WEAK AND STRONG CONVERGENCE THEOREMS
FOR EXTENDED HYBRID MAPPINGS IN HILBERT SPACES

WATARU TAKAHASHI, NGAI-CHING WONG, AND JEN-CHIH YAO

Abstract. Let $C$ be a closed convex subset of a real Hilbert space $H$. A mapping $U : C \to H$ is called extended hybrid if there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that
\[
\alpha(1 + \gamma)\|UX - UY\|^2 + (1 - \alpha(1 + \gamma))\|x - Uy\|^2 \\
\leq (\beta + \alpha\gamma)\|UX - Y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\
- (\alpha - \beta)\gamma\|x - UX\|^2 - \gamma\|x - Y\|^2
\]
for all $x, y \in C$. In this paper, we first show that the class of extended hybrid mappings contains the class of strict pseudo-contractions defined by Browder and Petryshyn [6]. We also obtain some important properties for extended hybrid mappings and strict pseudo-contractions in a Hilbert space. Using these results, we prove weak convergence theorems of Baillon's type [3] and of Mann's type [19] for extended hybrid mappings in a Hilbert space. Finally, we get strong convergence theorems of Halpern's type [9] and of the hybrid methods [22] and [30] for these mappings.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. A mapping $T : C \to H$ is said to be nonexpansive if
\[
\|Tz - Ty\| \leq \|x - y\|
\]
for all $x, y \in C$. A mapping $T : C \to H$ is said to be a strict pseudo-contraction [6] if there exists a real number $k$ with $0 \leq k < 1$ such that
\[
\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2
\]
for all $x, y \in C$. We also call such a mapping $T$ a $k$-strict pseudo-contraction. A $k$-strict pseudo-contraction $T : C \to H$ is nonexpansive if $k = 0$. A mapping $T : C \to H$ is said to be nonspreading [17] and hybrid [28] if
\[
2\|Tz - Ty\|^2 \leq \|Tx - Ty\|^2 + \|Ty - Tz\|^2
\]
and
\[
3\|Tz - Ty\|^2 \leq \|x - y\|^2 + \|Tx - Ty\|^2 + \|Ty - Tz\|^2
\]

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for all \( x, y \in C \), respectively; see also [11], [12], [13] and [16]. We know from [28] that a nonexpansive mapping, a nonspreading mapping and a hybrid mapping are deduced from a firmly nonexpansive mapping. A mapping \( T : C \to H \) is said to be \textit{firmly nonexpansive} [5], [8] if 
\[
\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle
\]
for all \( x, y \in C \). A firmly nonexpansive mapping \( F \) can be deduced from an equilibrium problem in a Hilbert space; see, for instance, [4] and [7]. Recently, Kocourek, Takahashi and Yao [15] considered a broad class of nonlinear mappings in a Hilbert space which contains the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings: A mapping \( T : C \to H \) is called \textit{generalized hybrid} [15] if there are \( \alpha, \beta \in \mathbb{R} \) such that
\[
(1.4) \quad \alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta)\|x - y\|^2
\]
for all \( x, y \in C \). We call such a mapping an \((\alpha, \beta)\)-\textit{generalized hybrid} mapping. For example, an \((\alpha, \beta)\)-generalized hybrid mapping is nonexpansive for \( \alpha = 1 \) and \( \beta = 0 \), nonspreading for \( \alpha = 2 \) and \( \beta = 1 \), and hybrid for \( \alpha = \frac{1}{2} \) and \( \beta = \frac{1}{2} \).

Hojo, Takahashi and Yao [10] also introduced a class of nonlinear mappings in a Hilbert space which contains the class of generalized hybrid mappings: A mapping \( U : C \to H \) is called \textit{extended hybrid} if there are \( \alpha, \beta, \gamma \in \mathbb{R} \) such that
\[
(1.5) \quad \alpha(1 + \gamma)\|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma))\|x - Uy\|^2
\]
\[
\leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2
\]
\[
- (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2
\]
for all \( x, y \in C \).

In this paper, we first show that the class of extended hybrid mappings contains the class of strict pseudo-contractions in a Hilbert space. We also obtain some important properties for extended hybrid mappings and strict pseudo-contractions in a Hilbert space. Using these results, we prove weak convergence theorems of Baillon’s type [3] and of Mann’s type [19] for extended hybrid mappings in a Hilbert space. Finally, we get strong convergence theorems of Halpern’s type [9] and of the hybrid methods [22], [30] for these mappings.

2. Preliminaries

Let \( H \) be a (real) Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| \). We denote the strong convergence and the weak convergence of \( \{x_n\} \) to \( x \in H \) by \( x_n \to x \) and \( x_n \rightharpoonup x \), respectively. From [27], we know the following basic equality: For \( x, y \in H \) and \( \lambda \in \mathbb{R} \), we have
\[
(2.1) \quad \|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2.
\]
Furthermore, we know that for \( x, y, u, v \in H \),
\[
(2.2) \quad 2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2.
\]

Let \( C \) be a nonempty closed convex subset of \( H \) and let \( T \) be a mapping from \( C \) into \( H \). Then, we denote by \( F(T) \) the set of fixed points of \( T \). A mapping \( T : C \to H \) with \( F(T) \neq \emptyset \) is called \textit{quasi-nonexpansive} if \( \|x - Ty\| \leq \|x - y\| \) for
all $x \in F(T)$ and $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping $T$ is closed and convex; see Ito and Takahashi [14]. It is not so difficult to show this fact in a Hilbert space. In fact, to show that $F(T)$ is closed, let us take a sequence $\{z_n\} \subset F(T)$ such that $z_n \rightarrow z_0$. Since $C$ is closed and convex, $C$ is weakly closed and hence $z_0 \in C$. We also have

$$||z_0 - Tz_0|| \leq ||z_0 - z_n|| + ||z_n - Tz_0|| \leq 2||z_0 - z_n||$$

for $n \in \mathbb{N}$. Tending $n \rightarrow \infty$, we have that $z_0 \in F(T)$ and hence $F(T)$ is closed.

To show that $F(T)$ is convex, let us take $z_1, z_2 \in F(T)$ and $\lambda \in [0, 1]$, and put $z_0 = \lambda z_1 + (1-\lambda)z_2$. Then we have from (2.1) that

$$||z_0 - Tz_0||^2 = ||\lambda z_1 + (1-\lambda)z_2 - Tz_0||^2$$

$$= ||\lambda(z_1 - Tz_0) + (1-\lambda)(z_2 - Tz_0)||^2$$

$$= \lambda||z_1 - Tz_0||^2 + (1-\lambda)||z_2 - Tz_0||^2 - \lambda(1-\lambda)||z_1 - z_2||^2$$

$$\leq \lambda||z_1 - z_0||^2 + (1-\lambda)||z_2 - z_0||^2 - \lambda(1-\lambda)||z_1 - z_2||^2$$

$$= \lambda(1-\lambda)^2||z_1 - z_2||^2 + \lambda^2(1-\lambda)||z_1 - z_2||^2 - \lambda(1-\lambda)||z_1 - z_2||^2$$

$$= \lambda(1-\lambda)(1-\lambda - 1)||z_1 - z_2||^2 = 0$$

and hence $z_0 \in F(T)$. So, $F(T)$ is convex.

Let $C$ be a nonempty closed convex subset of $H$ and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $||x - z|| = \text{inf}_{y \in C} ||x - y||$. We denote such a correspondence by $z = P_Cx$. $P_C$ is called the metric projection of $H$ onto $C$. It is known that $P_C$ is nonexpansive and

$$\langle x - P_Cx, P_Cx - u \rangle \geq 0$$

for all $x \in H$ and $u \in C$. Furthermore, we know that

$$(2.3) \quad ||P_Cx - P_Cy||^2 \leq \langle x - y, P_Cx - P_Cy \rangle$$

for all $x, y \in H$; see [27] for more details. The following lemma was proved by Takahashi and Toyoda [31].

**Lemma 2.1.** Let $D$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $P$ be the metric projection of $H$ onto $D$ and let $\{x_n\}$ be a sequence in $H$. If $||x_{n+1} - u|| \leq ||x_n - u||$ for all $u \in D$ and $n \in \mathbb{N}$, then $\{Px_n\}$ converges strongly.

