TRANSITION PROBABILITIES OF NORMAL STATES DETERMINE THE JORDAN STRUCTURE OF A QUANTUM SYSTEM

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Abstract. Let $\Phi : \mathfrak{S}(M_1) \to \mathfrak{S}(M_2)$ be a bijection (not assumed affine nor continuous) between the sets of normal states of two quantum systems, modelled on the self-adjoint parts of von Neumann algebras M_1 and M_2, respectively. This paper concerns with the situation when Φ preserves (or partially preserves) one of the following three notions of "transition probability" on the normal state spaces: the transition probability P_U introduced by Uhlmann, the transition probability P_R introduced by Raggio, and an "asymmetric transition probability" P_0 (as introduced in this article).

It is shown that the two systems are isomorphic, i.e. M_1 and M_2 are Jordan *-isomorphic, if Φ preserves all pairs with zero Uhlmann (respectively, Raggio or asymmetric) transition probability, in the sense that for any normal states μ and ν, we have

$$P\big(\Phi(\mu), \Phi(\nu)\big) = 0 \quad \text{if and only if} \quad P(\mu, \nu) = 0,$$

where P stands for P_U (respectively, P_R or P_0). Furthermore, as an extension of Wigner's theorem, it is shown that there is a Jordan *-isomorphism $\Theta : M_2 \to M_1$ satisfying

$$\Phi = \Theta^*|_{\mathfrak{S}(M_1)}$$

if and only if Φ preserves the "asymmetric transition probability". This is also equivalent to Φ preserving the Raggio transition probability. Consequently, if Φ preserves the Raggio transition probability, it will preserve the Uhlmann transition probability as well. As another application, the sets of normal states equipped with either the usual metric, the Bures metric or "the metric induced by the self-dual cone" are complete Jordan *-invariants for the underlying von Neumann algebras.

1. Introduction

Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two (complex) Hilbert spaces and $T : \mathfrak{H}_1 \to \mathfrak{H}_2$ be a bijective map (not assumed linear nor continuous). Wigner's theorem states that if T preserves the transition probability, in the sense that

$$|\langle T(\xi), T(\eta)\rangle|^2 = |\langle \xi, \eta\rangle|^2 \qquad (\xi, \eta \in \mathfrak{H}_1),$$

then there exist a unitary or an anti-unitary $S : \mathfrak{H}_1 \to \mathfrak{H}_2$ and a unimodular complex-valued function f on $\mathfrak{H}_2$ such that $T(\xi) = f(\xi)S(\xi)$ ($\xi \in \mathfrak{H}_1$). Uhlhorn's theorem, as a generalization of Wigner's theorem, states that if $\dim \mathfrak{H}_1 \geq 3$ and T preserves pairs with

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$$\langle T(\xi), T(\eta)\rangle = 0 \quad \text{if and only if} \quad \langle \xi, \eta\rangle = 0 \qquad (\xi, \eta \in \mathfrak{H}_1),$$

then there exist a unitary or an anti-unitary S and a function $g : \mathfrak{H}_2 \to \mathbb{C} \setminus \{0\}$ such that $T(\xi) = g(\xi)S(\xi)$ $(\xi \in \mathfrak{H}_1)$.

Let A be a (complex) C^*-algebra and $\mu, \nu \in A^*$ be pure states of A. The *transition probability* between μ and ν is defined to be the quantity

$$P(\mu, \nu) := \mu(\mathbf{s}_\nu),$$

where $\mathbf{s}_\nu$ is the support projection of ν in A^{**}. It is well-known that $P(\mu, \nu) = P(\nu, \mu)$, i.e., $\mu(\mathbf{s}_\nu) = \nu(\mathbf{s}_\mu)$, for pure states μ and ν (see e.g. [4]). Suppose that $\pi : A \to \mathcal{L}(\mathfrak{H})$ is a *-representation of A and $\xi \in \mathfrak{H}$. As usual, we denote the vector state of ξ by

$$\omega_\xi(x) := \langle \pi(x)\xi, \xi\rangle \qquad (x \in A). \tag{1.1}$$

In the case when $A = \mathcal{L}(\mathfrak{H})$ and $\pi : \mathcal{L}(\mathfrak{H}) \to \mathcal{L}(\mathfrak{H})$ is the default representation, the functionals ω_ξ and ω_η are pure normal states of $\mathcal{L}(\mathfrak{H})$ (where $\xi, \eta \in \mathfrak{H}$) and we have

$$P(\omega_\xi, \omega_\eta) = |\langle \xi, \eta\rangle|^2.$$

In this setting, Wigner's (respectively, Uhlhorn's) theorem can be interpreted as structural results concerning bijections between the pure normal state spaces of $\mathcal{L}(\mathfrak{H}_1)$ and $\mathcal{L}(\mathfrak{H}_2)$ that preserve (respectively, partially preserve) the transition probability. Several proofs of Wigner's theorem were given in the literature (see e.g. [11] or [24, Theorem 1]). Moreover, Wigner's theorem and Uhlhorn's theorem have been extended to the setting of indefinite inner product spaces by Molnár (see [18, Theorem 1] and [19, Corollary 1]). Through our study, we will also give another proof for Wigner's theorem (see Corollary 3.3) that only requires the relation between projections and closed faces (as given in [4, Theorem 3.35]).

On the other hand, Shultz provided a throughout study of transition probability preserving bijections between pure state spaces of general C^*-algebras. Under some extra conditions, such maps are induced by the dual maps of *-isomorphisms or Jordan *-isomorphisms of the underlying C^*-algebras (see e.g., [3, 4, 24] for details). Related considerations of maps between pure state spaces of C^*-algebras preserving transition probability or other properties can also be found in, e.g., [5, 16, 26].

However, the pure state setting of transition probability is inappropriate to be adapted to the case of general von Neumann algebras. Unlike $\mathcal{L}(\mathfrak{H})$, a general von Neumann algebra may not have any pure normal state at all. Therefore, people are looking for suitable notions of transition probability on the space $\mathfrak{S}(M)$ of all normal states on a von Neumann algebra M (see e.g. [2, 6, 22, 25, 29]). Here, by a normal state on M, we mean a norm one positive normal linear functional on M, and it is different from the notion of "physical states" as introduced in [9].

Let $\mathbf{R}(M)$ denote the collection of all (unitary equivalence classes of) faithful unital *-representations of a von Neumann algebra M. For any $\mu, \nu \in \mathfrak{S}(M)$ and $(\mathfrak{H}, \pi) \in \mathbf{R}(M)$, we set $\mathfrak{H}(\mu) := \{\xi \in \mathfrak{H} : \omega_\xi = \mu\}$ (could be empty). The quantity

$$P_U(\mu,\nu) := \sup\{|\langle \xi, \eta\rangle|^2 : \xi \in \mathfrak{H}(\mu), \eta \in \mathfrak{H}(\nu), (\mathfrak{H},\pi) \in \mathbf{R}(M)\}$$

is well-defined and is called the *Uhlmann transition probability* of μ and ν ([28]). The Uhlmann transition probability is related to the so-called *Bures distance* d_B through the formula

$$\mathrm{d}_B(\mu,\nu) := \sqrt{2 - 2\sqrt{P_U(\mu,\nu)}}. \tag{1.2}$$

This metric d_B is in general different from the usual distance d_1 on $\mathfrak{S}(M)$ as given by

$$\mathrm{d}_1(\mu,\nu) := \|\mu - \nu\|.$$

In [21], Raggio defined another transition probability. Suppose that $(M, \mathfrak{H}, \mathfrak{P}, J)$ is the standard form for M as introduced by Haagerup in [12] (see Section 2 below for a brief exploration). By [12, Lemma 2.10], for any $\mu \in \mathfrak{S}(M)$, there is a unique $\xi_\mu \in \mathfrak{P}$ satisfying

$$\mu = \omega_{\xi_\mu}. \tag{1.3}$$

If $\mu, \nu \in \mathfrak{S}(M)$, the positive real number

$$P_R(\mu,\nu) := \langle \xi_\mu, \xi_\nu\rangle$$

is called the *Raggio transition probability* of μ and ν. As in the Uhlmann case, the Raggio transition probability induces a metric d_2 on $\mathfrak{S}(M)$ through the relation

$$\mathrm{d}_2(\mu,\nu) := \sqrt{2 - 2\sqrt{P_R(\mu,\nu)}} \qquad (\mu,\nu \in \mathfrak{S}(M)). \tag{1.4}$$

This metric coincides with the one induced from $\mathfrak{H}$, namely,

$$\mathrm{d}_2(\mu,\nu) = \|\xi_\mu - \xi_\nu\| \qquad (\mu,\nu \in \mathfrak{S}(M)). \tag{1.5}$$

In [21, Corollary 1], the following relation between the Raggio and the Uhlmann transition probabilities was presented:

$$P_U(\mu,\nu) \leq P_R(\mu,\nu) \leq P_U(\mu,\nu)^{1/2} \qquad (\mu,\nu \in \mathfrak{S}(M)). \tag{1.6}$$

In addition, there is a more naïve extension of the "transition probability":

$$P_0(\mu,\nu) := \mu(\mathbf{s}_\nu) \qquad (\mu,\nu \in \mathfrak{S}(M)). \tag{1.7}$$

Strictly speaking, P_0 is not a transition probability, because unlike the two extensions above, P_0 is asymmetric, and $P_0(\mu,\nu) = 1$ is equivalent to $\mathbf{s}_\mu \leq \mathbf{s}_\nu$ instead of $\mu = \nu$ (c.f. [22, p.325]). Nevertheless, abusing the language, we still call P_0 the "*asymmetric transition probability*". It seems to be conceptual clear and technically easier to work with it.

