# NONSURJECTIVE ZERO PRODUCT PRESERVERS BETWEEN MATRICES OVER AN ARBITRARY FIELD

CHI-KWONG LI, MING-CHENG TSAI, YA-SHU WANG AND NGAI-CHING WONG

ABSTRACT. In this paper, we give concrete descriptions of additive or linear disjointness preservers between matrix algebras over an arbitrary field $\mathbb{F}$ of different sizes. In particular, we show that a linear map $\Phi : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ preserving zero products carries the form

$$\Phi(A) = S \begin{pmatrix} R \otimes A & 0 \\ 0 & \Phi_0(A) \end{pmatrix} S^{-1},$$

for some invertible matrices $R$ in $M_k(\mathbb{F})$, $S$ in $M_r(\mathbb{F})$ and a zero product preserving linear map $\Phi_0 : M_n(\mathbb{F}) \to M_{r-nk}(\mathbb{F})$ with range consisting of nilpotent matrices. Here, either $R$ or $\Phi_0$ can be vacuous. The structure of $\Phi_0$ could be quite arbitrary. We classify $\Phi_0$ with some additional assumption. When $\Phi(I_n)$ has a zero nilpotent part, especially when $\Phi(I_n)$ is diagonalizable, we have $\Phi_0(X)\Phi_0(Y) = 0$ for all $X, Y$ in $M_n(\mathbb{F})$, and we give more information about $\Phi_0$ in this case. Similar results for double zero product preservers and orthogonality preservers are obtained.

## 1. INTRODUCTION

There are considerable interests in studying preserver problems for matrices or operators; see, for example, [6, 13, 15–18, 21, 23, 24], and the references therein. Many preserver problems are connected to the study in those maps $\Phi$ of matrices or operators preserving zero products, i.e.,

$$\Phi(A)\Phi(B) = 0 \quad \text{whenever} \quad AB = 0.$$

See, for example, [1, 3–5, 7, 12, 14]. It is usually expected that $\Phi$ gives rise to an algebra or a Jordan homomorphism. Most studies focus on surjective linear maps because general maps may not have nice structure. Even for (necessarily nonsurjective) linear preservers between two matrix algebras of different sizes, the results can be very complicated and intractable.

Denote by $M_n = M_n(\mathbb{F})$ the algebra of $n \times n$ matrices over a field $\mathbb{F}$. The classical results of Jacobson, Rickart, Kaplansky, Herstien, etc. (see, e.g., [9,10]), together with the Skolem-Noether theorem, ensure that every *surjective* zero product preserving linear map $\Phi : M_n \to M_r$ is a scalar multiple of an inner algebra isomorphism, $A \mapsto \alpha S^{-1}AS$, for a nonzero scalar $\alpha$ and an invertible $S$ in $M_n$ (and thus $n = r$). See, e.g., [6, Theorems 2.6 and 3.1].

*Date*: March 13, 2020.

2000 *Mathematics Subject Classification.* 08A35, 15A86, 46L10, 46L40, 47B48.

*Key words and phrases.* ring, Jordan, and algebra homomorphisms; zero product, orthogonality and idempotent preservers; matrix algebras.

The situation is quite different when $\Phi$ is not surjective. For example, the map $A \mapsto \begin{pmatrix} 0 & f(A) \\ 0 & 0 \end{pmatrix}$ defined by any linear (or even non-linear) map $f$ preserves zero products, but it does not contain much useful information about the domain and range.

In this paper, we give concrete descriptions of the structures of additive or linear zero product preservers $\Phi$ between matrix algebras of different sizes. It turns out that such a map is always a sum of ring homomorphism and degenerate map with range space consisting of nilpotent matrices. The first map admits a concrete description, and the second map could be quite arbitrary. Nevertheless, we obtain additional information of the second map under some mild assumption so that our structure theorem can be applied to the study of related problems effectively.

Our paper is organized as follows. In Section 2, we fix the notations and collect some known facts we will use in this paper.