Let $C$ be a nonempty closed convex subset of $H$. Then, we know that a mapping $T : C \rightarrow H$ is called generalized hybrid [15] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$(2.4) \quad \alpha||Tx - Ty||^2 + (1-\alpha)||x - Ty||^2 \leq \beta||Tx - y||^2 + (1-\beta)||x - y||^2$$

for all $x, y \in C$. We can show that if $x = Tx$, then for any $y \in C$,

$$\alpha||x - Ty||^2 + (1-\alpha)||x - Ty||^2 \leq \beta||x - y||^2 + (1-\beta)||x - y||^2$$

and hence

$$(2.5) \quad ||x - Ty|| \leq ||x - y||.$$
This means that an \((\alpha, \beta)\)-generalized hybrid mapping with a fixed point is quasi-nonexpansive. A mapping \(S : C \to H\) is \textit{super hybrid} \cite{15, 33} if there are \(\alpha, \beta, \gamma \in \mathbb{R}\) such that

\[
\alpha \|Sx - Sy\|^2 + (1 - \alpha + \gamma)\|x - Sy\|^2 \\
\leq (\beta + (\beta - \alpha)\gamma)\|Sx - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 \\
+ (\alpha - \beta)\gamma\|x - Sx\|^2 + \gamma\|y - Sy\|^2
\]

for all \(x, y \in C\). We call such a mapping an \((\alpha, \beta, \gamma)\)-\textit{super hybrid} mapping. An \((\alpha, \beta, 0)\)-super hybrid mapping is \((\alpha, \beta)\)-generalized hybrid. So, the class of super hybrid mappings contains the class of generalized hybrid mappings. Kocourek, Takahashi and Yao \cite{15} also proved the following fixed point theorem for super hybrid mappings in a Hilbert space.

\textbf{Theorem 2.2.} Let \(C\) be a nonempty bounded closed convex subset of a Hilbert space \(H\) and let \(\alpha, \beta\) and \(\gamma\) be real numbers with \(\gamma \geq 0\). Let \(S : C \to C\) be an \((\alpha, \beta, \gamma)\)-super hybrid mapping. Then, \(S\) has a fixed point in \(C\). In particular, if \(S : C \to C\) is an \((\alpha, \beta)\)-generalized hybrid mapping, then \(S\) has a fixed point in \(C\).

We also know a fixed point theorem \cite{10} for generalized hybrid non-self mappings in a Hilbert space.

\textbf{Theorem 2.3.} Let \(C\) be a nonempty bounded closed convex subset of a Hilbert space \(H\) and let \(\alpha\) and \(\beta\) be real numbers. Let \(T\) be an \((\alpha, \beta)\)-generalized hybrid mapping of \(C\) into \(H\) with \(\alpha - \beta \geq 0\). Suppose that there exists \(m > 1\) such that for any \(x \in C\), \(Tx = x + ty - x\) for some \(y \in C\) and \(t\) with \(1 \leq t \leq m\). Then, \(T\) has a fixed point in \(C\).

To prove one of our main results, we need the following lemma \cite{2}:

\textbf{Lemma 2.4.} Let \(\{s_n\}\) be a sequence of nonnegative real numbers, let \(\{\alpha_n\}\) be a sequence of \([0, 1]\) with \(\sum_{n=1}^{\infty} \alpha_n = \infty\), let \(\{\beta_n\}\) be a sequence of nonnegative real numbers with \(\sum_{n=1}^{\infty} \beta_n < \infty\), and let \(\{\gamma_n\}\) be a sequence of real numbers with \(\limsup_{n \to \infty} \gamma_n \leq 0\). Suppose that

\[
s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n\gamma_n + \beta_n
\]

for all \(n = 1, 2, ..., \). Then \(\lim_{n \to \infty} s_n = 0\).

Let \(l^\infty\) be the Banach space of bounded sequences with supremum norm. Let \(\mu\) be an element of \((l^\infty)^*\) (the dual space of \(l^\infty\)). Then, we denote by \(\mu(f)\) the value of \(\mu\) at \(f = (x_1, x_2, x_3, ...) \in l^\infty\). Sometimes, we denote by \(\mu_n(x_n)\) the value \(\mu(f)\). A linear functional \(\mu\) on \(l^\infty\) is called a \textit{mean} if \(\mu(e) = \|\mu\| = 1\), where \(e = (1, 1, 1, ...)\). A mean \(\mu\) is called a \textit{Banach limit} on \(l^\infty\) if \(\mu_n(x_{n+1}) = \mu_n(x_n)\). We know that there exists a Banach limit on \(l^\infty\). If \(\mu\) is a Banach limit on \(l^\infty\), then for \(f = (x_1, x_2, x_3, ...) \in l^\infty\),

\[
\liminf_{n \to \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \to \infty} x_n.
\]

In particular, if \(f = (x_1, x_2, x_3, ...) \in l^\infty\) and \(x_n \to a \in \mathbb{R}\), then we have \(\mu(f) = \mu_n(x_n) = a\). For the proof of existence of a Banach limit and its other elementary properties, see \cite{25}.
Let \( \alpha, \beta, \gamma \in \mathbb{R} \) such that
\[
\alpha(1 + \gamma)\|Ux - Uy\|^2 + (1 - \alpha(1 + \gamma))\|x - Uy\|^2 \\
\leq (\beta + \alpha\gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\
- (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2
\]
for all \( x, y \in C \) and such a mapping \( U \) is called \((\alpha, \beta, \gamma)\)-extended hybrid. In [10], the authors derived a relation between the class of generalized hybrid mappings and the class of extended hybrid mappings in a Hilbert space.

**Theorem 3.1.** Let \( C \) be a nonempty closed convex subset of a Hilbert space \( H \) and let \( \alpha, \beta, \gamma \) be real numbers with \( \gamma \neq -1 \). Let \( T \) and \( U \) be mappings of \( C \) into \( H \) such that \( U = \frac{1}{1 + \gamma}T + \frac{2}{1 + \gamma}I \), where \( Ix = x \) for all \( x \in H \). Then, for \( 1 + \gamma > 0 \), \( T : C \to H \) is an \((\alpha, \beta, \gamma)\)-generalized hybrid mapping if and only if \( U : C \to H \) is an \((\alpha, \beta, \gamma)\)-extended hybrid mapping. In this case, \( F(T) = F(U) \).

In this section, we first prove a fixed point theorem for strict pseudo-contractions in a Hilbert space.

**Theorem 3.2.** Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( k \) be a real number with \( 0 \leq k < 1 \) and let \( U : C \to H \) be a \( k \)-strict pseudo-contraction. Then, \( U \) is a \((1, 0, -k)\)-extended hybrid mapping and \( F(U) \) is closed and convex. If, in addition, \( C \) is bounded and \( U \) is a mapping of \( C \) into itself, then \( F(U) \) is nonempty.

**Proof.** Let \( U : C \to H \) be a \( k \)-strict pseudo-contraction. Then, \( 0 \leq k < 1 \) and
\[
\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|(I - U)x - (I - U)y\|^2
\]
for all \( x, y \in C \). So, we have from (2.2) that for all \( x, y \in C \),
\[
\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\| (I - U)x - (I - U)y\|^2 \\
= \|x - y\|^2 + k \|x - y - (Ux - Uy)\|^2 \\
= \|x - y\|^2 + k (\|x - y\|^2 + \|Ux - Uy\|^2 - 2(x - y, Ux - Uy)) \\
= \|x - y\|^2 + k (\|x - y\|^2 + \|Ux - Uy\|^2 \\
- \|x - Uy\|^2 - \|y - Ux\|^2 + \|x - Ux\|^2 + \|y - Uy\|^2)
\]
and hence
\[
(3.3) \quad (1 - k)\|Ux - Uy\|^2 + k \|x - Uy\|^2 \leq -k \|Ux - y\|^2 \\
+ (1 + k) \|x - y\|^2 + k \|x - Ux\|^2 + k \|y - Uy\|^2.
\]
Putting \( \alpha = 1 \), \( \beta = 0 \) and \( \gamma = -k \) in (3.1), we get (3.3). Then, \( U \) is a \((1,0,-k)\)-extended hybrid mapping. Furthermore, putting \( T = (1 - k)U + kI \), where \( Ix = x \) for all \( x \in H \), we have that
\[
U = \frac{1}{1 - k} T + \frac{-k}{1 - k} I.
\]
Using \( 1 + \gamma = 1 - k > 0 \) and Theorem 3.1, we have that \( T \) is a \((1,0)\)-generalized hybrid mapping, i.e., a nonexpansive mapping. So, \( F(T) \) is closed and convex. From \( F(T) = F(U) \), \( F(U) \) is also closed and convex. Since \( C \) is a bounded closed convex set and \( T \) is a nonexpansive mapping of \( C \) into itself, \( F(T) \) is nonempty; see [27]. Hence \( F(U) \) is nonempty.

In general, we have the following fixed point theorem for extended hybrid mappings in a Hilbert space.

**Theorem 3.3.** Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( \alpha, \beta, \gamma \) be real numbers. Let \( U: C \to H \) be an \((\alpha, \beta, \gamma)\)-extended hybrid mapping with \( 1 + \gamma > 0 \). Then \( F(U) \) is closed and convex. If, in addition, \( C \) is bounded, \( 0 \leq -\gamma < 1 \) and \( U \) is a mapping of \( C \) into itself, then \( F(U) \neq \emptyset \).