Notice that two normal states $\mu, \nu \in \mathfrak{S}(M)$ are orthogonal, i.e., having orthogonal support projections, exactly when they have zero transition probability in any (and equivalently, all) of the above three settings (see (3.10) in Section 3).

The main concern of this article is on those bijections from the normal state space of one von Neumann algebra to that of another preserving either one of the three transition probabilities above (but not assumed to be affine nor continuous). We obtain two analogues of Wigner's theorem for bijections between normal state spaces of general quantum systems (which are modelled on self-adjoint elements of von Neumann algebras). Furthermore, several weak analogues of Uhlhorn's theorem for normal state spaces of general quantum systems were also obtained.

More precisely, we verified that the normal state space equipped with either the Uhlmann transition probability, the Raggio transition probability or the "asymmetric transition probability", completely identifies the underlying quantum system (see Theorems 3.2 and 3.4). It is shown that a bijection between normal state spaces preserving either the "asymmetric transition probability" (as defined in (1.7)) or the Raggio transition probability was shown to be induced by a Jordan *-isomorphism (see Theorems 3.2(b) and 3.8). Consequently, bijections between normal state spaces preserving the Raggio transition probability will preserve the Uhlmann transition probability (see Corollary 3.9). The result concerning the "asymmetric transition probability" can be regarded as an extension of the original Wigner's theorem because of Corollary 3.3.

This study highlighted the importance of the Raggio and the Uhlmann transition probability in quantum mechanics and it also established a strong relation between these two notions of transition probability. On the other hand, the notion of "asymmetric transition probability" that defined in (1.7) seems to be conceptually clearer and easier to implement in physics, although it is not strictly speaking a transition probability. Furthermore, Theorem 3.2(b) implies that the datum of measurements of observables associated with support projections of states at all other states is sufficient to determine the quantum system completely.

In developing our main results, we also obtained that several metric spaces associated with the sets of normal states of von Neumann algebras (without any algebraic structure) are complete Jordan *-invariants for the underlying algebras (see Corollary 3.11).

2. Notations and Preliminaries

Throughout this article, M, M_1 and M_2 are (complex) von Neumann algebras. We denote by $\mathfrak{S}(M)$ and $\mathcal{P}(M)$ the normal state space of M and the set of all projections in M, respectively.

Suppose that $\mathfrak{H}$ is a (complex) Hilbert space with M being a (unital) von Neumann subalgebra of $\mathcal{L}(\mathfrak{H})$. Let J be a conjugate linear isometric involution on $\mathfrak{H}$ and $\mathfrak{P} \subseteq \mathfrak{H}$ be a cone which is *self-dual*, in the sense that

$$\mathfrak{P} = \{\eta \in \mathfrak{H} : \langle \eta, \xi \rangle \geq 0, \text{ for any } \xi \in \mathfrak{P}\}.$$

Then $(M, \mathfrak{H}, \mathfrak{P}, J)$ is called *the standard form of the von Neumann algebra M* (see [12]) if the following conditions are satisfied:

(1) $JMJ = M'$;
(2) $JcJ = c^*$ for any $c \in \mathcal{Z}(M)$;
(3) $J\xi = \xi$ for any $\xi \in \mathfrak{P}$;
(4) $aa^{\mathfrak{t}}(\mathfrak{P}) \subseteq \mathfrak{P}$ for any $a \in M$;

here, M' is the commutant of M in $\mathcal{L}(\mathfrak{H})$, $\mathcal{Z}(M) := M \cap M'$ and $a^{\mathfrak{t}} := JaJ$. We put

$$S_{\mathfrak{H}} := \{\xi \in \mathfrak{H} : \|\xi\| = 1\} \quad \text{and} \quad S_{\mathfrak{P}} := \mathfrak{P} \cap S_{\mathfrak{H}}.$$

It is easy to check that

$$(M', \mathfrak{H}, \mathfrak{P}, J) \tag{2.1}$$

is the standard form of M', and for any $p \in \mathcal{P}(M) \cap \mathcal{Z}(M)$,

$$(pM, p\mathfrak{H}, p\mathfrak{P}, J|_{p\mathfrak{H}}) \tag{2.2}$$

is the standard form of pM.

Remark 2.1. Suppose that $\{x_i\}_{i\in\mathfrak{I}}$ is a net in M that WOT-converges to $x \in M$, when considered as operators in $\mathcal{L}(\mathfrak{H})$. Then, as $\omega_\xi(x_i) \to \omega_\xi(x)$ ($\xi \in \mathfrak{P}$) and $\{\omega_\xi : \xi \in \mathfrak{P}\} = M_*^+$, we know that $\{x_i\}_{i\in\mathfrak{I}}$ weak*-converges to x. Thus, the WOT on $M \subseteq \mathcal{L}(\mathfrak{H})$ coincides with the weak*-topology. Moreover, if $\{e_i\}_{i\in\mathfrak{I}}$ is an increasing net in $\mathcal{P}(M)$ with $e_i \uparrow e_0 \in \mathcal{P}(M)$, then $e_i y e_i \xrightarrow{w^*} e_0 y e_0$ ($y \in M$), because $\omega_\xi(e_i y e_i) = \langle y e_i \xi, e_i \xi\rangle \to \omega_\xi(e_0 y e_0)$ ($\xi \in \mathfrak{P}$).

The following proposition can be regarded as a result of Dye, because all the ingredients for its proof are already in [10] (and a similar discussion can be found in [23]), although it is not explicitly stated in any literature, as far as we know.

Proposition 2.2. *(Dye) Let M_1 and M_2 be two von Neumann algebras. Suppose that there is an* orthoisomorphism $\Gamma : \mathcal{P}(M_1) \to \mathcal{P}(M_2)$, *i.e. Γ is bijective, and for any $p, q \in \mathcal{P}(M_1)$,*

$$p\,q = 0 \quad \textit{if and only if} \quad \Gamma(p)\Gamma(q) = 0.$$

M_1 and M_2 are Jordan *-isomorphic.

Proof: By [10, Lemma 1], the bijection Γ is an order isomorphism that sends central projections to central projections. Let e_k be the central projection in M_k such that $\mathrm{e}_k M_k$ is the type I_2 part of M_k ($k = 1, 2$).

We first show that $\Gamma(\mathrm{e}_1) = \mathrm{e}_2$ and $(1 - \mathrm{e}_1)M_1$ is Jordan *-isomorphic to $(1 - \mathrm{e}_2)M_2$. In fact, as Γ is an order isomorphism with $\Gamma(1 - \mathrm{e}_1) \in \mathcal{Z}(M_2)$, it restricts to an orthoisomorphism from $\mathcal{P}((1 - \mathrm{e}_1)M_1)$ onto $\mathcal{P}(\Gamma(1 - \mathrm{e}_1)M_2)$. The absence of non-zero type I_2 summand in $(1 - \mathrm{e}_1)M_1$ and the Corollary in [10, p. 83] ensure that $\Gamma|_{\mathcal{P}((1-\mathrm{e}_1)M_1)}$ extends to a Jordan *-isomorphism from $(1-\mathrm{e}_1)M_1$ onto $\Gamma(1-\mathrm{e}_1)M_2$. Hence, $\Gamma(1-\mathrm{e}_1)M_2$ does not have a non-zero type I_2 summand neither. This means $\Gamma(1 - \mathrm{e}_1)\mathrm{e}_2 = 0$, or equivalently,

$$\Gamma(1 - \mathrm{e}_1) \le 1 - \mathrm{e}_2.$$

Similarly, $\Gamma^{-1}(1 - \mathrm{e}_2) \le 1 - \mathrm{e}_1$. By [10, Lemma 1], one has $1 - \Gamma(\mathrm{e}_1) = \Gamma(1 - \mathrm{e}_1) = 1 - \mathrm{e}_2$.