We provide in Section 3 concrete structures of additive/linear zero product preservers between matrix algebras over an arbitrary field $\mathbb{F}$. In particular, we show that a linear map $\Phi : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ preserving zero products carries the form

$$A \mapsto S^{-1} \begin{pmatrix} R \otimes A & 0 \\ 0 & \Phi_0(A) \end{pmatrix} S,$$

for some invertible matrices $R$ in $M_k(\mathbb{F})$, $S$ in $M_r(\mathbb{F})$, and a zero product preserving linear map $\Phi_0 : M_n(\mathbb{F}) \to M_{r-nk}(\mathbb{F})$ with range consisting of nilpotent matrices. When the nilpotent part of $\Phi(I_n)$ is trivial, especially when $\Phi(I_n)$ is diagonalizable, $\Phi_0(X)\Phi_0(Y) = 0$ for all $X, Y$ in $M_n(\mathbb{F})$. A full description of such maps $\Phi_0$ is given. In particular, if $\Phi$ is surjective, we must have $n = r$, $\Phi_0 = 0$, and $R = \alpha I_n$ for a nonzero scalar $\alpha$. This reduces to the stated result in the beginning of the introduction.

In Section 4, we describe the structures of linear maps between matrices preserving idempotents, double zero products, range orthogonality, or double orthogonality. Similar results on additive zero product preservers on the Jordan algebras of self-adjoint or symmetric matrices are also obtained.

To end this paper, we outline in Section 5 some open problems for future studies.

## 2. Notations and preliminaries

Denote by $\mathbb{F}$ the underlying field, and denote by $M_n = M_n(\mathbb{F})$ the algebra of $n \times n$ matrices over the field $\mathbb{F}$. We note that some results below might hold in a more general setting of finite or infinite dimensional Banach algebras or $C^*$-algebras. However, the description of the preservers will be more concrete in the matrix case, while the operator algebra technique might not work for the general case $\mathbb{F} \neq \mathbb{R}$ or $\mathbb{C}$.

Let $E_{ij}$ be the matrix (of an appropriate size depending on context) with the $(i, j)$th entry being 1 and all others being 0. Let $\delta_{ij}$ be the Kronecker delta symbol, i.e., $\delta_{jk} = 1$ when $j = k$, and 0 else. We write $I_n$ and $0_n$, or simply $I$ or 0, for the identity and zero matrices in $M_n$, respectively. Sometimes, 0 can refer to a zero rectangular matrix.

If $\tau$ is an endomorphism of the underlying field $\mathbb{F}$, we write $A_\tau$ for the matrix $[\tau(a_{ij})]$ when $A = [a_{ij}]$. We write $A^{\mathrm{t}} = [a_{ji}]$ for the transpose of $A$. When $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we also write $A^* = \overline{A^{\mathrm{t}}} = [\overline{a_{ji}}]$ for the adjoint, i.e. the conjugate transpose, of $A$. If $\mathbb{F} = \mathbb{R}$, we have $A^{\mathrm{t}} = A^*$.

We call a square matrix $A$ *symmetric* if $A^{\mathrm{t}} = A$, *self-adjoint* if $A^* = A$, *orthogonal* if $A^{-1} = A^{\mathrm{t}}$, *unitary* if $A^{-1} = A^*$, an *idempotent* if $A^2 = A$, and a *projection* if $A = A^2 = A^*$. We say that two idempotents (resp. projections) $A, B$ are *disjoint* (resp. *orthogonal*) to each other if $AB = BA = 0$. We call a complex matrix $A$ *normal* if $A^*A = AA^*$, and call $\frac{A^*+A}{2}$ and $\frac{A-A^*}{2\mathrm{i}}$ the *real* and the *imaginary* parts of $A$, respectively. Here, $\mathrm{i} = \sqrt{-1}$.

We call an additive map $\Phi : M_n \to M_r$ between matrices (maybe of different sizes) over a field $\mathbb{F}$,

- a *ring homomorphism* (resp. *ring anti-homomorphism*) if $\Phi(AB) = \Phi(A)\Phi(B)$ (resp. $\Phi(B)\Phi(A)$) for all $A, B$ in $M_n$;
- an *algebra homomorphism* (resp. *algebra anti-homomorphism*) if it is a linear ring homomorphism (resp. ring anti-homomorphism);
- a *Jordan homomorphism* if $\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)$, or equivalently, $\Phi(A^2) = \Phi(A)^2$, when $\mathbb{F}$ has characteristics not equal 2, for all $A, B$ in $M_n$;
- a *zero product preserver* if $\Phi(A)\Phi(B) = 0$ whenever $AB = 0$;
- a *double zero product preserver* if $\Phi(A)\Phi(B) = \Phi(B)\Phi(A) = 0$ whenever $AB = BA = 0$;