**Proof.** Let \( U: C \to H \) be an \((\alpha, \beta, \gamma)\)-extended hybrid mapping with \( 1 + \gamma > 0 \). Putting \( T = (1 + \gamma)U - \gamma I \), we have
\[
U = \frac{1}{1 + \gamma} T + \frac{\gamma}{1 + \gamma} I.
\]
From Theorem 3.1, we have that \( T \) is an \((\alpha, \beta)\)-generalized hybrid mapping of \( C \) into \( H \). If \( F(U) \neq \emptyset \), then \( F(T) \neq \emptyset \) from \( F(U) = F(T) \). Then we have from (2.5) that \( T: C \to H \) is quasi-nonexpansive. So, we have that \( F(T) \) is closed and convex and hence \( F(U) \) is closed and convex. If \( F(U) = \emptyset \), it is obvious that \( F(U) \) is closed and convex. Let \( U: C \to C \) be an \((\alpha, \beta, \gamma)\)-extended hybrid mapping with \( 0 \leq -\gamma < 1 \). We note that if \( 0 \leq -\gamma < 1 \), then \( 1 + \gamma > 0 \). Since \( 0 \leq -\gamma < 1 \) and \( T = (1 + \gamma)U - \gamma I \), we have from Theorem 3.1 that \( T \) is an \((\alpha, \beta)\)-generalized hybrid mapping of \( C \) into itself. Using Theorem 2.2, we have \( F(T) \neq \emptyset \). So, \( F(U) \neq \emptyset \).

Using Theorem 3.3, we have the following fixed point theorem.
Theorem 3.4. Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $k$ be a real number with $0 < k < 1$. Let $U : C \to H$ be a mapping such that

\begin{equation}
2\|Ux - Uy\|^2 \leq \|x - y\|^2 + \|Ux - y\|^2 + k(\|x - (I - U)y\|^2 - 2\langle x - Uy, y - Uy \rangle)
\end{equation}

for all $x, y \in C$. Then, $F(U)$ is closed and convex. In addition, if $C$ is bounded and $U$ is a mapping $C$ into itself, then $F(U) \neq \emptyset$.

Proof. Using (2.2), we have that the inequality (3.4) is equivalent to

\begin{equation}
2(1 - k)\|Ux - Uy\|^2 \leq (1 - 2k)\|Ux - y\|^2 + 2k\|x - y\|^2 + k\|x - Uy\|^2.
\end{equation}

On the other hand, putting $\alpha = 2$, $\beta = 1$ and $\gamma = -k$ in (3.1), we get this inequality (3.5). So, $U$ is a $(1, 2, -k)$-extended hybrid mapping. Using $0 \leq k < 1$ and Theorem 3.3, we have the desired result. $\square$

For example, taking $k = \frac{1}{2}$ in (3.4), we obtain that

\begin{equation}
2\|Ux - Uy\|^2 \leq 2\|x - y\|^2 + \|x - Uy\|^2 + \|y - Uy\|^2
\end{equation}

for all $x, y \in C$. Using Theorem 3.4, we have that such a mapping $U$ has a fixed point in $C$ if $C$ is bounded, closed and convex. Furthermore, $F(U)$ is closed and convex.

We also have the following important result for extended hybrid mappings in a Hilbert space.

Theorem 3.5. Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\alpha, \beta, \gamma$ be real numbers and let $U : C \to H$ be an $(\alpha, \beta, \gamma)$-extended hybrid mapping with $1 + \gamma > 0$. Then, $I - U$ is demiclosed, i.e., $x_n \to z$ and $x_n - Ux_n \to 0$ imply $z \in F(U)$.

Proof. Since $U : C \to H$ is extended hybrid, there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

\begin{align*}
\alpha(1 + \gamma)\|Ux - Uy\|^2 &\leq (\beta + \alpha \gamma)\|Ux - y\|^2 + (1 - \alpha(1 + \gamma))\|x - y\|^2 \\
&\leq (\beta + \alpha \gamma)\|Ux - y\|^2 + (1 - (\beta + \alpha \gamma))\|x - y\|^2 \\
&\leq (\alpha - \beta)\gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2
\end{align*}

for all $x, y \in C$. Suppose $x_n \to z$ and $x_n - Ux_n \to 0$. Let us consider

\begin{align*}
\alpha(1 + \gamma)\|Ux_n - Uz\|^2 &\leq (\beta + \alpha \gamma)\|Ux_n - z\|^2 + (1 - \alpha(1 + \gamma))\|x_n - z\|^2 \\
&\leq (\beta + \alpha \gamma)\|Ux_n - z\|^2 + (1 - (\beta + \alpha \gamma))\|x_n - z\|^2 \\
&\leq (\alpha - \beta)\gamma\|x_n - Ux_n\|^2 - \gamma\|Uz\|^2.
\end{align*}

From this inequality, we have

\begin{align*}
\alpha(1 + \gamma)\|Ux_n - x_n + x_n - Uz\|^2 &\leq (\beta + \alpha \gamma)\|Ux_n - x_n + x_n - z\|^2 + (1 - (\beta + \alpha \gamma))\|x_n - z\|^2 \\
&\leq (\beta + \alpha \gamma)\|Ux_n - x_n + x_n - z\|^2 + (1 - (\beta + \alpha \gamma))\|x_n - z\|^2.
\end{align*}
We apply a Banach limit $\mu$ to both sides of the inequality. Then, we have
\[
\alpha(1 + \gamma)\mu_n\|Ux_n - x_n + z\|^2 + (1 - \alpha(1 + \gamma))\mu_n\|x_n - z\|^2 \\
\leq (\beta + \alpha\gamma)\mu_n\|Ux_n - x_n + z\|^2 + (1 - (\beta + \alpha\gamma))\mu_n\|x_n - z\|^2 \\
- (\alpha - \beta)\gamma\mu_n\|x_n - Ux_n\|^2 - \gamma\mu_n\|z - Uz\|^2.
\]
We know from the properties of $\mu$ that
\[
\mu_n\|Ux_n - x_n + z\|^2 \\
= \mu_n\|Ux_n - x_n\|^2 + \|x_n - z\|^2 + 2\mu_n\langle Ux_n - x_n, x_n - z \rangle \\
= \mu_n\|x_n - z\|^2
\]
and
\[
\mu_n\|Ux_n - x_n + z\|^2 = \mu_n\|x_n - z\|^2. \quad \text{So, we have}
\]
\[
\alpha(1 + \gamma)\mu_n\|x_n - Uz\|^2 + (1 - \alpha(1 + \gamma))\mu_n\|x_n - Uz\|^2 \\
\leq (\beta + \alpha\gamma)\mu_n\|x_n - z\|^2 + (1 - (\beta + \alpha\gamma))\mu_n\|x_n - z\|^2 \\
- \gamma\|z - Uz\|^2
\]
and hence
\[
\mu_n\|x_n - Uz\|^2 \leq \mu_n\|x_n - z\|^2 - \gamma\|z - Uz\|^2.
\]
From $\mu_n\|x_n - z\|^2 = \mu_n\|x_n - z + z - Uz\|^2 = \mu_n\|x_n - Uz\|^2 + \|z - Uz\|^2$, we also have
\[
\mu_n\|x_n - z\|^2 + \|z - Uz\|^2 \leq \mu_n\|x_n - z\|^2 - \gamma\|z - Uz\|^2.
\]
Hence, we obtain $(1 + \gamma)\|z - Uz\|^2 \leq 0$. Since $1 + \gamma > 0$, we have $\|z - Uz\|^2 \leq 0$. Then, $Uz = z$. This implies that $I - U$ is demiclosed. \hfill $\Box$

Using Theorems 3.2 and 3.6, we have the following result obtained by Marino and Xu [20]; see also [1].

**Corollary 3.6.** Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $k$ be a real number with $0 \leq k < 1$ and $U : C \to H$ be a $k$-strict pseudo-contraction. Then, $I - U$ is demiclosed, i.e., $x_n \to z$ and $x_n - Ux_n \to 0$ imply $z \in F(U)$.

**Proof.** We know from Theorem 3.2 that a $k$-strict pseudo-contraction $U : C \to H$ is $(1,0,\cdot,k)$-entended hybrid. Furthermore, $0 \leq k < 1$ implies $1 + \gamma = 1 - k > 0$. So, we have the desired result from Theorem 3.6. \hfill $\Box$

4. **Nonlinear ergodic theorem**

In this section, using the technique developed in [24], [29] and [32], we prove a nonlinear ergodic theorem of Baillon’s type [3] for extended hybrid mappings in a Hilbert space. For proving it, we need the following two lemmas proved by Takahashi and Yao and Kocourek [33] and Hojo, Takahashi and Yao [10].
Lemma 4.1. Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to H$ be a generalized hybrid mapping. Suppose that there exists \( \{x_n\} \subseteq C \) such that $x_n \to z$ and $x_n - Tx_n \to 0$. Then, $z \in F(T)$.