It remains to show that e_1M_1 is Jordan *-isomorphic to e_2M_2. Indeed, because $\Gamma(\mathrm{e}_1) = \mathrm{e}_2$, the map Γ restricts to an orthoisomorphism from $\mathcal{P}(\mathrm{e}_1M_1)$ onto $\mathcal{P}(\mathrm{e}_2M_2)$. Since

$$\Gamma\big(\mathcal{Z}(\mathrm{e}_1M_1) \cap \mathcal{P}(\mathrm{e}_1M_1)\big) = \mathcal{Z}(\mathrm{e}_2M_2) \cap \mathcal{P}(\mathrm{e}_2M_2),$$

Γ induces an orthoisomorphism from $\mathcal{P}(\mathcal{Z}(\mathrm{e}_1M_1))$ onto $\mathcal{P}(\mathcal{Z}(\mathrm{e}_2M_2))$, and the Corollary in [10, p. 83] implies that $\mathcal{Z}(\mathrm{e}_1M_1)$ is *-isomorphic to $\mathcal{Z}(\mathrm{e}_2M_2)$. The conclusion now follows from the fact that $\mathrm{e}_kM_k = \mathcal{Z}(\mathrm{e}_kM_k) \otimes \mathrm{M}_2(\mathbb{C})$ $(k = 1, 2)$. □

Our next proposition is very likely to be known as well. However, since we do not find it explicitly stated anywhere, we present its proof here.

Proposition 2.3. *Let $\Theta : M_2 \to M_1$ be a Jordan *-isomorphism. Then $\Theta^*(\mathfrak{S}(M_1)) = \mathfrak{S}(M_2)$ and $\Theta^*|_{\mathfrak{S}(M_1)}$ preserves both the Raggio and the Uhlmann transition probabilities.*

Proof: By [14, Theorem 10], for $k \in \{1, 2\}$, there exists a central projection $\mathrm{e}_k \in \mathcal{P}(M_k) \cap \mathcal{Z}(M_k)$ such that Θ restricts to a *-isomorphism $\Theta_i : \mathrm{e}_2M_2 \to \mathrm{e}_1M_1$ and to a (complex linear) *-anti-isomorphism $\Theta_a : \mathrm{f}_2M_2 \to \mathrm{f}_1M_1$, where $\mathrm{f}_k := 1 - \mathrm{e}_k$ $(k = 1, 2)$. Since all *-isomorphisms and *-anti-isomorphisms between von Neumann algebras are both isometric and weak*-continuous, one has $\Theta^*(\mathfrak{S}(M_1)) = \mathfrak{S}(M_2)$. Notice also that

$$\mathfrak{S}(M_k) = \left\{\big(t\mu_k, (1-t)\nu_k\big) : t \in [0, 1]; \mu_k \in \mathfrak{S}(\mathrm{e}_kM_k); \nu_k \in \mathfrak{S}(\mathrm{f}_kM_k)\right\}.$$

Clearly, $\mathbf{R}(M_k) = \mathbf{R}(\mathrm{e}_kM_k) \times \mathbf{R}(\mathrm{f}_kM_k)$ in the canonical way, and Θ_i induces a bijection from $\mathbf{R}(\mathrm{e}_1M_1)$ onto $\mathbf{R}(\mathrm{e}_2M_2)$. We claim that Θ_a also induces a bijection from $\mathbf{R}(\mathrm{f}_1M_1)$ onto $\mathbf{R}(\mathrm{f}_2M_2)$. Indeed, suppose that $(\psi, \mathfrak{K})$ is a *-representation of f_1M_1. Denote by $\overline{\mathfrak{K}}$ the conjugate Hilbert space of $\mathfrak{K}$ and define

$$\overline{\psi}(x)\overline{\xi} := \overline{\psi(x^*)\xi} \qquad \big(x \in \mathrm{f}_1M_1; \xi \in \mathfrak{K}\big).$$

Then $(\overline{\psi}, \overline{\mathfrak{K}})$ is a (complex linear) *-anti-representation of f_1M_1. Thus, $(\overline{\psi} \circ \Theta_a, \overline{\mathfrak{K}})$ is a *-representation of f_2M_2, and it is not difficult to see that the map from $\mathbf{R}(\mathrm{f}_1M_1)$ to $\mathbf{R}(\mathrm{f}_2M_2)$ as given by

$$(\psi, \mathfrak{K}) \mapsto (\overline{\psi} \circ \Theta_a, \overline{\mathfrak{K}})$$

is bijective as claimed.

Now, if $\mu \in \mathfrak{S}(\mathrm{e}_1M_1)$, $\nu \in \mathfrak{S}(\mathrm{f}_1M_1)$, $(\phi, \mathfrak{H}) \in \mathbf{R}(\mathrm{e}_1M_1)$, $(\psi, \mathfrak{K}) \in \mathbf{R}(\mathrm{f}_1M_1)$, $\xi \in \mathfrak{H}$ and $\eta \in \mathfrak{K}$ satisfying $\mu = \omega_\xi$ and $\nu = \omega_\eta$, then

$$t\mu + (1-t)\nu = \omega_{(\sqrt{t}\xi, \sqrt{1-t}\eta)} \qquad (t \in [0, 1]),$$

where $(\sqrt{t}\xi, \sqrt{1-t}\eta) \in \mathfrak{H} \oplus \mathfrak{K}$. In addition, we have

$$\langle\overline{\psi}(y)\overline{\eta_1}, \overline{\eta_1}\rangle = \langle\psi(y)\eta_1, \eta_1\rangle \quad \text{and} \quad |\langle\overline{\eta_1}, \overline{\eta_2}\rangle| = |\langle\eta_1, \eta_2\rangle| \quad (y \in \mathrm{f}_1M_1; \eta_1, \eta_2 \in \mathfrak{K}).$$

From these, it is not hard to check that $\Theta^*|_{\mathfrak{S}(M_1)}$ preserves P_U.

On the other hand, in order to show the map $\Theta^*|_{\mathfrak{S}(M_1)}$ preserving P_R, it suffices to verify that the map Γ from $S_{\mathfrak{P}_1}$ to $S_{\mathfrak{P}_2}$ defined by

$$\Gamma : \xi_\mu \mapsto \xi_{\Theta^*(\mu)} \qquad (\mu \in \mathfrak{S}(M_1))$$

(see (1.3)) preserves the inner products.

Let $k \in \{1, 2\}$ and $(M_k, \mathfrak{H}_k, \mathfrak{P}_k, J_k)$ be the standard form of M_k. As in (2.2) and (2.1), we know that

$$\left(\mathrm{e}_k M_k, \mathrm{e}_k\mathfrak{H}_k, \mathrm{e}_k\mathfrak{P}_k, J_k|_{\mathrm{e}_k\mathfrak{H}_k}\right) \quad \text{and} \quad \left(\mathrm{f}_k M_k', \mathrm{f}_k\mathfrak{H}_k, \mathrm{f}_k\mathfrak{P}_k, J_k|_{\mathrm{f}_k\mathfrak{H}_k}\right)$$

are the standard forms of $\mathrm{e}_k M_k$ and $\mathrm{f}_k M_k'$, respectively. Firstly, as Θ_i is a *-isomorphism, the restriction of Γ to $\mathrm{e}_1 S_{\mathfrak{P}_1}$ preserves the inner products (thanks to the uniqueness of the standard form). Secondly, we note that

$$\mu \mapsto \mu' := \omega_{\xi_\mu}|_{\mathrm{f}_1 M_1'} \qquad (\mu \in \mathfrak{S}(\mathrm{f}_1 M_1))$$

is a bijection from $\mathfrak{S}(\mathrm{f}_1 M_1)$ onto $\mathfrak{S}(\mathrm{f}_1 M_1')$ (here we regard $\omega_{\xi_\mu} \in \mathfrak{S}(\mathcal{L}(\mathfrak{H}_1))$) such that

$$\xi_{\mu'} = \xi_\mu \qquad (\mu \in \mathfrak{S}(\mathrm{f}_1 M_1)).$$

Moreover, since $x \mapsto J_1\Theta_a(x^*)J_1$ is a *-isomorphism from $\mathrm{f}_2 M_2$ onto $\mathrm{f}_1 M_1'$, we know that the restriction of Γ to $\mathrm{f}_1 S_{\mathfrak{P}_1}$ also preserves the inner products. Finally, the required conclusion concerning Γ follows from the fact that

$$\xi_{t\mu+(1-t)\nu} = \sqrt{t}\xi_\mu + \sqrt{1-t}\xi_\nu \quad (t \in [0, 1]; \mu, \nu \in \mathfrak{S}(M_1)).$$

