned}$$

This, together with the hypothesis, tells us that $2c\mu(\Delta_d) \leq 2e\mu(\Delta_d)$, and this contradicts with $e < c$.

(b) $\Rightarrow$). This implication is clear if we set $h_{\Delta,l} := \frac{1}{\mu(\Delta)}\chi_\Delta$.

$\Leftarrow$). Fix any $l \in \{1,...,n\}$ and set $f_l := \chi_{E_l}\cdot f$. Let $\Delta \in \Omega$ with $\Delta \subseteq E_l$ and $\mu(\Delta) > 0$. Consider $h_{\Delta,l} \in L^1(\mu)_+^{\mathbf{1}}$ to be the element as in the statement. The equality

$$\|f - h_{\Delta,l}\|_{L^1(\mu)} = \|f - f_l\|_{L^1(\mu)} + \|f_l - h_{\Delta,l}\|_{L^1(\mu)} = \big(1 - c_l\mu(E_l)\big) + \|f_l - h_{\Delta,l}\|_{L^1(\mu)}$$

implies $\|f_l - h_{\Delta,l}\|_{L^1(\mu)} = 1 + c_l\mu(E_l) - 2c_l\mu(\Delta)$. Thus, we conclude from part (a) that $f_l = c_l\chi_{E_l}$. Since $\sum_{k=1}^n c_k\mu(E_k) = 1$ and $\int_X f\,d\mu = 1$, we know that $f = \sum_{k=1}^n f_k$. □

Lemma 4. *Let (X,Ω,μ) be a semi-finite measure space. If $\Lambda : L^1(\mu)_+^{\mathbf{1}} \to L^1(\mu)_+^{\mathbf{1}}$ is a bijection satisfying*

$$\|\Lambda(f) - \Lambda(g)\| = \|f - g\| \quad \text{and} \quad \mu\big(\operatorname{supp} g \setminus \operatorname{supp}\Lambda(g)\big) = 0 \qquad (f,g \in L^1(\mu)_+^{\mathbf{1}}), \tag{2.1}$$

then Λ is the identity map.

Proof: Since the set of positive simple functions with norm one is dense in $L^1(\mu)_+^{\mathbf{1}}$, we only need to show that $\Lambda\big(\sum_{k=1}^n c_k\chi_{E_k}\big) = \sum_{k=1}^n c_k\chi_{E_k}$ for any positive scalars $c_1,\dots,c_n$, and disjoint subsets $E_1,...,E_n \in \Omega$ with $\mu(E_i) > 0$ $(i = 1,...n)$ satisfying $\sum_{k=1}^n c_k\mu(E_k) = 1$. Let us set $f := \sum_{k=1}^n c_k\chi_{E_k}$ and fix an arbitrary integer $l \in \{1,...,n\}$. Let us also denote

$$\Lambda(f)_l := \Lambda(f)\cdot\chi_{E_l} \quad \text{and} \quad g := \chi_{E_l}/\mu(E_l).$$

Then $\|f - g\|_{L^1(\mu)} = 2 - 2c_l\mu(E_l)$. By (2.1), we have $\mu\big(E_l \setminus \operatorname{supp}\Lambda(g)\big) = 0$, which implies

$$\|\Lambda(g) - \Lambda(f)\| = \|\Lambda(g) - \Lambda(f)_l\| + \|\Lambda(f) - \Lambda(f)_l\| \geq 2 - 2\|\Lambda(f)_l\|.$$

Therefore, the first equality of (2.1) implies $c_l\mu(E_l) \leq \|\Lambda(f)_l\|$. Furthermore, since

$$\sum\nolimits_{k=1}^n c_k\mu(E_k) = 1 = \|\Lambda(f)\| = \sum\nolimits_{k=1}^n \|\Lambda(f)_k\|,$$

we conclude that $\int_{E_l}\Lambda(f)\,d\mu = \|\Lambda(f)_l\| = c_l\mu(E_l)$ $(l = 1,...,n)$.