Lemma 4.2. Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T$ be a generalized hybrid mapping from $C$ into itself. Suppose that \( \{T^n x\} \) is bounded for some $x \in C$. Define $S_n x = \frac{1}{n} \sum_{k=1}^{n} T^k x$. Then, \( \lim_{n \to \infty} \|S_n x - TS_n x\| = 0 \). In particular, if $C$ is bounded, then

\[
\lim_{n \to \infty} \sup_{x \in C} \|S_n x - TS_n x\| = 0.
\]

Theorem 4.3. Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\alpha$, $\beta$ and $\gamma$ be real numbers and let $U : C \to C$ be an $(\alpha, \beta, \gamma)$-extended hybrid mapping such that $0 \leq -\gamma < 1$ and $F(U) \neq \emptyset$. Let $P$ be the metric projection of $H$ onto $F(U)$. Then, for any $x \in C$,

\[
S_n x = \frac{1}{n} \sum_{k=1}^{n} ((1 + \gamma)U - \gamma I)^k x
\]

converges weakly to $z \in F(U)$, where $z = \lim_{n \to \infty} PT^n x$ and $T = (1 + \gamma)U - \gamma I$.

Proof. Put $T = (1 + \gamma)U - \gamma I$. Since $0 \leq -\gamma < 1$, we have from Theorem 3.1 that $T$ is an $(\alpha, \beta, \gamma)$-generalized hybrid mapping of $C$ into itself, i.e.,

\[
(4.1) \quad \alpha \|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta \|Tx - y\|^2 + (1 - \beta)\|x - y\|^2
\]

for all $x, y \in C$. Since $T$ is a generalized hybrid mapping and $F(T) = F(U) \neq \emptyset$, $T$ is quasi-nonexpansive. So, $F(T)$ is closed and convex. Let $x \in C$ and $u \in F(T)$. Then, we have $\|T^{n+1}x - u\| \leq \|T^n x - u\|$. Putting $D = F(T)$ in Lemma 2.1, we have that $\lim_{n \to \infty} PT^n x$ converges strongly. Put $z = \lim_{n \to \infty} PT^n x$. Let us show $S_n x \to z$. Since \( \{T^n x\} \) is bounded, so is \( \{S_n x\} \). Let \( \{S_n x\} \) be a subsequence of \( \{S_n x\} \) such that \( S_n x \to v \). By Lemma 4.2, we know $\lim_{n \to \infty} \|S_n x - TS_n x\| = 0$. Using Lemma 4.1, we have $v = TV$. To show $S_n x \to z$, it is sufficient to prove $z = v$. From $v \in F(T)$, we have

\[
\langle v - z, T^k x - PT^k x \rangle = \langle v - PT^k x, T^k x - PT^k x \rangle + \langle PT^k x - z, T^k x - PT^k x \rangle
\]

\[
\leq \langle PT^k x - z, T^k x - PT^k x \rangle
\]

\[
\leq \|PT^k x - z\| \|T^k x - PT^k x\|
\]

\[
\leq \|PT^k x - z\| L
\]

for all $k \in \mathbb{N}$, where $L = \sup\{\|T^k x - PT^k x\| : k \in \mathbb{N}\}$. Summing these inequalities from $k = 1$ to $n_i$ and dividing by $n_i$, we have

\[
\left\langle v - z, S_{n_i} x - \frac{1}{n_i} \sum_{k=1}^{n_i} PT^k x \right\rangle \leq \frac{1}{n_i} \sum_{k=1}^{n_i} \|PT^k x - z\| L.
\]

Since $S_{n_i} x \to v$ as $i \to \infty$ and $PT^n x \to z$ as $n \to \infty$, we have $\langle v - z, v - z \rangle \leq 0$. This implies $z = v$. Therefore, \( \{S_n x\} \) converges weakly to $z \in F(T) = F(U)$, where $z = \lim_{n \to \infty} PT^n x$. So, we get the desired result. \( \square \)

Using Theorem 4.3, we obtain the following corollary.
Corollary 4.4. Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $k$ be a real number with $0 \leq k < 1$ and $U : C \to C$ be a $k$-strict pseudo-contraction and $F(U) \neq \emptyset$. Let $P$ be the metric projection of $H$ onto $F(U)$. Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{m=1}^{n} ((1-k)U + kI)^m x$$

converges weakly to $z \in F(U)$, where $z = \lim_{n \to \infty} PT^n x$ and $T = (1-k)U + kI$.

Proof. We know from Theorem 3.2 that a $k$-strict pseudo-contraction $U : C \to C$ is $(1,0,k)$-extended hybrid. Furthermore, $0 \leq k < 1$ and $-\gamma = k$ imply $0 \leq -\gamma < 1$. So, we have the desired result from Theorem 4.3. □

5. Weak convergence theorem of Mann’s type

In this section, we prove a weak convergence theorem of Mann’s type [19] for extended hybrid mappings in a Hilbert space. Before proving the theorem, we need the following lemma proved by Takahashi, Yao and Kocourek [33].

Lemma 5.1. Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $P_C$ be the metric projection of $H$ onto $C$. Let $\alpha$, $\beta$ and $\gamma$ be real numbers with $\gamma \neq -1$ and let $S : C \to H$ be an $(\alpha, \beta, \gamma)$-super hybrid mapping with $F(S) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = P_C \{\alpha_n x_n + (1 - \alpha_n)(\frac{1}{1+\gamma} Sx_n + \frac{\gamma}{1+\gamma} x_n)\}, \quad n \in \mathbb{N}.$$ 

Then, the sequence $\{x_n\}$ converges weakly to an element $v$ of $F(S)$, where $v = \lim_{n \to \infty} P_{F(S)} x_n$ and $P_{F(S)}$ is the metric projection of $H$ onto $F(S)$.

Theorem 5.2. Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $P_C$ be the metric projection of $H$ onto $C$. Let $\alpha$, $\beta$ and $\gamma$ be real numbers. Let $U : C \to H$ be an $(\alpha, \beta, \gamma)$-extended hybrid mapping such that $1 + \gamma > 0$ and $F(U) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \to \infty} \alpha_n (1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = P_C \{\alpha_n x_n + (1 - \alpha_n)((1+\gamma)U x_n - \gamma x_n)\}, \quad n \in \mathbb{N}.$$ 

Then, the sequence $\{x_n\}$ converges weakly to an element $v$ of $F(U)$, where $v = \lim_{n \to \infty} P_{F(U)} x_n$ and $P_{F(U)}$ is the metric projection of $H$ onto $F(U)$.

Proof. Put $T = (1 + \gamma)U - \gamma I$. Then, we have from $1 + \gamma > 0$ and Theorem 3.1 that $T : C \to H$ is an $(\alpha, \beta)$-generalized hybrid mapping and $F(U) = F(T) \neq \emptyset$. Furthermore, we have that

$$x_{n+1} = P_C \{\alpha_n x_n + (1 - \alpha_n)T x_n\}, \quad n \in \mathbb{N}.$$ 

Using Lemma 5.1 with $\gamma = 0$, we have that $\{x_n\}$ converges weakly to an element $v$ of $F(T)$, where $v = \lim_{n \to \infty} P_{F(T)} x_n$ and $P_{F(T)}$ is the metric projection of $H$ onto $F(T) = F(U)$. □

As direct consequences of Theorem 5.2, we obtain the following results.
Corollary 5.3. Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $P_C$ be the metric projection of $H$ onto $C$. Let $\gamma$ be a real number with $1 + \gamma > 0$ and let $U : C \to H$ be an $(2, 1, \gamma)$-extended hybrid mapping, i.e.,
\[
2(1 + \gamma)\|Ux - Uy\|^2 - (1 + 2\gamma)\|x - Uy\|^2 \\
\leq (1 + 2\gamma)\|Ux - y\|^2 - 2\gamma\|x - y\|^2 \\
- \gamma\|x - Ux\|^2 - \gamma\|y - Uy\|^2
\]
for all $x, y \in C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and
\[
x_{n+1} = P_C\{\alpha_n x_n + (1 - \alpha_n)((1 + \gamma)Ux_n - \gamma x_n)\}, \quad n \in \mathbb{N}.
\]
If $F(U) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element $v$ of $F(U)$, where $v = \lim_{n \to \infty} P_{F(U)}x_n$ and $P_{F(U)}$ is the metric projection of $H$ onto $F(U)$.

Corollary 5.4. Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $P_C$ be the metric projection of $H$ onto $C$. Let $\gamma$ be a real number with $1 + \gamma > 0$ and let $U : C \to H$ be an $(\frac{3}{2}, 1, \gamma)$-extended hybrid mapping, i.e.,
\[
3(1 + \gamma)\|Ux - Uy\|^2 - (1 + 3\gamma)\|x - Uy\|^2 \\
\leq (1 + 3\gamma)\|Ux - y\|^2 + (1 - 3\gamma)\|x - y\|^2 \\
- 2\gamma\|x - Ux\|^2 - 2\gamma\|y - Uy\|^2
\]
for all $x, y \in C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n \to \infty} \alpha_n(1 - \alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and
\[
x_{n+1} = P_C(\alpha_n x_n + (1 - \alpha_n)((1 + \gamma)Ux_n - \gamma x_n)), \quad n \in \mathbb{N}.
\]
If $F(U) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element $v$ of $F(U)$, where $v = \lim_{n \to \infty} P_{F(U)}x_n$ and $P_{F(U)}$ is the metric projection of $H$ onto $F(U)$.