□

3. The main results

Set $\mathcal{P}_\sigma(M) := \{\mathbf{s}_\mu : \mu \in \mathfrak{S}(M)\}$. For any $p \in \mathcal{P}(M)$, it follows from Zorn's Lemma that there is an orthogonal family $\{p_i\}_{i\in\mathfrak{I}}$ in $\mathcal{P}_\sigma(M)$ satisfying

$$p = \sum_{i\in\mathfrak{I}} p_i \tag{3.1}$$

(the convergence is taken in the weak*-topology). We write

$$F_0(p) := \{\nu \in \mathfrak{S}(M) : \nu(p) = 0\}.$$

Obviously, $F_0(p)$ coincides with the closed face $(1-p)\mathfrak{S}(M)(1-p) \cap \mathfrak{S}(M)$ of $\mathfrak{S}(M)$. Moreover, since

$$\nu(\mathbf{s}_\mu) = 0 \quad \text{if and only if} \quad \mathbf{s}_\nu\mathbf{s}_\mu = 0, \tag{3.2}$$

we have

$$F_0(\mathbf{s}_\mu) \;=\; \{\nu \in \mathfrak{S}(M) : \mathbf{s}_\nu\mathbf{s}_\mu = 0\} \qquad (\mu \in \mathfrak{S}(M)). \tag{3.3}$$

If $p = \sum_{i\in\mathfrak{I}} p_i$ is as in (3.1), then

$$F_0(p) = \bigcap_{i\in\mathfrak{I}} F_0(p_i). \tag{3.4}$$

The reader may consult [27] for more explorations between projections and their associated faces.

We say that a map $\Phi : \mathfrak{S}(M_1) \to \mathfrak{S}(M_2)$ is *biorthogonality preserving* if for any $\mu, \nu \in \mathfrak{S}(M_1)$, one has

$$\mathbf{s}_\mu \mathbf{s}_\nu = 0 \quad \text{if and only if} \quad \mathbf{s}_{\Phi(\mu)} \mathbf{s}_{\Phi(\nu)} = 0.$$

Lemma 3.1. *Let M_1 and M_2 be two von Neumann algebras. Suppose that $\Phi : \mathfrak{S}(M_1) \to \mathfrak{S}(M_2)$ is a biorthogonality preserving bijection.*

(a) There exists an orthoisomorphism $\check{\Phi} : \mathcal{P}(M_1) \to \mathcal{P}(M_2)$ such that

$$\Phi(F_0(p)) = F_0(\check{\Phi}(p)) \quad (p \in \mathcal{P}(M_1)) \quad \textit{and} \quad \check{\Phi}(\mathbf{s}_\mu) = \mathbf{s}_{\Phi(\mu)} \quad (\mu \in \mathfrak{S}(M_1)). \tag{3.5}$$

(b) If there is a Jordan *-isomorphism $\Theta : M_2 \to M_1$ satisfying $\Phi = \Theta^*|_{\mathfrak{S}(M_1)}$, then

$$\Phi(\nu)(\mathbf{s}_{\Phi(\mu)}) = \nu(\mathbf{s}_\mu) \quad (\mu, \nu \in \mathfrak{S}(M_1)).$$

In this case, we have $\check{\Phi} = \Theta^{-1}|_{\mathcal{P}(M_1)}$.

Proof. (a) We denote by $\mathcal{F}(M_k)$ the set of all closed faces of $\mathfrak{S}(M_k)$ $(k = 1, 2)$. The bijectivity of Φ and (3.3) tell us that Φ is biorthogonality preserving if and only if

$$\Phi(F_0(\mathbf{s}_\mu)) = F_0(\mathbf{s}_{\Phi(\mu)}) \qquad (\mu \in \mathfrak{S}(M_1)). \tag{3.6}$$

Let $p \in \mathcal{P}(M_1)$, and $p := \sum_{i \in \mathfrak{I}} \mathbf{s}_{\mu_i}$ be a decomposition as in (3.1) for a family $\{\mu_i\}_{i \in \mathfrak{I}}$ in $\mathfrak{S}(M_1)$ with its elements having disjoint support projections. By the hypothesis, elements in $\{\Phi(\mu_i)\}_{i \in \mathfrak{I}}$ have disjoint support projections, and hence $\sum_{i \in \mathfrak{I}} \mathbf{s}_{\Phi(\mu_i)}$ converges in the weak*-topology to a projection $\check{\Phi}(p) \in \mathcal{P}(M_2)$. Since Φ is injective, (3.4) and (3.6) imply

$$\Phi(F_0(p)) = \bigcap_{i \in \mathfrak{I}} \Phi(F_0(\mathbf{s}_{\mu_i})) = \bigcap_{i \in \mathfrak{I}} F_0(\mathbf{s}_{\Phi(\mu_i)}) = F_0\big(\sum_{i \in \mathfrak{I}} \mathbf{s}_{\Phi(\mu_i)}\big) = F_0(\check{\Phi}(p)).$$

Moreover, the map $F_0 : p \mapsto F_0(p)$ is a bijection from $\mathcal{P}(M_k)$ onto $\mathcal{F}(M_k)$ for $k = 1, 2$ ([4, Theorem 3.35]). These show that $\check{\Phi}(p)$ is independent of the choice of $\{\mu_i\}_{i \in \mathfrak{I}}$, and that Φ induces a map $\Phi^{\mathcal{F}} : \mathcal{F}(M_1) \to \mathcal{F}(M_2)$.

In the same way, Φ^{-1} induces a map from $\mathcal{F}(M_2)$ to $\mathcal{F}(M_1)$ which is clearly the inverse of $\Phi^{\mathcal{F}}$. Therefore, $\Phi^{\mathcal{F}}$ is a bijection, and the bijectivity of the map $\check{\Phi} : \mathcal{P}(M_1) \to \mathcal{P}(M_2)$ follows from the bijectivity of F_0.

Suppose now that $p, q \in \mathcal{P}(M_1)$ satisfying $pq = 0$. Then for any $p', q' \in \mathcal{P}_\sigma(M_1)$ with $p' \leq p$ and $q' \leq q$, one has $p'q' = 0$. Hence, from the hypothesis concerning Φ and the definition of $\check{\Phi}$, we conclude that $\check{\Phi}(p)\check{\Phi}(q) = 0$. Again, by considering Φ^{-1}, we know that $\check{\Phi}$ is an orthoisomorphism.

(b) By the second equality in (3.5),

$$\mu(\Theta(\check{\Phi}(\mathbf{s}_\mu))) = \mu(\Theta(\mathbf{s}_{\Phi(\mu)})) = \Theta^*(\mu)(\mathbf{s}_{\Theta^*(\mu)})) = 1.$$

Thus, $\mathbf{s}_\mu \leq \Theta(\check{\Phi}(\mathbf{s}_\mu)) = \Theta(\mathbf{s}_{\Theta^*(\mu)})$. Conversely, as $\mu(\mathbf{s}_\mu) = 1$, one has $\Theta^*(\mu)(\Theta^{-1}(\mathbf{s}_\mu)) = 1$, which means that $\mathbf{s}_{\Theta^*(\mu)} \leq \Theta^{-1}(\mathbf{s}_\mu)$. These give

$$\mathbf{s}_\mu = \Theta(\check{\Phi}(\mathbf{s}_\mu)) = \Theta(\mathbf{s}_{\Theta^*(\mu)}), \tag{3.7}$$

and hence $\Phi(\nu)(\mathbf{s}_{\Phi(\mu)}) = \nu(\mathbf{s}_\mu)$. On the other hand, due to the construction of $\check{\Phi}$ in the argument for part (a), Equality (3.7) also produces the second conclusion. □

Theorem 3.2. *Let M_1 and M_2 be von Neumann algebras, and let $\Phi : \mathfrak{S}(M_1) \to \mathfrak{S}(M_2)$ be a bijection.*

(a) If Φ is biorthogonality preserving, then M_1 and M_2 are Jordan *-isomorphic.

(b) There is a Jordan *-isomorphism $\Theta : M_2 \to M_1$ satisfying $\Phi = \Theta_*|_{\mathfrak{S}(M_1)}$ if and only if Φ preserves the "asymmetric transition probability" P_0, namely,

$$P_0\big(\Phi(\mu), \Phi(\nu)\big) = P_0(\mu, \nu) \qquad (\mu, \nu \in \mathfrak{S}(M_1)).$$

Proof: (a) This follows directly from Lemma 3.1(a) and Proposition 2.2.

(b) Suppose that such a Jordan *-isomorphism Θ exists. Then the displayed equality in Lemma 3.1(b) tells us that $P_0\big(\Phi(\nu), \Phi(\mu)\big) = P_0(\nu, \mu)$ $(\mu, \nu \in \mathfrak{S}(M_1))$.