In case $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, we say that $\Phi$ is

- an *algebra* or a *ring* or a *Jordan *(anti)-homomorphism* if $\Phi$ is an algebra, a ring or a Jordan (anti-)homomorphism satisfying that $\Phi(A^*) = \Phi(A)^*$ for all $A$ in $M_n$;
- a *range orthogonality preserver* if $\Phi(A)^*\Phi(B) = 0$ whenever $A^*B = 0$;
- a *double orthogonality preserver* if $\Phi(A)^*\Phi(B) = \Phi(B)\Phi(A)^* = 0$ whenever $A^*B = BA^* = 0$.

The following lemmas collects some known results.

**Lemma 2.1.** *Let $\mathbb{F}$ be any field.*

(a) *Every $A \in M_n(\mathbb{F})$ is similar to a direct sum $R \oplus N$ of an invertible matrix $R$ and a nilpotent matrix $N$ such that $N$ is a direct sum of upper triangular Jordan blocks for the eigenvalue zero of $A$. Here, either $R$ or $N$ can be vacuous.*
(b) *Every $A \in M_n(\mathbb{F})$ is a linear sum of three idempotents.*
(c) *Every non-invertible $A \in M_n(\mathbb{F})$ is a product of idempotents.*
(d) *If $n \geq 2$, then the ring $M_n(\mathbb{F})$ is generated by its idempotents.*
(e) *Every symmetric $A \in M_n(\mathbb{R})$ is a real linear sum of mutually disjoint symmetric rank one idempotents.*
(f) *Every self-adjoint (resp. normal) $A \in M_n(\mathbb{C})$ is a real (resp. complex) linear sum of mutually orthogonal rank one projections.*

*Proof.* (b) is [20, Theorem 1], while (c) is [8, Theorem]. Assertion (d) is a consequence of (c) and the fact that every matrix can be written as a sum of rank one matrices. Moreover, (e) and (f) are just standard textbook results. See, e.g., [11].

(a) is not new either. We sketch a proof here for easy reference. Let the characteristic polynomial of $A$ be $x^{n-s}f(x)$ where $f(x)$ is a polynomial of degree $s$ with no factor $x$. Since $x^{n-s}$ and $f(x)$ are relative prime, there are polynomials $p(x), q(x)$ in $\mathbb{F}[x]$ such that $1 = p(x)x^{n-s} + q(x)f(x)$. Hence, $I_n = p(A)A^{n-s} + q(A)f(A)$. It follows that the kernel spaces $\ker A^{n-s}$ and $\ker f(A)$ have zero intersection. This together with the observation, $f(A)(p(A)A^{n-s}x) = A^{n-s}(q(A)f(A)x) = 0$ for all $x \in \mathbb{F}^n$, concludes that $\mathbb{F}^n = \ker f(A) \oplus \ker A^{n-s}$. It is not difficult to see that the matrix $N$ representing the restriction of $A$ acting on $\ker A^{n-s}$ is a nilpotent matrix, while the matrix $R$ representing the restriction of $A$ acting on $\ker f(A)$ is invertible. Moreover, after a similarity transformation, $N$ can be arranged to be a direct sum of upper triangular Jordan blocks for the eigenvalue zero of $A$. Now, $A$ is similar to the direct sum of $R$ and $N$. There are, of course, cases in which either $R$ or $N$ is vacuous. ∎

One can derive the following results from Lemma 2.1(f), or find a proof from, e.g., [2]. We will work on the general case when the underlying field $\mathbb{F}$ is arbitrary in Theorem 4.5.