Taking $\gamma = -\frac{1}{2}$ in Corollaries 5.3 and 5.4, we obtain two mappings such that
\[
2\|Ux - Uy\|^2 \leq 2\|x - y\|^2 + \|x - Ux\|^2 + \|y - Uy\|^2
\]
and
\[
3\|Ux - Uy\|^2 + \|x - Uy\|^2 + \|y - Ux\|^2 \\
\leq 5\|x - y\|^2 + 2\|x - Ux\|^2 + 2\|y - Uy\|^2
\]
for all $x, y \in C$, respectively. We can apply Corollaries 5.3 and 5.4 for such mappings and then obtain weak convergence theorems in a Hilbert space. Next, we prove a weak convergence theorem of Mann’s type for a class of non-self mappings containing the class of nonexpansive mappings in a Hilbert space. For proving it, we state the following lemma proved by Takahashi, Yao and Kocourek [33].

Lemma 5.5. Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\gamma$ be a real number with $\gamma \neq -1$ and let $S : C \to H$ be a mapping such that
\[
\|Sx - Sy\|^2 + 2\gamma\langle x - y, Sx - Sy \rangle \leq (1 + 2\gamma)\|x - y\|^2
\]
for all \( x, y \in C \). Let \( \{\alpha_n\} \) be a sequence of real numbers such that \( 0 \leq \alpha_n \leq 1 \) and \( \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty \). Suppose \( \{x_n\} \) is a sequence generated by \( x_1 = x \in C \) and
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) P_C\left(\frac{1}{1 + \gamma} S x_n + \frac{\gamma}{1 + \gamma} x_n\right), \quad n = 1, 2, \ldots.
\]
If \( F(S) \neq \emptyset \), then the sequence \( \{x_n\} \) converges weakly to an element \( v \) of \( F(S) \), where \( v = \lim_{n \to \infty} P_{F(S)} x_n \) and \( P_{F(S)} \) is the metric projection of \( H \) onto \( F(S) \).

**Theorem 5.6.** Let \( H \) be a Hilbert space, let \( C \) be a nonempty closed convex subset of \( H \) and let \( P_C \) be the metric projection of \( H \) onto \( C \). Let \( \alpha, \beta \) and \( \gamma \) be real numbers. Let \( \gamma \) be a real number with \( \gamma > 0 \) and let \( U : C \to H \) be a mapping with \( F(U) \neq \emptyset \) such that
\[
\|Ux - Uy\|^2 \leq \|x - y\|^2 - \gamma (\|I - U\| x - (I - U)y)^2
\]
for all \( x, y \in C \). Let \( \{\alpha_n\} \) be a sequence of real numbers such that \( 0 \leq \alpha_n \leq 1 \) and \( \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty \). Suppose \( \{x_n\} \) is a sequence generated by \( x_1 = x \in C \) and
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n) P_C((1 + \gamma)Ux_n - \gamma x_n), \quad n \in \mathbb{N}.
\]
Then the sequence \( \{x_n\} \) converges weakly to an element \( v \) of \( F(U) \), where \( v = \lim_{n \to \infty} P_{F(U)} x_n \) and \( P_{F(U)} \) is the metric projection of \( H \) onto \( F(U) \).

**Proof.** We have that for any \( x, y \in C \),
\[
\|Ux - Uy\|^2 \leq \|x - y\|^2 - \gamma (\|I - U\| x - (I - U)y)^2
\]
\[\iff \|Ux - Uy\|^2 \leq \|x - y\|^2 - \gamma (\|x - y\|^2 + \|Ux - Uy\|^2 - \|Ux - x\|^2 + \|y - Uy\|^2)
\]
\[\iff (1 + \gamma)\|Ux - Uy\|^2 - \gamma \|x - Uy\|^2
\]
\[\leq \gamma \|Ux - Uy\|^2 + (1 - \gamma)\|x - y\|^2 - \gamma \|Ux - x\|^2 - \gamma \|y - Uy\|^2.
\]
Thus, \( U \) is a \((1, 0, \gamma)\)-extended hybrid mapping with \( 1 + \gamma > 0 \). Put \( T = (1+\gamma)U - \gamma I \). Then, we have from Theorem 3.1 that \( T : C \to H \) is an \((1, 0)\)-generalized hybrid mapping, i.e., a nonexpansive mapping and \( F(U) = F(T) \neq \emptyset \). Using Lemma 5.5 with \( \gamma = 0 \) or Reich’s theorem [23], we have that \( \{x_n\} \) converges weakly to an element \( v \) of \( F(T) \), where \( v = \lim_{n \to \infty} F_{F(T)} x_n \) and \( F_{F(T)} \) is the metric projection of \( H \) onto \( F(T) = F(U) \).

As a direct consequence of Theorem 5.6, we have the following corollary.

**Corollary 5.7.** Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( k \) be a real number with \( 0 \leq k < 1 \) and \( U : C \to C \) be a \( k \)-strict pseudo-contraction and \( F(U) \neq \emptyset \). Let \( P \) be the metric projection of \( H \) onto \( F(U) \). Let \( \{\alpha_n\} \) be a sequence of real numbers such that \( 0 \leq \alpha_n \leq 1 \) and \( \sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty \). Suppose \( \{x_n\} \) is a sequence generated by \( x_1 = x \in C \) and
\[
x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\{(1 - k)Ux_n + kx_n\}, \quad n \in \mathbb{N}.
\]
Then the sequence \( \{x_n\} \) converges weakly to an element \( v \) of \( F(U) \), where \( v = \lim_{n \to \infty} F_{F(U)} x_n \) and \( F_{F(U)} \) is the metric projection of \( H \) onto \( F(U) \).
Proof. We know from Theorem 3.2 that a $k$-strict pseudo-contraction $U : C \to H$ is $(1,0,\gamma)$-extended hybrid. Furthermore, $0 \leq k < 1$ and $-\gamma = k$ imply $1 + \gamma > 0$. So, we have the desired result from Theorem 5.6.

Using Corollary 5.7, we prove a weak convergence theorem of Mann’s type for strict pseudo-contractions which was obtained by Marino and Xu [20]; see also [1].

**Theorem 5.8.** Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $k$ be a real number with $0 \leq k < 1$ and $U : C \to C$ be a $k$-strict pseudo-contraction such that $F(U) \neq \emptyset$. Let $\{\beta_n\}$ be a sequence of real numbers such that $k < \beta_n < 1$ and $\sum_{n=1}^{\infty}(\beta_n - k)(1 - \beta_n) = \infty$. Suppose that $\{x_n\}$ is a sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)Ux_n, \quad n \in \mathbb{N}.$$  

Then the sequence $\{x_n\}$ converges weakly to an element $v$ of $F(U)$.

Proof. We have that for any $n \in \mathbb{N}$,

$$y_n = \beta_n x_n + (1 - \beta_n)Ux_n = \frac{\beta_n - k}{1 - k}x_n + (1 - \frac{\beta_n - k}{1 - k})\{(1 - k)Ux_n + kx_n\}.$$  

Putting $\alpha_n = \frac{\beta_n - k}{1 - k}$, we have from $1 > \beta_n > k$ that $1 - k > \beta_n - k > 0$ and hence $1 > \frac{\beta_n - k}{1 - k} = \alpha_n > 0$. Furthermore, we have that

$$\sum_{n=1}^{\infty}(\beta_n - k)(1 - \beta_n) = \infty$$ 

$$\iff \sum_{n=1}^{\infty}(1 - k)\alpha_n(1 - k)(1 - \alpha_n) = \infty$$ 

$$\iff (1 - k)^2 \sum_{n=1}^{\infty}\alpha_n(1 - \alpha_n) = \infty$$ 

$$\iff \sum_{n=1}^{\infty}\alpha_n(1 - \alpha_n) = \infty.$$  

From Corollary 5.7, we have the desired result.

6. **Strong convergence theorems**

In this section, we first prove a strong convergence theorem of Halpern’s type [9] for extended hybrid mappings in a Hilbert space.

**Theorem 6.1.** Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\gamma$ be a real number with $1 + \gamma > 0$ and let $U : C \to H$ be a mapping such that

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 - \gamma\|(I - U)x - (I - U)y\|^2.$$
for all \(x, y \in C\). Let \(\{\alpha_n\} \subset [0, 1]\) be a sequence of real numbers such that \(\alpha_n \to 0\), \(\sum_{n=1}^{\infty} \alpha_n = \infty\) and \(\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty\). Suppose \(\{x_n\}\) is a sequence generated by \(x_1 = x \in C\), \(u \in C\) and

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n)P_{C'}\{(1 + \gamma)Ux_n - \gamma x_n\}, \quad n \in \mathbb{N}.
\]

If \(F(U) \neq \emptyset\), then the sequence \(\{x_n\}\) converges strongly to an element \(v\) of \(F(U)\), where \(v = P_{F(U)}u\) and \(P_{F(U)}\) is the metric projection of \(H\) onto \(F(U)\).