For the converse implication, we first note that because $\mu(\mathbf{s}_\nu) = \Phi(\mu)(\mathbf{s}_{\Phi(\nu)})$ $(\mu, \nu \in \mathfrak{S}(M))$, the map Φ is biorthogonality preserving (see (3.2)). Consider $\check{\Phi} : \mathcal{P}(M_1) \to \mathcal{P}(M_2)$ to be the map as in Lemma 3.1. Let M_1^0 be the real linear span of $\mathcal{P}(M_1)$ in M_1. We want to extend $\check{\Phi}$ to M_1^0 by setting

$$\check{\Phi}(x) := \sum_{k=1}^{n} r_k \check{\Phi}(p_k),$$

when $x = \sum_{k=1}^n r_k p_k$ for some $n \in \mathbb{N}$, $r_1, ..., r_n \in \mathbb{R}$ and $p_1, ..., p_n \in \mathcal{P}(M_1)$. To verify this extension being well-defined, let us consider $p_k = \sum_{i \in \mathfrak{I}_k} \mathbf{s}_{\mu_{k,i}}$ to be a decomposition as in (3.1) $(k = 1, ..., n)$. By the construction of $\check{\Phi}$ in the proof of Lemma 3.1(a), for any $\mu \in \mathfrak{S}(M_1)$,

$$\Phi(\mu)\left(\sum_{k=1}^{n} r_k \check{\Phi}(p_k)\right) \;=\; \sum_{k=1}^{n} r_k \sum_{i\in\mathfrak{I}_k} \Phi(\mu)\big(\mathbf{s}_{\Phi(\mu_{k,i})}\big) \;=\; \sum_{k=1}^{n} r_k \sum_{i\in\mathfrak{I}_k} \mu\big(\mathbf{s}_{\mu_{k,i}}\big) \;=\; \mu(x).$$

Thus, the surjectivity of Φ implies that $\check{\Phi}(x)$ is independent of the choices of $r_1, ..., r_n$ nor $p_1, ..., p_n$. Obviously, $\check{\Phi}$ is a linear map on M_1^0 satisfying

$$\nu\big(\check{\Phi}(x)\big) = \Phi^{-1}(\nu)(x) \qquad (x \in M_1^0; \nu \in \mathfrak{S}(M_2)). \tag{3.8}$$

This implies $\|\check{\Phi}(x)\| = \|x\|$ $(x \in M_1^0)$, and $\check{\Phi}$ extends to an isometry from M_1^{sa} onto M_2^{sa}.

Now, one may employ either [7, Theorem 2.2] or [20, Theorem A.4] to conclude that $\check{\Phi}$ is a Jordan isomorphism. However, we would like to present the argument for this particular case here for completeness. Indeed, as $\check{\Phi}$ preserves orthogonality, we have

$$\check{\Phi}(x^2) = \sum_{k=1}^{n} r_k^2 \check{\Phi}(p_k) = \check{\Phi}\left(\sum_{k=1}^{n} r_k p_k\right)^2.$$

The continuity of $\check{\Phi}$ ensures that $\check{\Phi}(x^2) = \check{\Phi}(x)^2$ $(x \in M_1^{\mathrm{sa}})$, as is required.

Finally, if we set $\Theta : M_2^{\rm sa} \to M_1^{\rm sa}$ to be the inverse of $\check{\Phi}$, then Relation (3.8) implies that $\Phi = \Theta_*|_{\mathfrak{S}(M_2)}$. $\square$

Notice that in the proof of part (b) above, the only non-trivial fact that is not proved in this article is the bijection between projections and closed faces as given in [4, Theorem 3.35] (which is needed for Lemma 3.1). In the following, we will use this part (b) to give an alternative and "almost self-contained" proof (except that [4, Theorem 3.35] is required) of Wigner's theorem. Because of this, one may regard Theorem 3.2(b) as an extension of Wigner's theorem. Note also that the first part of the proof of this corollary is similar to that of [24, Theorem 1], but instead of showing the extension to be affine and using [15, Corollary 5], we show that the extension preserves P_0 and use our Theorem 3.2(b) to obtain the conclusion.

Corollary 3.3. *(Wigner) Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be two Hilbert spaces and let*

$$\mathfrak{S}_{\rm p}(\mathcal{L}(\mathfrak{H}_k)) := \{\omega_\xi : \xi \in S_{\mathfrak{H}_k}\} \qquad (k = 1, 2).$$

If $\Phi : \mathfrak{S}_{\rm p}(\mathcal{L}(\mathfrak{H}_1)) \to \mathfrak{S}_{\rm p}(\mathcal{L}(\mathfrak{H}_2))$ is a bijection that preserves the transition probability, there is a Jordan *-isomorphism $\Theta : \mathcal{L}(\mathfrak{H}_2) \to \mathcal{L}(\mathfrak{H}_1)$ with $\Phi = \Theta_*|_{\mathfrak{S}_{\rm p}(\mathcal{L}(\mathfrak{H}_1))}$.

Proof: For $k \in \{1, 2\}$ and $\xi \in S_{\mathfrak{H}_k}$, we know that $\mathbf{s}_{\omega_\xi}$ is the projection from $\mathfrak{H}_k$ onto $\mathbb{C} \cdot \xi$. Through diagonalisation of positive trace-class operators, for each $\mu \in \mathfrak{S}(\mathcal{L}(\mathfrak{H}_1))$, we can find a countable (could be finite) orthonormal family $\{\xi_i\}_{i\in\mathfrak{I}}$ in $S_{\mathfrak{H}_1}$ and a family $\{t_i\}_{i\in\mathfrak{I}}$ in $(0, 1]$ with $\sum_{i\in\mathfrak{I}} t_i = 1$ such that $\mu = \sum_{i\in\mathfrak{I}} t_i\omega_{\xi_i}$ (converges in norm). In this case, we propose to set

$$\bar{\Phi}(\mu) := \sum_{i\in\mathfrak{I}} t_i\Phi(\omega_{\xi_i})$$

(again, it is the norm limit). For any finite orthonormal sequence $\{\zeta_k\}_{k=1}^N$ in $S_{\mathfrak{H}_1}$, one has, by the hypothesis,

$$\sum_{i\in\mathfrak{I}} t_i\Phi(\omega_{\xi_i})\left(\sum_{k=1}^N \mathbf{s}_{\Phi(\omega_{\zeta_k})}\right) = \sum_{k=1}^N\sum_{i\in\mathfrak{I}} t_i\omega_{\xi_i}(\mathbf{s}_{\omega_{\zeta_k}}) = \mu\left(\sum_{k=1}^N \mathbf{s}_{\omega_{\zeta_k}}\right). \tag{3.9}$$

Since Φ is surjective, the above tells us that the value of $\sum_{i\in\mathfrak{I}} t_i\Phi(\omega_{\xi_i})$ on any finite rank projection in $\mathcal{P}(\mathcal{L}(\mathfrak{H}_2))$ is independent of the decomposition $\mu = \sum_{i\in\mathfrak{I}} t_i\omega_{\xi_i}$. Thus, $\bar{\Phi}(\mu)$ is well-defined.

On the other hand, Relation (3.9) also tells us that $\bar{\Phi} : \mathfrak{S}(\mathcal{L}(\mathfrak{H}_1)) \to \mathfrak{S}(\mathcal{L}(\mathfrak{H}_2))$ is an injection. The surjectivity of $\bar{\Phi}$ follows from the surjectivity of Φ. Furthermore, (3.9) implies that $\bar{\Phi}(\mu)(\mathbf{s}_{\Phi(\omega_\eta)}) = \mu(\mathbf{s}_{\omega_\eta})$ $(\eta \in S_{\mathfrak{H}_1})$.