Now, suppose that l is a fixed integer in $\{1,\dots,n\}$ and $\Delta \in \Omega$ satisfying $\Delta \subseteq E_l$ as well as $\mu(\Delta) > 0$. If we set $h := \Lambda\big(\frac{1}{\mu(\Delta)}\chi_\Delta\big)\cdot\chi_\Delta$, then the two equalities in Relation (2.1) imply that $h = \Lambda\big(\frac{1}{\mu(\Delta)}\chi_\Delta\big)$ as elements in $L^1(H)$ and that

$$\|\Lambda(f) - h\| = \big\|\Lambda(f) - \Lambda\big(\chi_\Delta/\mu(\Delta)\big)\big\| = \|f - \chi_\Delta/\mu(\Delta)\| = 2 - 2c_l\mu(\Delta).$$

Therefore, Lemma 3(b) gives the required conclusion $\Lambda(f) = f$. □

For a W^*-algebra M, we denote by $\mathcal{P}(M)$ the set of projections. A map $\Psi : \mathcal{P}(M) \to \mathcal{P}(N)$ is called an *orthoisomorphism* if for any $p,q \in \mathcal{P}(M)$, one has $\Psi(p)\cdot\Psi(q) = 0$ if and only if $p\cdot q = 0$.

Proof of Proposition 2: Note that for normal states f and g of a W^*-algebra with support projections $\mathbf{s}_f$ and $\mathbf{s}_g$ respectively, one has

$$\|f - g\| = 2 \quad \text{if and only if} \quad \mathbf{s}_f\cdot\mathbf{s}_g = 0.$$

Thus, it follows from [16, Lemma 3.1(a)] that the metric preserving bijection Φ produces an orthoisomorphism $\check{\Phi} : \mathcal{P}(M) \to \mathcal{P}(N)$ such that

$$\check{\Phi}(\mathbf{s}_f) = \mathbf{s}_{\Phi(f)} \qquad (f \in (M_*)_+^{\mathbf{1}}).$$

By the corollary in [6, p.18], we know that $\check{\Phi}$ extends to a Jordan *-isomorphism $\Theta : M \to N$, which is automatically weak-*-continuous (see, e.g., [18, Corollary 4.1.23]). Therefore, both M and N are abelian, and the map Θ is a *-isomorphism.

Hence, $M = L^\infty(X, \Omega, \mu)$ for a semi-finite measure space (X, Ω, μ) (see, e.g, [18, Proposition 1.18.1]), and its predual M_* equals $L^1(X, \Omega, \mu)$. Consider $\Psi := \Theta_*^{-1}|_{L^1(X,\Omega,\mu)^{\mathbf{1}}_+}$. If we set $\Lambda := \Psi^{-1} \circ \Phi$, then Λ is a bijection from $L^1(X, \Omega, \mu)^{\mathbf{1}}_+$ onto itself satisfying the two relations in (2.1). Consequently, Lemma 4 tells us that $\Phi = \Psi$ as required. $\square$

Now, we will give the proof of the parts of Theorem 1 concerning the invariants $L^1(G)^{\mathbf{1}}_+$ and $M(G)^{\mathbf{1}}_+$. In fact, we have more precise statements for them as follows. In these statements, $*$ is the convolution product.

Theorem 5. *Let G and H be locally compact groups with Haar measures μ_G and μ_H that define the norms on $L^1(G)$ and $L^1(H)$, respectively.*

(a) If $\Phi : L^1(G)^{\mathbf{1}}_+ \to L^1(H)^{\mathbf{1}}_+$ is a bijection satisfying $\Phi(f * g) = \Phi(f) * \Phi(g)$ and $\|\Phi(f) - \Phi(g)\| = \|f - g\|$ $(f, g \in L^1(G)^{\mathbf{1}}_+)$, then there exist a homeomorphic group isomorphism $\phi : H \to G$ and a constant $c > 0$ such that $\Phi(f)(t) = cf(\phi(t))$ for every $f \in L^1(G)^{\mathbf{1}}_+$ and μ_H-almost every $t \in H$.

(b) If $\Phi : M(G)^{\mathbf{1}}_+ \to M(H)^{\mathbf{1}}_+$ is a bijection satisfying $\Phi(\alpha * \beta) = \Phi(\alpha) * \Phi(\beta)$ and $\|\Phi(\alpha) - \Phi(\beta)\| = \|\alpha - \beta\|$ $(\alpha, \beta \in M(G)^{\mathbf{1}}_+)$, then there exists a homeomorphic group isomorphism $\phi : H \to G$ such that $\Phi(\alpha)(E) = \alpha(\phi(E))$, for any $\alpha \in M(G)$ and compact subset $E \subseteq G$.