**Proof.** As in the proof of Theorem 5.6, we have that \(U\) is a \((1, 0, \gamma)\)-extended hybrid mapping of \(C\) into \(H\). Put \(T = (1 + \gamma)U - \gamma I\). Then, we have from Theorem 3.1 that \(T\) is a \((1, 0, \gamma)\)-generalized hybrid mapping of \(C\) into \(H\), i.e., \(T\) is a nonexpansive mapping of \(C\) into \(H\). Furthermore, we have \(F(U) = F(T)\). From Wittmann’s theorem [35], we obtain \(x_n \to P_{F(P_C T)}u\); see also Takahashi [26]. Let us show \(F(P_C T) = F(T) = F(U)\). We know \(F(T) = F(U)\). It is obvious that \(F(T) \subset F(P_C T)\). We show \(F(P_C T) \subset F(T)\). If \(P_C T v = v\), we have from the property of \(P_C\) that for \(u \in F(T)\),

\[
2\|v - u\|^2 = 2\|P_C T v - u\|^2 \leq 2\langle T v - u, P_C T v - u \rangle = \|T v - u\|^2 + \|P_C T v - u\|^2 - \|T v - P_C T v\|^2
\]

and hence

\[
2\|v - u\|^2 \leq \|v - u\|^2 + \|v - u\|^2 - \|T v - v\|^2.
\]

Then, we have \(0 \leq -\|T v - v\|^2\) and hence \(T v = v\). This completes the proof. \(\square\)

As a direct consequence of Theorem 6.1, we have the following corollary.

**Corollary 6.2.** Let \(H\) be a Hilbert space and let \(C\) be a nonempty closed convex subset of \(H\). Let \(k\) be a real number with \(0 \leq k < 1\) and \(U : C \to C\) be a \(k\)-strict pseudo-contraction with \(F(U) \neq \emptyset\). Let \(P\) be the metric projection of \(H\) onto \(F(U)\). Let \(\{\alpha_n\} \subset [0, 1]\) be a sequence of real numbers such that \(\alpha_n \to 0\), \(\sum_{n=1}^{\infty} \alpha_n = \infty\) and \(\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty\). Suppose \(\{x_n\}\) is a sequence generated by \(x_1 = x \in C, u \in C\) and

\[
x_{n+1} = \alpha_n u + (1 - \alpha_n)\{(1 - k)Ux_n + kx_n\}, \quad n \in \mathbb{N}.
\]

Then the sequence \(\{x_n\}\) converges strongly to an element \(v\) of \(F(U)\), where \(v = P_{F(U)}u\) and \(P_{F(U)}\) is the metric projection of \(H\) onto \(F(U)\).

Next, using an idea of mean convergence and the method of the proof in [18], we prove a strong convergence theorem of Halpern’s type for extended hybrid mappings in a Hilbert space.

**Theorem 6.3.** Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\) and let \(\alpha, \beta, k\) be real numbers. Let \(U : C \to C\) be a \((\alpha, \beta, -k)\)-extended hybrid mapping such that \(0 \leq k < 1\) and \(F(U) \neq \emptyset\) and let \(P\) be the metric projection of \(H\) onto \(F(U)\). Suppose \(\{x_n\}\) is a sequence generated by \(x_1 = x \in C, u \in C\) and

\[
\begin{align*}
x_{n+1} &= \alpha_n u + (1 - \alpha_n)z_n, \\
z_n &= \frac{1}{n} \sum_{m=1}^{n} ((1 - k)U + kI)^m x_n
\end{align*}
\]
for all \( n = 1, 2, ..., \) where \( 0 \leq \alpha_n \leq 1, \) \( \alpha_n \rightarrow 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty. \) Then \( \{x_n\} \) converges strongly to \( Pu. \)

**Proof.** For an \((\alpha, \beta, -k)\)-extended hybrid mapping \( U : C \rightarrow C, \) define
\[
T = (1 - k)U + kI.
\]

Then, we have from Theorem 3.1 that \( T : C \rightarrow C \) is an \((\alpha, \beta)\)-generalized hybrid mapping such that \( F(T) = F(U). \) Since \( F(T) = F(U) \) is nonempty, we take \( q \in F(T). \) Put \( r = \|u - q\|. \) We define
\[
D = \{ y \in H : \|y - q\| \leq r \} \cap C.
\]

Then \( D \) is a nonempty bounded closed convex subset of \( C. \) Furthermore, \( D \) is \( T\)-invariant and contains \( u. \) Thus we may assume that \( C \) is bounded without loss of generality. Since \( T \) is quasi-nonexpansive, we have that for all \( q \in F(T) \) and \( n = 1, 2, 3, ..., \)

\[
\|z_n - q\| = \left\| \frac{1}{n} \sum_{m=1}^{n} T^m x_n - q \right\| \leq \frac{1}{n} \sum_{m=1}^{n} \|T^m x_n - q\|
\]

(6.1)

\[
\leq \frac{1}{n} \sum_{m=1}^{n} \|x_n - q\| = \|x_n - q\|.
\]

Let us show \( \limsup_{n \rightarrow \infty} \langle u - Pu, z_n - Pu \rangle \leq 0. \) Since \( \{z_n\} \) is bounded, there exists a subsequence \( \{z_{n_i}\} \) of \( \{z_n\} \) with \( z_{n_i} \rightarrow v. \) We may assume without loss of generality

\[
\limsup_{n \rightarrow \infty} \langle u - Pu, z_n - Pu \rangle = \lim_{i \rightarrow \infty} \langle u - Pu, z_{n_i} - Pu \rangle.
\]

By Lemma 4.2, we have \( \lim_{n \rightarrow \infty} \|z_n - Tz_n\| = 0. \) Using Lemma 4.1, we have \( v \in F(T). \) Since \( P \) is the metric projection of \( H \) onto \( F(T), \) we have
\[
\lim_{i \rightarrow \infty} \langle u - Pu, z_{n_i} - Pu \rangle = \langle u - Pu, v - Pu \rangle \leq 0.
\]

This implies
\[
\limsup_{n \rightarrow \infty} \langle u - Pu, z_n - Pu \rangle \leq 0.
\]

Since \( x_{n+1} - Pu = (1 - \alpha_n)(z_n - Pu) + \alpha_n(u - Pu), \) from (6.1) we have
\[
\|x_{n+1} - Pu\|^2 = \|(1 - \alpha_n)(z_n - Pu) + \alpha_n(u - Pu)\|^2
\]

\[
\leq (1 - \alpha_n)^2 \|z_n - Pu\|^2 + 2\alpha_n \langle u - Pu, x_{n+1} - Pu \rangle
\]

\[
\leq (1 - \alpha_n) \|x_n - Pu\|^2 + 2\alpha_n \langle u - Pu, x_{n+1} - Pu \rangle.
\]

Putting \( s_n = \|x_n - Pu\|^2, \) \( \beta_n = 0 \) and \( \gamma_n = 2\langle u - Pu, x_{n+1} - Pu \rangle \) in Lemma 2.4, from \( \sum_{n=1}^{\infty} \alpha_n = \infty \) and (6.2) we have
\[
\lim_{n \rightarrow \infty} \|x_n - Pu\| = 0.
\]

This completes the proof. \( \square \)
7. Strong convergence theorems by hybrid methods

In this section, using the hybrid method by Nakajo and Takahashi [22], we first prove a strong convergence theorem for extended hybrid self-mappings in a Hilbert space. The method of the proof is due to Nakajo and Takahashi [22] and Marino and Xu [20].

**Theorem 7.1.** Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $\alpha$, $\beta$, and $k$ be real numbers and let $U : C \to H$ be an $(\alpha, \beta, -k)$-extended hybrid mapping such that $k < 1$ and $F(U) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and

$$
\begin{align*}
&y_n = \alpha_n x_n + (1 - \alpha_n)(1 - k)U x_n + kx_n, \\
&C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - k)^2\alpha_n(1 - \alpha_n)\|x_n - Ux_n\|^2\}, \\
&Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n}x, \quad \forall n \in \mathbb{N},
\end{align*}
$$

where $P_{C_n \cap Q_n}$ is the metric projection of $H$ onto $C_n \cap Q_n$ and $\{\alpha_n\} \subset (-\infty, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = R_{F(U)}x$, where $P_{F(U)}$ is the metric projection of $H$ onto $F(U)$.