Let $\nu \in \mathfrak{S}(\mathcal{L}(\mathfrak{H}_1))$ and $\nu = \sum_{j\in\mathfrak{J}} r_j\omega_{\eta_j}$ be the decomposition of ν similar to that of μ in the above. As $\{\mathbf{s}_{\omega_{\eta_j}}\}_{j\in\mathfrak{J}}$ (and hence $\{\mathbf{s}_{\Phi(\omega_{\eta_j})}\}_{j\in\mathfrak{J}}$) is a countable (possibly finite) orthogonal family and $r_j > 0$ for all $j \in \mathfrak{J}$, we have

$$\mathbf{s}_\nu = \sum_{j\in\mathfrak{J}} \mathbf{s}_{\omega_{\eta_j}} \quad \text{as well as} \quad \mathbf{s}_{\bar{\Phi}(\nu)} = \sum_{j\in\mathfrak{J}} \mathbf{s}_{\Phi(\omega_{\eta_j})}$$

(the convergences are in the weak*-topology). Hence,

$$\bar{\Phi}(\mu)(\mathbf{s}_{\bar{\Phi}(\nu)}) \;=\; \bar{\Phi}(\mu)\left(\sum_{j\in\mathfrak{J}} \mathbf{s}_{\Phi(\omega_{\eta_j})}\right) \;=\; \sum_{j\in\mathfrak{J}} \mu(\mathbf{s}_{\omega_{\eta_j}}) \;=\; \mu(\mathbf{s}_\nu).$$

Finally, Theorem 3.2(b) gives a Jordan *-isomorphism $\Theta : \mathcal{L}(\mathfrak{H}_1) \to \mathcal{L}(\mathfrak{H}_2)$ satisfying $\Phi = \Theta_* \mid_{\mathfrak{S}(\mathcal{L}(\mathfrak{H}_1))}$. □

On the other hand, Theorem 3.2(a) can be regarded as an extension of a weak form of Uhlhorn's theorem for normal state spaces of von Neumann algebras. In particular, we have the following.

Theorem 3.4. *Let M_1 and M_2 be von Neumann algebras. Then M_1 and M_2 are Jordan *-isomorphic if and only if there is a bijection $\Phi : \mathfrak{S}(M_1) \to \mathfrak{S}(M_2)$ satisfying any one of the following conditions:*

(1) Φ preserves the usual metric d_1;
(2) Φ preserves pairs with zero Raggio transition probabilities, i.e. for any $\mu, \nu \in \mathfrak{S}(M_1)$,

$$P_R\big(\Phi(\mu), \Phi(\nu)\big) = 0 \quad \textit{if and only if} \quad P_R(\mu,\nu) = 0;$$

(3) Φ preserves pairs with zero Uhlmann transition probabilities;
(4) Φ preserves pairs with zero "asymmetric transition probabilities".

Proof: Notice that "the only if part" follows from Theorem 3.2(b), Proposition 2.3 as well as the fact that Jordan *-isomorphisms are isometric. Conversely, we claim that in each of the four cases in the statement, Φ is biorthogonality preserving, and thus Theorem 3.2(a) applies.

Indeed, the assertion for the case of Φ preserving the usual metric d_1 follows from the well-known fact that $\mathbf{s}_\mu \mathbf{s}_\nu = 0$ if and only if $\|\mu - \nu\| = 2$.

Suppose that Φ preserves pairs with zero Raggio transition probabilities. By (1.4) and (1.5), we know that $P_R(\mu,\nu) = 0$ if and only if $\|\xi_\mu - \xi_\nu\|^2 = 2$. On the other hand, it follows from [12, Lemma 2.10(2)] that $\|\xi_\mu - \xi_\nu\|^2 = 2$ if and only if $\|\mu - \nu\| = 2$, because $\|\xi_\mu - \xi_\nu\|\|\xi_\mu + \xi_\nu\| = \sqrt{4 - 4\langle \xi_\mu, \xi_\nu\rangle^2}$. Thus, the assertion for the second case follows from that of the first case.

The assertions for the third case follows from Relation (1.6) and the second case, while that of the fourth case follows from Relation (3.2). □

As seen in the above, for any $\mu, \nu \in \mathfrak{S}(M_1)$, one has

$$\mathbf{s}_\mu \mathbf{s}_\nu = 0 \quad\Longleftrightarrow\quad P_U(\mu,\nu) = 0 \quad\Longleftrightarrow\quad P_R(\mu,\nu) = 0 \quad\Longleftrightarrow\quad P_0(\mu,\nu) = 0, \tag{3.10}$$

which are also equivalent to $\|\mu - \nu\| = 2$. In particular, we obtained an alternative proof of [1, Lemma 1.8].

One may wonder if it is possible to get a stronger conclusion for Theorem 3.2(a) (and hence a stronger conclusion for Theorem 3.4) similar to that of Theorem 3.2(b). However, the following example shows that it is impossible even in the case when $M_1 = M_2 = \mathcal{L}(\ell^2)$.

Example 3.5. Let $M = \mathcal{L}(H)$ where H is a Hilbert space with $\dim H \geq 2$. Consider $\sim$ to be the equivalence relation in $\mathfrak{S}(M)$ defined by

$$\mu \sim \nu \quad \text{if and only if} \quad \mathbf{s}_\mu = \mathbf{s}_\nu.$$

Denote by $\mathcal{C}$ the set of equivalence classes of $\mathfrak{S}(M)$ under $\sim$. Suppose that ζ_1 and ζ_2 are two orthogonal elements in $S_{\mathfrak{H}}$, and $\mathrm{e}_{\zeta_k} \in \mathcal{P}(M)$ is the orthogonal projection onto $\mathbb{C} \cdot \zeta_k$ ($k = 1, 2$). For any $t \in (0, 1)$, if we set $\mu_t := t\omega_{\zeta_1} + (1-t)\omega_{\zeta_2}$, then $\mathbf{s}_{\mu_t} = \mathrm{e}_{\zeta_1} + \mathrm{e}_{\zeta_2}$. Hence, $\{\mu_t : t \in (0, 1)\} \subseteq C_0$ for an element $C_0 \in \mathcal{C}$. Choose any bijection $\Phi_0 : C_0 \to C_0$ satisfying

$$\Phi_0(\mu_t) = (1-t)\omega_{\zeta_1} + t\omega_{\zeta_2} \qquad (t \in (0, 1)),$$

and define a bijection $\Phi : \mathfrak{S}(M) \to \mathfrak{S}(M)$ by setting $\Phi|_{C_0} = \Phi_0$ as well as

$$\Phi(\mu) = \mu \qquad (\mu \in \mathfrak{S}(M) \setminus C_0).$$

From the definition of Φ, we know that $\mathbf{s}_{\Phi(\mu)} = \mathbf{s}_\mu$ ($\mu \in \mathfrak{S}(M)$), and Φ is biorthogonality preserving. However, since

$$\|\omega_{\zeta_1} - \mu_t\| = 2 - 2t \quad \text{and} \quad \|\Phi(\omega_{\zeta_1}) - \Phi(\mu_t)\| = 2t \qquad (t \in (0, 1)),$$

Φ is not induced by any continuous map from M_* to itself.

Nevertheless, in the case when the bijection Φ actually preserves the Raggio transition probability, we will see in Theorem 3.8 below that the conclusion as in Theorem 3.2(b) holds. In order to obtain this result, we need some more preparations.

Recall that a normed space X is said to be *strictly convex* if for any $x, y \in X$, the condition $\|x + y\| = \|x\| + \|y\|$ implies that x and y are linearly dependent. Clearly, any Hilbert space is strictly convex. Let us recall the following well-known fact in Banach spaces theory.

Lemma 3.6. *Suppose that X_1 and X_2 are real Banach spaces such that X_2 is strictly convex. If K is a convex subset of X_1 and $f : K \to X_2$ is a metric preserving map, then f is automatically an affine map.*

For the benefit of the reader, we sketch a proof here. In fact, in order to show

$$f\big(tx + (1-t)y\big) = tf(x) + (1-t)f(y)$$

for any $x \neq y$ in K and $t \in (0, 1)$, we may assume (by "shifting" K and f if necessary) that $y = 0$ and that $f(0) = 0$. In this case, we have $\|f(x)\| = \|f(x) - f(0)\| = \|x\|$ and

$$\|f(x) - f(tx)\| = \|x - tx\| = (1-t)\|f(x)\| = \|f(x)\| - t\|x\| = \|f(x)\| - \|f(tx)\|. \tag{3.11}$$

The strict convexity gives $f(x) - f(tx) \in \mathbb{R} \cdot f(tx)$. This, together with the last two equalities in (3.11), establishes the required relation: $f(tx) = tf(x)$.