**Lemma 2.2.** *Let $\theta : M_n(\mathbb{C}) \to M_r(\mathbb{C})$ be a complex linear map.*

(a) *$\theta$ is a Jordan homomorphism if and only if $\theta$ sends idempotents to idempotents.*

(b) *$\theta$ is a Jordan *-homomorphism if and only if $\theta$ sends projections to projections.*

## 3. Additive and Linear Maps Preserving Zero Products

Let $\mathbb{F}$ be any field and $M_n = M_n(\mathbb{F})$ the algebra of $n \times n$ matrices over the field $\mathbb{F}$. We study those additive/linear maps $\Phi : M_n \to M_r$ preserving zero products, i.e.,

$$\Phi(A)\Phi(B) = 0_r \qquad \text{whenever } A, B \in M_n \text{ satisfy } AB = 0.$$

By Lemma 2.1(a), there is an invertible matrix $S$ in $M_r$ such that

$$S^{-1}\Phi(I_n)S = R \oplus N,$$

where $R$ in $M_s$ is invertible, and $N$ in $M_{r-s}$ is nilpotent such that $N$ is a direct sum of upper triangular zero Jordan blocks (for the eigenvalue zero of $\Phi(I_n)$). Furthermore, the size $\nu$ of the largest zero Jordan block of $N$ is the *nil index* of the nilpotent matrix $N$, which is the smallest nonnegative integer $\nu$ such that $N^\nu = 0$. If $S_1^{-1}\Phi(I_n)S_1 = R_1 \oplus N_1$ is another direct sum of an invertible matrix $R_1$ and a nilpotent matrix $N_1$, we see that $N_1$ is similar to $N$ and has nil index $\nu$.

Using the above decomposition of $\Phi(I_n)$, we can state the main theorem of this section.

**Theorem 3.1.** *Let the underlying field $\mathbb{F}$ be arbitrary. Let $\Phi : M_n \to M_r$ be an additive map preserving zero products. Assume that $S^{-1}\Phi(I_n)S = R \oplus N$, where $S$ in $M_r$ and $R$ in $M_s$ are invertible, and $N$ in $M_{r-s}$ is nilpotent of nil index $\nu$. Then, $k = s/n$ is a nonnegative integer and $\Phi$ has the form*

$$A \mapsto S \begin{pmatrix} R\Phi_1(A) & 0 \\ 0 & \Phi_0(A) \end{pmatrix} S^{-1} = S \begin{pmatrix} \Phi_1(A)R & 0 \\ 0 & \Phi_0(A) \end{pmatrix} S^{-1} \tag{3.1}$$

*where $\Phi_1 : M_n \to M_s$ is a unital ring homomorphism, and $\Phi_0 : M_n \to M_{r-nk}$ is a zero product preserving additive map into nilpotent matrices such that the product of any $\nu + 1$ of them is zero. If $N = 0$, then*

$$\Phi_0(X)\Phi_0(Y) = 0_{r-nk} \quad \textit{for all} \;\; X, Y \in M_n.$$

Here, either $R$ (and thus $\Phi_1$) or $\Phi_0$ can be vacuous. Similar conventions also apply to other results in this paper. The proof of Theorem 3.1 will be given in Subsection 3.1. We will make some remarks below to put the theorem in perspective.

First, the theorem states that the map $\Phi$ can be decomposed as the sum of the map $A \mapsto S(R\Phi_1(A) \oplus 0_{r-s})S^{-1}$ and $A \mapsto S(0_s \oplus \Phi_0(A))S^{-1}$, where the former one is closely related to a ring homomorphism and the latter one is a zero product preserving map with ranges lying in the set of nilpotent matrices. In particular, one can use the canonical form $\Phi(I_n)$ to do the additive decomposition of the map $\Phi$.

Second, it is interesting that $\Phi_1$ is actually a unital ring homomorphism. When $\Phi_1$ is linear we will show in Subsection 3.2 that $\Phi_1$ has the form

$$A \mapsto S_1(I_k \otimes A)S_1^{-1}$$

for some invertible $S_1$ in $M_s$. Moreover, because $R\Phi_1(A) = \Phi_1(A)R$ for all $A$, we see that $R = S_1(R_1 \otimes I_n)S_1^{-1}$ for some invertible $R_1$ in $M_k$.

Third, in the statement of Theorem 3.1 not much is said about the map $\Phi_0$. In Subsection 3.3, we will show that the structure of $\Phi_0$ can be quite wild in general. Anyway, we will provide more information about the map in Subsection 3.3. Moreover, as we will see in subsequent discussion, in many useful applications of Theorem 3.1 one has $\Phi_0 = 0$.