**Proof.** Put $T = (1 - k)U + k1$. We have $U = \frac{1}{1-k}T + \frac{-k}{1-k}J$. So, we have from Theorem 3.1 that $T$ is an $(\alpha, \beta)$-generalized hybrid mapping of $C$ onto $H$ and $F(U) = F(T)$. Since $F(T)$ is closed and convex, $F(U)$ is closed and convex. So, there exists the metric projection of $H$ onto $F(U)$. Furthermore, we have

$$
y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n
$$

for all $n \in \mathbb{N}$. For any $z \in H$, the inequality

$$
\|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - k)^2\alpha_n(1 - \alpha_n)\|x_n - Ux_n\|^2
$$

is equivalent to

$$
2\langle x_n - y_n, z \rangle \leq \|x_n\|^2 - \|y_n\|^2 - (1 - k)^2\alpha_n(1 - \alpha_n)\|x_n - Ux_n\|^2.
$$

So, we have that $C_n$, $Q_n$ and $C_n \cap Q_n$ are closed and convex for all $n \in \mathbb{N}$. We next show that $C_n \cap Q_n$ is nonempty. Let $z \in F(T) = F(U)$. Since $T$ is quasi-nonexpansive, we have that

$$
\|y_n - z\|^2 = \|\alpha_n x_n + (1 - \alpha_n)Tx_n - z\|^2
$$

and

$$
= \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|Tx_n - z\|^2 - \alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2
$$

$$
\leq \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|x_n - x_n\|^2
$$

$$
= \|x_n - z\|^2 - (1 - k)^2\alpha_n(1 - \alpha_n)\|Ux_n - x_n\|^2.
$$

So, we have $z \in C_n$ and hence $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show by induction that $F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. From $F(T) \subset Q_1$, it follows that $F(T) \subset C_1 \cap Q_1$. Suppose that $F(T) \subset C_k \cap Q_k$ for some $k \in \mathbb{N}$. From $x_{k+1} = P_{C_k \cap Q_k}x$, we have

$$
\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0, \quad \forall z \in C_k \cap Q_k.
$$

Since $F(T) \subset C_k \cap Q_k$, we also have

$$
\langle x_{k+1} - z, x - x_{k+1} \rangle \geq 0, \quad \forall z \in F(T).
$$
This implies $F(T) \subset Q_{k+1}$. So, we have $F(T) \subset C_{k+1} \cap Q_{k+1}$. By induction, we have $F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This means that $\{x_n\}$ is well-defined. Since $x_n \in C$ and $\langle x_n - x_n, x - x_n \rangle = 0$, we have $x_n \in Q_n$. Furthermore, from the definition of $Q_n$, we have $x_n = P_{Q_n} x$. Using $x_n = P_{Q_n} x$ and $x_{n+1} = P_{C_n \cap Q_n} x \subset Q_n$, we have from (2.2) that

\begin{align*}
0 & \leq 2\langle x - x_n, x_n - x_{n+1} \rangle \\
& = \|x - x_{n+1}\|^2 - \|x - x_n\|^2 - \|x_n - x_{n+1}\|^2 \\
& \leq \|x - x_{n+1}\|^2 - \|x - x_n\|^2.
\end{align*}

So, we get that

\begin{equation}
\|x - x_n\|^2 \leq \|x - x_{n+1}\|^2.
\end{equation}

Furthermore, since $x_n = P_{Q_n} x$ and $z \in F(T) \subset Q_n$, we have

\begin{equation}
\|x - x_n\|^2 \leq \|x - z\|^2.
\end{equation}

So, we have that $\lim_{n \to \infty} \|x - x_n\|^2$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{Tx_n\}$ is also bounded. From (7.1), we also have

\begin{equation}
\|x_n - x_{n+1}\|^2 \leq \|x - x_{n+1}\|^2 - \|x - x_n\|^2
\end{equation}

and hence

\begin{equation}
\|x_n - x_{n+1}\| \to 0.
\end{equation}

From $x_{n+1} \in C_n$, we have that

\begin{equation}
\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 - \alpha_n(1 - \alpha_n)\|x_n - Tx_n\|^2.
\end{equation}

On the other hand, we know

\begin{equation}
\|y_n - x_{n+1}\|^2 = \|\alpha_n x_n + (1 - \alpha_n)Tx_n - x_{n+1}\|^2 \\
= \alpha_n\|x_n - x_{n+1}\|^2 + (1 - \alpha_n)\|Tx_n - x_{n+1}\|^2 \\
- \alpha_n(1 - \alpha_n)\|x_n - Tx_n\|^2.
\end{equation}

From (7.5) and (7.6), we have

\((1 - \alpha_n)\|Tx_n - x_{n+1}\|^2 \leq (1 - \alpha_n)\|x_n - x_{n+1}\|^2.\)

Since $1 - \alpha_n > 0$, we have $\|Tx_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2$ and hence

\[\|Tx_n - x_{n+1}\| \to 0.\]

From

\[\|Tx_n - x_n\|^2 = \|Tx_n - x_{n+1}\|^2 + 2\langle Tx_n - x_{n+1}, x_{n+1} - x_n \rangle + \|x_{n+1} - x_n\|^2,\]

we also have

\begin{equation}
\|Tx_n - x_n\| \to 0.
\end{equation}

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to z^*$. From (7.7) and Lemma 4.1, we have $z^* \in F(T)$. Put $z_0 = P_{F(T)} x$. Since $z_0 = P_{F(T)} x \subset C_n \cap Q_n$ and $x_{n+1} = P_{C_n \cap Q_n} x$, we have that

\begin{equation}
\|x - x_{n+1}\|^2 \leq \|x - z_0\|^2.
\end{equation}
Since \( \| \cdot \|^2 \) is weakly lower semicontinuous, from \( x_n \to z^* \) we have that
\[
\| x - z^* \|^2 = \| x \|^2 - 2\langle x, z^* \rangle + \| z^* \|^2 \\
\leq \liminf_{i \to \infty} (\| x \|^2 - 2\langle x, x_{n_i} \rangle + \| x_{n_i} \|^2)
\]
\[
= \liminf_{i \to \infty} \| x - x_{n_i} \|^2 \\
\leq \| x - z_0 \|^2.
\]
From the definition of \( z_0 \), we have \( z^* = z_0 \). So, we obtain \( x_n \to z_0 \). We finally show that \( x_n \to z_0 \). Since
\[
\| z_0 - x_n \|^2 = \| z_0 - x \|^2 + \| x - x_n \|^2 + 2\langle z_0 - x, x_n \rangle,
\]
we have
\[
\limsup_{n \to \infty} \| z_0 - x_n \|^2 = \limsup_{n \to \infty} (\| z_0 - x \|^2 + \| x - x_n \|^2 + 2\langle z_0 - x, x_n \rangle)
\]
\[
\leq \limsup_{n \to \infty} (\| z_0 - x \|^2 + \| x - z_0 \|^2 + 2\langle z_0 - x, x_n \rangle)
\]
\[
= \| z_0 - x \|^2 + \| x - z_0 \|^2 + 2\langle z_0 - x, x - z_0 \rangle = 0.
\]
So, we obtain \( \lim_{n \to \infty} \| z_0 - x_n \| = 0 \). Hence, \( \{ x_n \} \) converges strongly to \( z_0 \). This completes the proof. \( \square \)

Using Theorem 7.1, we can prove the following theorem obtained by Marino and Xu [20].

**Theorem 7.2.** Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( k \) be a real number with \( 0 \leq k < 1 \) and let \( U : C \to C \) be a \( k \)-strict pseudo contraction such that \( F(U) \neq \emptyset \). Let \( \{ x_n \} \subset C \) be a sequence generated by \( x_1 = x \in C \) and

\[
\begin{align*}
y_n &= \beta_n x_n + (1 - \beta_n) U x_n, \\
C_n &= \{ z \in C : \| y_n - z \|^2 \leq \| x_n - z \|^2 - (\beta_n - k)(1 - \beta_n) \| x_n - U x_n \|^2 \}, \\
Q_n &= \{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \}, \\
x_{n+1} &= P_{C_n \cap Q_n} x, \quad \forall n \in \mathbb{N},
\end{align*}
\]
where \( P_{C_n \cap Q_n} \) is the metric projection of \( H \) onto \( C_n \cap Q_n \) and \( \{ \beta_n \} \subset (-\infty, 1) \). Then, \( \{ x_n \} \) converges strongly to \( z_0 = R_{F(U)} x \), where \( P_{F(U)} \) is the metric projection of \( H \) onto \( F(U) \).

**Proof.** We first know that a \((1,0,k)\)-extended hybrid mapping with \( 0 \leq k < 1 \) is a \( k \)-strict pseudo contraction. We also have that for any \( n \in \mathbb{N} \),
\[
y_n = \beta_n x_n + (1 - \beta_n) U x_n
\]
\[
= \frac{\beta_n - k}{1 - k} x_n + (1 - \frac{\beta_n - k}{1 - k}) \{ (1 - k) U x_n + k x_n \}.
\]
Putting \( \alpha_n = \frac{\beta_n - k}{1 - k} \), we have from \( 1 > \beta_n \) that \( 1 - k > \beta_n - k \) and hence \( 1 > \frac{\beta_n - k}{1 - k} = \alpha_n \). Furthermore, we have that for any \( n \in \mathbb{N} \) and \( z \in C \),
\[
\| y_n - z \|^2 \leq \| x_n - z \|^2 - (\beta_n - k)(1 - \beta_n) \| x_n - U x_n \|^2
\]
Hence, we have
\[ \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - k)\alpha_n(1 - k)(1 - \alpha_n)\|x_n - Ux_n\|^2 \]
\[ \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - k)^2\alpha_n(1 - \alpha_n)\|x_n - Ux_n\|^2. \]

From Theorem 7.1, we have the desired result. \( \square \)

Next, we prove a strong convergence theorem by the shrinking projection method [30] for extended hybrid non-self mappings in a Hilbert space.