Proof. (a) Note that $L^1(G)^{\mathbf{1}}_+$ and $L^1(H)^{\mathbf{1}}_+$ are the normal state spaces of the abelian W^*-algebras $M = L^\infty(G)$ and $N = L^\infty(H)$, respectively. From Proposition 2 we know that Φ can be extended to a surjective (complex) linear isometry from $L^1(G)$ onto $L^1(H)$. Now, the multiplicative assumption on Φ tells us that the extension is a Banach algebra isomorphism. By [24, Theorem 1], one obtains a homeomorphic group isomorphism $\phi : H \to G$, a continuous character $\theta : H \to \mathbb{T}$ and a constant $c > 0$ satisfying

$$\Phi(f)(t) = c\theta(t)f\big(\phi(t)\big)$$

for every $f \in L^1(G)^{\mathbf{1}}_+$ and μ_H-almost every $t \in H$. As $\Phi(\chi_E) \in L^1(G)_+$ for arbitrary measurable subset $E \subseteq G$ with $\mu_G(E) = 1$, we know that $\theta(t) \geq 0$ (or equivalently, $\theta(t) = 1$) for μ_H-almost every $t \in H$. Thus, the continuity of θ tells us that $\theta(t) = 1$ for all $t \in H$.

(b) Note that $M(G)^{\mathbf{1}}_+$ and $M(H)^{\mathbf{1}}_+$ are the normal state spaces of the abelian W^*-algebras $C_0(G)^{**}$ and $C_0(H)^{**}$, respectively. Following the same line of argument as in part (a), but with [24, Theorem 1] being replaced by the paragraph following the Corollary in [10], we can find a homeomorphic group isomorphism $\phi : H \to G$ and a continuous character $\theta : H \to \mathbb{T}$ with

$$\Phi(\alpha)(E) = \int_{\phi(E)} \theta(t) d\alpha(t)$$

for each $\alpha \in M(G)$ and each compact subset $E \subseteq G$. Since $\int_{\phi(E)} \theta(t) d\alpha(t) \geq 0$ for every compact subset $E \subseteq G$ and any $\alpha \in M(G)_+$, we know that $\theta(t) \geq 0$ for μ_H-almost all $t \in H$. Consequently, $\theta(t) = 1$ for all $t \in H$. $\square$

In order to present the other invariants in Theorem 1, we need to recall the notion of "left introverted subspace" from [5] (see [13] and [17] for more information). A closed subspace F of the C^*-algebra $C_b(G)$ of bounded continuous functions on a locally compact group G is said to be *left introverted* if for any $s \in G$, $a \in F$ and $f \in F^*$, one has

- $\lambda_s(a) \in F$;
- the function $f \odot a : t \mapsto f(\lambda_t(a))$ belongs to F;

here, $\lambda_s(a)(t) := a(s^{-1}t)$ $(t \in G)$. In this case, F^* is a Banach algebra under the product $\odot$ defined by $(f \odot g)(a) := f(g \odot a)$ $(f, g \in F^*; a \in F)$; see [5] for details.

Suppose that A is a left introverted C^*-subalgebra of $C_b(G)$. It is not hard to check that $(A^*)_+^{\mathbf{1}}$ is closed under $\odot$. Hence, $(A^*)_+^{\mathbf{1}}$ is a metric semigroup with the product $\odot$.

Examples of left introverted C^*-subalgebras of $C_b(G)$ are the space $\mathrm{AP}(G)$ of almost periodic continuous functions, the space $\mathrm{WAP}(G)$ of weakly almost periodic continuous functions, and the space $\mathrm{LUC}(G)$ of bounded left uniformly continuous functions. It follows from [20, Theorem 7] that $\mathrm{LUC}(G)$ is the largest left introverted closed subspace of $C_b(G)$. Moreover, $\mathrm{WAP}(G)$ (respectively, $\mathrm{AP}(G)$) is the largest left introverted closed subspaces of $C_b(G)$ with the multiplication, $\odot$, on the dual space being separately (respectively, jointly) weak-*-continuous on the unit sphere (see Theorems 5.6 and 5.8 of [12]).

Theorem 6. *Suppose that A and B are left introverted C^*-subalgebras of $C_b(G)$ and $C_b(H)$ containing $C_0(G)$ and $C_0(H)$, respectively. If there is a bijection $\Phi : (A^*)_+^{\mathbf{1}} \to (B^*)_+^{\mathbf{1}}$ satisfying*

$$\Phi(f \odot g) = \Phi(f) \odot \Phi(g) \quad \text{and} \quad \|\Phi(f) - \Phi(g)\| = \|f - g\| \qquad (f, g \in (A^*)_+^{\mathbf{1}}),$$

then G and H are isomorphic as topological groups.