Before we start the proof, we mention that when $n = r$, the special cases of Theorem 3.1, as well as Theorem 3.5, can be found in [21, Section 1]. In a more general context, Brešar and Šemrl study zero product preserving additive maps $\Phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ between matrices over a division ring $\mathbb{D}$. They show in [3, Theorem 5.2] that either $\Phi(A)\Phi(B) = 0$ for all $A, B$, or that there is a ring endomorphism $\Phi_1$ of $M_n(\mathbb{D})$ and a matrix $C$ in $M_n(\mathbb{D})$ such that $\Phi$ has the form

$$A \mapsto C\Phi_1(A) = \Phi_1(A)C.$$

However, the case $n < r$ is much more complicated as we shall see in the following.

## 3.1. **Proof of Theorem 3.1.**

We need the following (probably known) lemma to prove Theorem 3.1.

**Lemma 3.2.** *Let the underlying field $\mathbb{F}$ be arbitrary. Suppose $\Phi : M_n \to M_r$ is an additive map preserving zero products. Then*

$$\Phi(C)\Phi(AB) = \Phi(CA)\Phi(B) \quad \textit{for all} \;\; A, B, C \in M_n.$$

*Consequently,*

$$\Phi(I_n)\Phi(AB) = \Phi(A)\Phi(B) \quad \textit{for all} \;\; A, B \in M_n, \tag{3.2}$$

*and*

$$\Phi(I_n)\Phi(A) = \Phi(A)\Phi(I_n) \quad \textit{for all} \;\; A \in M_n. \tag{3.3}$$

(a) *If* $\Phi(I_n)$ *is invertible then* $A \mapsto \Phi(I_n)^{-1}\Phi(A)$ *is a ring homomorphism from* $M_n$ *into* $M_r$.

(b) *If* $\Phi(I_n)^\nu = 0$ *then the product of any* $\nu+1$ *elements from the range of* $\Phi$ *is zero, i.e.,*

$$\Phi(A_1)\Phi(A_2)\cdots\Phi(A_{\nu+1}) = 0 \quad \textit{for all} \ \ A_1, A_2, \ldots, A_{\nu+1} \in M_n.$$

*In particular, if* $\Phi(I_n) = 0$ *then the range of* $\Phi$ *has trivial multiplications, i.e.,*

$$\Phi(A)\Phi(B) = 0 \quad \textit{for all} \ \ A, B \in M_n.$$

*Proof.* We borrow from the proof of [6, Lemma 2.1]. The case $n = 1$ is obvious. Assume below that $n \geq 2$. Let $E = E^2$ in $M_n$. For any $B, C$ in $M_n$, consider

$$(C - CE)EB = CE(B - EB) = 0.$$

By the zero product preserving property, we have

$$(\Phi(C) - \Phi(CE))\Phi(EB) = \Phi(CE)(\Phi(B) - \Phi(EB)) = 0.$$

It follows

$$\Phi(C)\Phi(EB) = \Phi(CE)\Phi(EB) = \Phi(CE)\Phi(B).$$

Since $M_n$ is generated by its idempotents as a ring by Lemma 2.1(d),

$$\Phi(C)\Phi(AB) = \Phi(CA)\Phi(B), \quad A, B, C \in M_n.$$

Putting $C = I$, and putting $B = C = I$, respectively, we establish (3.2) and (3.3). It thus follows (a).

We now verify (b). By (3.2) and the assumption $\Phi(I_n)^\nu = 0$, we have

$$\begin{aligned} &\Phi(A_1)\Phi(A_2)\Phi(A_3)\cdots\Phi(A_{\nu+1}) = \Phi(I_n)\Phi(A_1A_2)\Phi(A_3)\cdots\Phi(A_{\nu+1}) = \cdots \\ &= \Phi(I_n)^\nu\Phi(A_1A_2A_3\cdots A_{\nu+1}) = 0 \quad \text{for all} \ \ A_1, A_2, \ldots, A_{\nu+1} \in M_n. \end{aligned}$$