**Theorem 7.3.** Let \( H \) be a Hilbert space and let \( C \) be a nonempty closed convex subset of \( H \). Let \( \alpha, \beta \) and \( k \) be real numbers and let \( U : C \to H \) be an \((\alpha, \beta, -k)\)-extended hybrid mapping such that \( k < 1 \) and \( F(U) \neq \emptyset \). Let \( C_1 = C \) and let \( \{x_n\} \subset C \) be a sequence generated by \( x_1 = x \in C \) and

\[
\begin{align*}
  y_n &= \alpha_n x_n + (1 - \alpha_n)(1 - k)Ux_n + kx_n, \\
  C_{n+1} &= \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - k)^2\alpha_n(1 - \alpha_n)\|Ux_n - x_n\|^2\}, \\
  x_{n+1} &= \Pi_{C_{n+1}}x, \quad \forall n \in \mathbb{N},
\end{align*}
\]

where \( \Pi_{C_{n+1}} \) is the metric projection of \( H \) onto \( C_{n+1} \), and \( \{\alpha_n\} \subset (-\infty, 1) \). Then, \( \{x_n\} \) converges strongly to \( z_0 = P_{F(U)}x \), where \( P_{F(U)} \) is the metric projection of \( H \) onto \( F(U) \).

**Proof.** Put \( T = (1 - k)U + kI \). Then, we have from Theorem 3.1 that \( T \) is an \((\alpha, \beta)\)-generalized hybrid mapping of \( C \) into \( H \) and \( F(U) = F(T) \). Since \( F(T) \) is closed and convex, so is \( F(U) \). Then, there exists the metric projection of \( H \) onto \( F(U) \). Furthermore, we have
\[ y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n \]
for all \( n \in \mathbb{N} \). We show that \( C_n \) are closed and convex, and \( F(T) \subset C_n \) for all \( n \in \mathbb{N} \). It is obvious from the assumption that \( C_1 = C \) is closed and convex, and \( F(T) \subset C_1 \). Suppose that \( C_k \) is closed and convex, and \( F(T) \subset C_k \) for some \( k \in \mathbb{N} \). As in the proof of Theorem 7.1, we know that for \( z \in C_k \), the inequality
\[ \|y_n - z\|^2 \leq \|x_n - z\|^2 - (1 - k)^2\alpha_n(1 - \alpha_n)\|x_n - Ux_n\|^2 \]
is equivalent to
\[ 2\langle x_n - y_n, z \rangle \leq \|x_n\|^2 - \|y_n\|^2 - (1 - k)^2\alpha_n(1 - \alpha_n)\|x_n - Ux_n\|^2. \]
Since \( C_k \) is closed and convex, so is \( C_{k+1} \). Take \( z \in F(T) \subset C_k \). Then we have from (2.2) that
\[ \|y_n - z\|^2 = \|\alpha_n x_n + (1 - \alpha_n)Tx_n - z\|^2 \]
\[ = \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|Tx_n - z\|^2 - \alpha_n(1 - \alpha_n)\|Tx_n - x_n\|^2 \]
\[ \leq \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 - (1 - k)^2\alpha_n(1 - \alpha_n)\|Ux_n - x_n\|^2. \]
Hence, we have \( z \in C_{k+1} \) and hence \( F(T) \subset C_{k+1} \). By induction, we have that \( C_n \) are closed and convex, and \( F(T) \subset C_n \) for all \( n \in \mathbb{N} \). Since \( C_n \) is closed and convex, there exists the metric projection \( \Pi_{C_n} \) of \( H \) onto \( C_n \). Thus, \( \{x_n\} \) is well-defined. Since \( \{C_n\} \) is a nonincreasing sequence of nonempty closed convex subsets of \( H \) with respect to inclusion, it follows that
\[ \emptyset \neq F(T) \subset \text{M- lim}_{n \to \infty} C_n = \bigcap_{n=1}^{\infty} C_n. \]
Put $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then, by Theorem 2.5 we have that \( \{P_{C_n}x\} \) converges strongly to \( x_0 = P_{C_0}x \), i.e.,

\[
x_n = P_{C_n}x \to x_0.
\]

To complete the proof, it is sufficient to show that \( x_0 = P_{F(T)}x \). Since \( x_n = P_{C_n}x \)
and \( x_{n+1} = P_{C_{n+1}}x \in C_{n+1} \subset C_n \), we have from (2.2) that

\[
0 \leq 2\langle x - x_n, x_n - x_{n+1} \rangle
= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 - \|x_n - x_{n+1}\|^2
\leq \|x - x_{n+1}\|^2 - \|x - x_n\|^2.
\]

Thus, we get that

\[
\|x - x_n\|^2 \leq \|x - x_{n+1}\|^2.
\]

Furthermore, since \( x_n = P_{C_n}x \) and \( z \in F(T) \subset C_n \), we have

\[
\|x - x_n\|^2 \leq \|x - z\|^2,
\]

from which it follows that \( \lim_{n \to \infty} \|x - x_n\|^2 \) exists. This implies that \( \{x_n\} \) is bounded. Hence, \( \{T x_n\} \) are also bounded. From (7.10), we have

\[
\|x_n - x_{n+1}\|^2 \leq \|x - x_{n+1}\|^2 - \|x - x_n\|^2.
\]

So, we have that

\[
\|x_n - x_{n+1}\| \to 0.
\]

From \( x_{n+1} \in C_{n+1} \), we also have that

\[
\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 - (1 - \alpha_n)^2 \|x_n - Ux_n\|^2
= \|x_n - x_{n+1}\|^2 - \alpha_n(1 - \alpha_n)\|x_n - Tx_n\|^2.
\]

On the other hand, we have from (2.2) that

\[
\|y_n - x_{n+1}\|^2 = \|\alpha_n x_n + (1 - \alpha_n)Tx_n - x_{n+1}\|^2
= \alpha_n\|x_n - x_{n+1}\|^2 + (1 - \alpha_n)\|Tx_n - x_{n+1}\|^2
- \alpha_n(1 - \alpha_n)\|x_n - Tx_n\|^2.
\]

From (7.14) and (7.15), we have

\[
(1 - \alpha_n)\|Tx_n - x_{n+1}\|^2 \leq (1 - \alpha_n)\|x_n - x_{n+1}\|^2.
\]

Since \( 1 - \alpha_n > 0 \), we have \( \|Tx_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 \) and hence

\[
\|Tx_n - x_{n+1}\| \to 0.
\]

Since

\[
\|Tx_n - x_n\|^2 = \|Tx_n - x_{n+1}\|^2 + 2\langle Tx_n - x_{n+1}, x_{n+1} - x_n \rangle + \|x_{n+1} - x_n\|^2,
\]

we also have

\[
\|Tx_n - x_n\| \to 0.
\]
From $x_n = P_{C_n}x \to x_0$, we have $x_n \to x_0$. Using (7.16) and Lemma 4.1 we have $x_0 \in F(T)$. Put $z_0 = P_{F(T)}x$. Since $z_0 = P_{F(T)}x \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}}x$, we have that

$$
\|x - x_{n+1}\|^2 \leq \|x - z_0\|^2.
$$

(7.17)

So, we have from $x_n = P_{C_n}x \to x_0$ that

$$
\|x - x_0\|^2 = \lim_{n \to \infty} \|x - x_n\|^2 \leq \|x - z_0\|^2.
$$

From the definition of $z_0$, we get $z_0 = x_0$. Hence, $\{x_n\}$ converges strongly to $z_0$. This completes the proof.

Using Theorem 7.3 and the metod of proof in Theorem 7.2, we have the following strong convergence theorem for strict pseud-contractions in a Hilbert space.

**Theorem 7.4.** Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $k$ be a real number with $0 \leq k < 1$ and let $U : C \to H$ be a $k$-strict pseudo-contraction such that $F(U) \neq \emptyset$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and

$$
\begin{align*}
\{y_n\} &= \beta_n x_n + (1 - \beta_n)U x_n, \\
C_{n+1} &= \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - (\beta_n - k)(1 - \beta_n)\|U x_n - x_n\|^2\}, \\
x_{n+1} &= P_{C_{n+1}}x, \quad \forall n \in \mathbb{N},
\end{align*}
$$

where $P_{C_{n+1}}$ is the metric projection of $H$ onto $C_{n+1}$, and $\{\beta_n\} \subset (-\infty, 1)$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(U)}x$, where $P_{F(U)}$ is the metric projection of $H$ onto $F(U)$.

**References**


WATANABE TAKAHASHI
Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Tokyo 152-8552, Japan
E-mail address: wataru@is.titech.ac.jp

NGAI-CHING WONG
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan
E-mail address: wong@math.nsysu.edu.tw

JEN-CHIH YAO
Center for General Education, Kaohsiung Medical University, Kaohsiung 80702, Taiwan
E-mail address: yaojc@kmu.edu.tw