Proof. Note that the double dual spaces A^{**} and B^{**} are both abelian W^*-algebras. The argument is similar to that in the proof of Theorem 5(a), except that we need to use [13, Theorem 1] instead of [24, Theorem 1]. ☐

It is easy to see that the left introverted C^*-algebras $\mathrm{WAP}(G)$ and $\mathrm{LUC}(G)$ contain $C_0(G)$, and the remaining parts of Theorem 1 follow.

Unlike $\mathrm{WAP}(G)$ and $\mathrm{LUC}(G)$, the intersection of the C^*-subalgebra $\mathrm{AP}(G)$ with $C_0(G)$ is $\{0\}$ unless G is compact. Thus, the argument for Theorem 6 does not work for $\mathrm{AP}(G)$. In fact, we have the following result. Let us recall some notation. As in [9], the *almost periodic compactification* (also known as the *Bohr compactification*), G^{ap}, of a locally compact group G is the spectrum of the abelian C^*-algebra $\mathrm{AP}(G)$, i.e., the weak-*-compact set of non-zero multiplicative linear functionals on $\mathrm{AP}(G)$. It is well-known that G^{ap} is a compact topological group under the weak-*-topology on $\mathrm{AP}(G)^*$.

Corollary 7. *Let G and H be locally compact groups. Then $(\mathrm{AP}(G)^*)_+^{\mathbf{1}} \cong (\mathrm{AP}(H)^*)_+^{\mathbf{1}}$ as metric semigroups if and only if $G^{ap} \cong H^{ap}$ as topological groups.*

Proof. It is well-known that $\mathrm{AP}(G) \cong C_0(G^{\mathrm{ap}})$ as ordered Banach algebras (see, e.g., [3, §4]). By Theorem 5(b) (notice that $C_0(G^{\mathrm{ap}})^* = M(G^{\mathrm{ap}})$), if $(\mathrm{AP}(G)^*)_+^{\mathbf{1}} \cong (\mathrm{AP}(H)^*)_+^{\mathbf{1}}$, then $G^{\mathrm{ap}} \cong H^{\mathrm{ap}}$. The converse is obvious. ☐

Note that the canonical group homomorphism sending G into G^{ap} is not injective, unless $\mathrm{AP}(G)$ separates points of G; e.g., when G is either abelian or compact. In the most extreme situation, G^{ap} is just a singleton set and such a group G is called *minimally almost periodic* in [21, 22]. For any minimally almost periodic group G, the metric semigroup $(\mathrm{AP}(G)^*)_+^{\mathbf{1}}$ is the trivial one (i.e., contains only one element).

3. Further questions and investigations

The Fourier algebra $A(G)$ and the Fourier-Stieltjes algebra $B(G)$ can be regarded as dual objects of $L^1(G)$ and $M(G)$, respectively. In fact, in the framework of locally compact quantum groups, $A(G)$ (respectively, $B(G)$) equals $L^1(\hat{G})$ (respectively, $M(\hat{G})$), where $\hat{G}$ is the "dual quantum group of G" (which is not a locally compact group unless G is abelian). In [23], Walter showed that $A(G)$ and $B(G)$ are both complete invariants of G up to opposition. Some related results can be found in [1, 2, 7, 15, 17]. On the other hand, Walter's result was extended to the quantum case by Daw and Le Pham (see [4]).

It is natural to ask if $A(G)^{\mathbf{1}}_{+}$ and $B(G)^{\mathbf{1}}_{+}$ are also complete invariants of G up to opposition. Let us state this as a conjecture as follows.

Conjecture 8. *Let G and H be locally compact groups, and H^{op} be the opposite group of H. If there is a metric preserving semigroup isomorphism from $A(G)^{\mathbf{1}}_{+}$ (respectively, $B(G)^{\mathbf{1}}_{+}$) onto $A(H)^{\mathbf{1}}_{+}$ (respectively, $B(H)^{\mathbf{1}}_{+}$), then either $G = H$ or $G = H^{op}$.*

Recently, we have found a proof for the corresponding result of the above conjecture in the case of "type I" locally compact quantum groups (see [14]). This can be used to obtain a positive answer for the above conjecture when G is either abelian or compact (or even when G is a compact quantum group). We are currently working on the general case.

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