Proposition 3.7. *Let $(M_k, \mathfrak{H}_k, J_k, \mathfrak{P}_k)$ be a von Neumann algebra in its standard form ($k = 1, 2$). There are canonical bijective correspondences (through restrictions) amongst the following:*

- *the set $\mathfrak{I}_\mathfrak{H}$ of complex linear isometries from $\mathfrak{H}_1$ onto $\mathfrak{H}_2$ sending $\mathfrak{P}_1$ onto $\mathfrak{P}_2$;*
- *the set $\mathfrak{I}_\mathfrak{P}$ of metric preserving surjections from $\mathfrak{P}_1$ onto $\mathfrak{P}_2$;*
- *the set $\mathfrak{I}_S$ of metric preserving surjections from $S_{\mathfrak{P}_1}$ onto $S_{\mathfrak{P}_2}$.*

Proof: For every $\rho \in \mathfrak{I}_\mathfrak{H}$, one clearly has $\rho|_{S_{\mathfrak{P}_1}} \in \mathfrak{I}_S$. The assignment $\rho \mapsto \rho|_{S_{\mathfrak{P}_1}}$ defines an injection $R : \mathfrak{I}_\mathfrak{H} \to \mathfrak{I}_S$, because $S_{\mathfrak{P}_1}$ generates $\mathfrak{H}_1$. Secondly, if $\chi \in \mathfrak{I}_S$, then

$$\langle \chi(\xi), \chi(\eta) \rangle = \langle \xi, \eta \rangle \in \mathbb{R}_+ \qquad (\xi, \eta \in S_{\mathfrak{P}_1}),$$

and it is easy to verify that the extension $\tilde{\chi} : t\xi \mapsto t\chi(\xi)$ ($\xi \in S_{\mathfrak{P}_1}, t \in \mathbb{R}_+$) belongs to $\mathfrak{I}_\mathfrak{P}$. This gives an injection $E : \mathfrak{I}_S \to \mathfrak{I}_\mathfrak{P}$. Furthermore, as elements in $\mathfrak{I}_\mathfrak{H}$ are affine, the composition $E \circ R : \mathfrak{I}_\mathfrak{H} \to \mathfrak{I}_\mathfrak{P}$ coincides with the restriction map $\rho \mapsto \rho|_{\mathfrak{P}_1}$. Thus, it remains to show that the restriction map from $\mathfrak{I}_\mathfrak{H}$ to $\mathfrak{I}_\mathfrak{P}$ is surjective.

Let us now consider $\varphi \in \mathfrak{I}_\mathfrak{P}$. Since the only extreme point in $\mathfrak{P}_k$ is the zero element, we know from Lemma 3.6 that $\varphi(0) = 0$. The metric preserving assumption now implies

$$\|\varphi(\xi)\| = \|\xi\| \quad \text{and} \quad \langle \varphi(\xi), \varphi(\eta) \rangle = \langle \xi, \eta \rangle \in \mathbb{R}_+ \qquad (\xi, \eta \in \mathfrak{P}_1). \tag{3.12}$$

For $k = 1, 2$, we denote by $\mathfrak{H}_k^{\mathrm{sa}}$ the real Hilbert space generated by $\mathfrak{P}_k$. As $\mathfrak{P}_k$ is a self-dual cone, if $\eta \in \mathfrak{H}_k^{\mathrm{sa}}$, there exist unique elements $\xi^+, \xi^- \in \mathfrak{P}_k$ with $\xi = \xi^+ - \xi^-$ and $\|\xi\|^2 = \|\xi^+\|^2 + \|\xi^-\|^2$ (see e.g. [13, Lemme I.1.2]).

Define $\tilde{\varphi} : \mathfrak{H}_1^{\mathrm{sa}} \to \mathfrak{H}_2^{\mathrm{sa}}$ by

$$\tilde{\varphi}(\xi) := \varphi(\xi^+) - \varphi(\xi^-) \qquad (\xi \in \mathfrak{H}_1^{\mathrm{sa}}).$$

For every $\xi, \eta \in \mathfrak{H}_1^{\mathrm{sa}}$, one knows from (3.12) and Lemma 3.6 that

$$\|\tilde{\varphi}(\xi) - \tilde{\varphi}(\eta)\|^2 \;=\; \|\varphi(\xi^+) + \varphi(\eta^-) - \varphi(\xi^-) - \varphi(\eta^+)\|^2 \;=\; \|\xi^+ - \xi^- - \eta^+ + \eta^-\|^2,$$

which means that $\tilde{\varphi}$ preserves metric. Hence,

$$\|\tilde{\varphi}(\xi)\|^2 = \|\xi\|^2 = \|\xi^+\|^2 + \|\xi^-\|^2 = \|\varphi(\xi^+)\|^2 + \|\varphi(\xi^-)\|^2,$$

and the uniqueness of $\tilde{\varphi}(\xi)^\pm$ produces

$$\tilde{\varphi}(\xi)^\pm = \varphi(\xi^\pm) \qquad (\xi \in \mathfrak{H}_1^{\mathrm{sa}}). \tag{3.13}$$

If $\psi := \varphi^{-1} : \mathfrak{P}_2 \to \mathfrak{P}_1$ and $\tilde{\psi}$ is defined in the same way as $\tilde{\varphi}$, then, by a similar property as (3.13) for $\tilde{\psi}$, we obtain that, for each $\zeta \in \mathfrak{H}_1^{\mathrm{sa}}$,

$$\tilde{\varphi}(\tilde{\psi}(\zeta)) \;=\; \varphi(\tilde{\psi}(\zeta)^+) - \varphi(\tilde{\psi}(\zeta)^-) \;=\; \varphi(\psi(\zeta^+)) - \varphi(\psi(\zeta^-)) \;=\; \zeta.$$

Consequently, $\tilde{\varphi}$ is surjective. It now follows from the Mazur-Ulam theorem that $\tilde{\varphi}$ is a linear isometry from $\mathfrak{H}_1^{sa}$ onto $\mathfrak{H}_2^{sa}$. Finally, the complexification, $\bar{\varphi}$, of $\tilde{\varphi}$ is an element in $\mathfrak{I}_\mathfrak{H}$ (note that linear isometries preserve inner products) satisfying $\bar{\varphi}|_{\mathfrak{P}_1} = \varphi$. $\square$

Recall that a projection $p \in \mathcal{P}(M) \setminus \{0\}$ is said to be *σ-finite* if any family of non-zero orthogonal subprojections of p is countable. It is easy to check that the set $\mathcal{P}_\sigma(M)$ (as in the beginning of this section) consists exactly of σ-finite projections and the sum of countably many orthogonal σ-finite projections is again σ-finite. We also recall that a von Neumann algebra is said to be *σ-finite* if its identity is a σ-finite projection.

Theorem 3.8. *Let M_1 and M_2 be two von Neumann algebras, and let $\Phi : \mathfrak{S}(M_1) \to \mathfrak{S}(M_2)$ be a bijection. Then Φ preserves the Raggio transition probability if and only if one can find a (necessarily unique) Jordan *-isomorphism $\Theta : M_2 \to M_1$ satisfying $\Phi = \Theta_*|_{\mathfrak{S}(M_1)}$.*

Proof: One direction of the equivalence follows from Proposition 2.3. For the opposite direction, we assume in the following that Φ preserves the Raggio transition probability.

Notice that because of Relation (3.10), the map Φ is biorthogonality preserving, and Lemma 3.1 gives an orthoisomorphism $\check{\Phi} : \mathcal{P}(M_1) \to \mathcal{P}(M_2)$. Moreover, by Relations (1.4) as well as (1.5), the map $\varphi : S_{\mathfrak{P}_1} \to S_{\mathfrak{P}_2}$ given by

$$\varphi(\xi_\mu) := \xi_{\Phi(\mu)} \qquad (\mu \in \mathfrak{S}(M_1)$$

is a metric preserving surjection, and Proposition 3.7 tells us that it extends to a complex linear isometry $\bar{\varphi} : \mathfrak{H}_1 \to \mathfrak{H}_2$ satisfying $\bar{\varphi}(\mathfrak{P}_1) = \mathfrak{P}_2$.

By considering finite sums of elements in $\mathcal{P}_\sigma(M_1)$, one obtains, through (3.1), an increasing net $\{e_i\}_{i \in \mathfrak{I}}$ of σ-finite projections such that $e_i \uparrow 1$. Let us put $f_i := \check{\Phi}(e_i)$ $(i \in \mathfrak{I})$. Then all f_i are σ-finite and $f_i \uparrow 1$ (because $\check{\Phi}$ is an orthoisomorphism).

By Corollary 2.5 and Lemma 2.6 of [12], the standard form for $e_i M_1 e_i$ is

$$\big(e_i M_1 e_i, e_i e_i^{\mathbf{t}} \mathfrak{H}_1, e_i e_i^{\mathbf{t}} \mathfrak{P}_1, e_i e_i^{\mathbf{t}} J_1 e_i e_i^{\mathbf{t}}\big)$$

(observe that $e_i^{\mathbf{t}} x e_i^{\mathbf{t}} \eta = x e_i^{\mathbf{t}} \eta = x\eta$, whenever $x \in e_i M_1 e_i, \eta \in e_i e_i^{\mathbf{t}} \mathfrak{H}_1$). In a similar way, $\big(f_i M_2 f_i, f_i f_i^{\mathbf{t}} \mathfrak{H}_2, f_i f_i^{\mathbf{t}} \mathfrak{P}_2, f_i f_i^{\mathbf{t}} J_2 f_i f_i^{\mathbf{t}}\big)$ is the standard form of $f_i M_2 f_i$.

We identify

$$\mathfrak{S}(e_i M_1 e_i) \cong e_i \mathfrak{S}(M_1) e_i \cap \mathfrak{S}(M_1) = F_0(1 - e_i)$$

and $\mathfrak{S}(f_i M_2 f_i) \cong F_0(1 - f_i)$ in the canonical ways. From this, the map Φ induces, through Lemma 3.1(a), a bijection $\Phi_i : \mathfrak{S}(e_i M_1 e_i) \to \mathfrak{S}(f_i M_2 f_i)$. For each $\mu \in \mathfrak{S}(e_i M_1 e_i)$, let $\xi^i_\mu \in S_{e_i e_i^{\mathbf{t}} \mathfrak{P}_1}$ be the element with $\mu(x) = \langle x \xi^i_\mu, \xi^i_\mu \rangle$ $(x \in e_i M_1 e_i)$. Then

$$\mu(y) = \mu(e_i y e_i) = \langle y \xi^i_\mu, \xi^i_\mu \rangle \qquad (y \in M_1),$$

and the uniqueness of the element ξ_μ in $\mathfrak{P}_1$ satisfying (1.3) implies that $\xi^i_\mu = \xi_\mu$. Hence, if $\varphi_i : S_{e_i e_i^{\mathbf{t}} \mathfrak{P}_1} \to S_{f_i f_i^{\mathbf{t}} \mathfrak{P}_2}$ is the bijection defined by $\varphi_i(\xi^i_\mu) := \xi^i_{\Phi_i(\mu)}$ $(\mu \in \mathfrak{S}(e_i M_1 e_i))$, we have $\varphi_i = \varphi|_{S_{e_i e_i^{\mathbf{t}} \mathfrak{P}_1}}$.

As in the beginning of the proof, $\psi_i := \bar{\varphi}|_{e_i e_i^{\mathbf{t}} \mathfrak{H}_1}$ is a bijective isometry from $e_i e_i^{\mathbf{t}} \mathfrak{H}_1$ to $f_i f_i^{\mathbf{t}} \mathfrak{H}_2$ with $\psi_i(e_i e_i^{\mathbf{t}} \mathfrak{P}_1) = f_i f_i^{\mathbf{t}} \mathfrak{P}_2$. Since both $e_i M_1 e_i$ and $f_i M_2 f_i$ are σ-finite, [8, Théorème 3.3] gives a Jordan *-isomorphism $\Lambda_i : e_iM_1e_i \to f_iM_2f_i$ such that for every $x \in e_iM_1e_i$ and $\xi \in S_{e_ie_i^{\mathfrak{t}}\mathfrak{P}_1}$, one has

$$\Phi(\omega_\xi)(\Lambda_i(x)) = \omega_{\varphi(\xi)}(\Lambda_i(x)) = \langle \Lambda_i(x)\varphi_i(\xi), \varphi_i(\xi)\rangle = \langle x\xi, \xi\rangle. \tag{3.14}$$

In particular, $\Phi_i = (\Lambda_i^{-1})^*|_{\mathfrak{S}(e_iM_1e_i)}$.

Again, as in the beginning of the proof, Φ_i is biorthogonality preserving and induces an orthoisomorphism $\check{\Phi}_i : \mathcal{P}(e_iM_1e_i) \to \mathcal{P}(f_iM_2f_i)$ satisfying Relation (3.5). It then follows from

$$\Phi\big(F_0(p) \cap \mathfrak{S}(e_iM_1e_i)\big) = F_0\big(\check{\Phi}(p)\big) \cap \mathfrak{S}(f_iM_2f_i) \qquad (p \in \mathcal{P}(e_iM_1e_i))$$

that $\check{\Phi}_i = \check{\Phi}|_{\mathcal{P}(e_iM_1e_i)}$. Thus, the second conclusion of Lemma 3.1(b) implies $\Lambda_i|_{\mathcal{P}(e_iM_1e_i)} = \check{\Phi}|_{\mathcal{P}(e_iM_1e_i)}$. From this, we know that whenever $i \leq j$, one has $\Lambda_j|_{\mathcal{P}(e_iM_1e_i)} = \Lambda_i|_{\mathcal{P}(e_iM_1e_i)}$, which ensures that

$$\Lambda_j|_{e_iM_1e_i} = \Lambda_i.$$

Set $M_1^e := \bigcup_{i\in\mathfrak{I}} e_iM_1e_i$ and $M_2^f := \bigcup_{i\in\mathfrak{I}} f_iM_2f_i$. The above allows us to define a Jordan *-isomorphism $\Lambda_0 : M_1^e \to M_2^f$ satisfying $\Lambda_0|_{e_iM_1e_i} = \Lambda_i$, and (3.14) gives

$$\omega_{\varphi(\xi)}(\Lambda_0(x)) = \omega_\xi(x) \qquad (x \in M_1^e, \xi \in \mathfrak{P}_1) \tag{3.15}$$

because φ is an isometry and $\bigcup_{i\in\mathfrak{I}} e_ie_i^{\mathfrak{t}}\mathfrak{P}_1$ is norm-dense in $\mathfrak{P}_1$. We thus know from $\{\omega_{\varphi(\xi)} : \xi \in \mathfrak{P}_1\} = (M_2)_*^+$ that Λ_0 is weak*-continuous.

On the other hand, since $e_iye_i \xrightarrow{w^*} y$ for any $y \in M_1$ (see e.g. Remark 2.1), M_1^e is weak*-dense in M_1. Hence, Λ_0 extends to a weak*-continuous complex linear map $\Lambda : M_1 \to M_2$ such that $\Lambda(M_1^+) \subseteq M_2^+$, $\Lambda(1) = 1$ and, because of (3.15),

$$\Phi(\mu)(\Lambda(x)) = \mu(x) \qquad (x \in M_1, \mu \in \mathfrak{S}(M_1)). \tag{3.16}$$

Similarly, Φ^{-1} induces a positive linear map $\Upsilon : M_2 \to M_1$ satisfying the corresponding property as (3.16). Clearly, Υ is the inverse of Λ, and Λ is an order isomorphism. By [15, Corollary 5], Λ is a Jordan *-isomorphism, and $\Theta := \Lambda^{-1}$ is the required map. $\square$

The proof above can be shorten quite a bit if [8, Théorème 3.3] holds for the non-σ-finite case. However, this seems to be open. Note that even in the later work of [13, Theorem VII.1.1], which generalised [8, Théorème 3.3] to the case of JBW^*-algebras, the σ-finite assumption was still imposed.

The following corollary is a direct consequence of Theorems 3.4 and 3.8 as well as Proposition 2.3.

Corollary 3.9. *If $\Phi : \mathfrak{S}(M_1) \to \mathfrak{S}(M_2)$ is a bijection preserving either the Raggio transition probability or the "asymmetric transition probability", then it preserves the Uhlmann transition probability as well.*

It is natural to ask if the converse of the above holds. This lead to the following question.

Question 3.10. If $\Phi : \mathfrak{S}(M_1) \to \mathfrak{S}(M_2)$ is a bijection preserving the Uhlmann transition probability, can one find a Jordan *-isomorphism $\Theta : M_2 \to M_1$ satisfying $\Phi = \Theta_*|_{\mathfrak{S}(M_1)}$?

Let us end this section with another application of our main results. Here, $d_{\|\cdot\|}$ denotes the metric on $\mathfrak{P}$ defined by the norm on $\mathfrak{H}$.

Corollary 3.11. *Let M be a von Neumann algebra. Each one of the following metric spaces: $(\mathfrak{P}, d_{\|\cdot\|})$, $(\mathfrak{S}(M), \mathrm{d}_B)$, $(\mathfrak{S}(M), \mathrm{d}_1)$ and $(\mathfrak{S}(M), \mathrm{d}_2)$ is a complete Jordan *-invariant for M.*

Proof: The fact that $(\mathfrak{S}(M), \mathrm{d}_1)$ is a complete Jordan *-invariant for M is already proved in Theorem 3.4. Moreover, it follows from Theorem 3.4 and Relation (1.4) (respectively, (1.2)) that $(\mathfrak{S}(M), \mathrm{d}_2)$ (respectively, $(\mathfrak{S}(M), \mathrm{d}_B)$) is a complete Jordan *-invariant. Consequently, $(\mathfrak{P}, d_{\|\cdot\|})$ is also a complete Jordan *-invariant because of Proposition 3.7. $\square$

Furthermore, the metric space $\{\xi \in \mathfrak{P} : \|\xi\| \leq 1\}$ (under the metric induced by the norm on $\mathfrak{H}$) is also a complete Jordan *-invariant for M. One can find the details of this, as well as its generalization to all non-commutative L_p-spaces ($p \in [1, +\infty]$), in our further work on this subject (namely, [17]).

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