■

*Proof of Theorem 3.1.* Replacing $\Phi$ by $S^{-1}\Phi(\cdot)S$, we can assume that $\Phi(I_n) = R \oplus N$. Let

$$\Phi(X) = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

where $Y_{11} \in M_s$. By (3.3) in Lemma 3.2, $\Phi(I)\Phi(X) = \Phi(X)\Phi(I)$. So,

$$RY_{11} = Y_{11}R, \quad RY_{12} = Y_{12}N, \quad NY_{21} = Y_{21}R \quad \text{and} \quad NY_{22} = Y_{22}N.$$

Without loss of generality, we can assume that $N = \sum_j d_j E_{j,j+1}$ with $d_j \in \{0, 1\}$ is a direct sum of upper triangular Jordan blocks of zero. If $Y_{12} = [v_1 \,|\, \cdots \,|\, v_{r-s}]$, where $v_1, \ldots, v_{r-s}$ are column vectors, then

$$[Rv_1 \,|\, \cdots \,|\, Rv_{r-s}] = [0 \,|\, d_1v_1 \,|\, \cdots \,|\, d_{r-s-1}v_{r-s-1}].$$

Thus, $v_1 = R^{-1}0 = 0$ and $v_j = d_{j-1}R^{-1}v_{j-1} = 0$ for $j = 2, \ldots, r-s$. Hence, $Y_{12} = 0$. Similarly, we can show that $Y_{21} = 0$. So, $\Phi(X)$ has the form $Y_{11} \oplus Y_{22}$. Thus, bringing back the similarity transformation, we can set up the additive maps $\Phi_1 : M_n \to M_s$ and $\Phi_0 : M_n \to M_{r-s}$ satisfying

$$(R^{-1} \oplus I_{r-s})S^{-1}\Phi(X)S = \Phi_1(X) \oplus \Phi_0(X). \tag{3.4}$$

Clearly, $\Phi_1(I) = R^{-1}R = I_s$. Moreover, $R\Phi_1(A) = \Phi_1(A)R$ for all $A$ in $M_n$. Suppose $A, B \in M_n$ such that $AB = 0_n$. Let $S^{-1}\Phi(A)S = A_1 \oplus A_2$ and $S^{-1}\Phi(B)S = B_1 \oplus B_2$. Since $\Phi(A)\Phi(B) = 0_r$, we have $A_1B_1 = 0_s$. Consequently,

$$\Phi_1(A)\Phi_1(B) = R^{-1}A_1R^{-1}B_1 = R^{-1}A_1B_1R^{-1} = 0_s.$$

By Lemma 3.2, $\Phi_1$ is a ring homomorphism, and $\Phi_0$ satisfies the said conclusion. ∎

### 3.2. **Algebra homomorphisms of matrices.**

In the following, we give a concrete description of algebra homomorphisms between matrix algebras.

**Theorem 3.3.** *Suppose $\Phi : M_n \to M_r$ is an algebra homomorphism between matrices over an arbitrary field $\mathbb{F}$.*

(a) *There exist a nonnegative integer $k$ with $t = r - nk \geq 0$, and an invertible matrix $S$ in $M_r$ such that $\Phi$ has the form*

$$A \mapsto S \begin{pmatrix} I_k \otimes A & 0 \\ 0 & 0_t \end{pmatrix} S^{-1}. \tag{3.5}$$

(b) *Assume $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If $\Phi(A)^* = \Phi(A)$ for every rank one projection $A$, then $S$ can be chosen such that $S^{-1} = S^*$.*

(c) *Assume $\mathbb{F} = \mathbb{C}$. If $\Phi(A)$ is symmetric for every rank one real symmetric idempotent $A$, then $S$ can be chosen to be complex orthogonal, i.e., $S^{-1} = S^{\mathrm{t}}$.*

*Proof.* If $\Phi$ is the zero map then the assertion is trivial. Assume that $\Phi$ is nonzero. Since $AI = IA = A$ for all $A$ in $M_n$, we see that $\Phi(I)$ is an idempotent matrix such that

$$\Phi(A) = \Phi(A)\Phi(I) = \Phi(I)\Phi(A) \quad \text{for all } A \in M_n.$$

Let $\Phi(I)$ have rank $m > 0$, and $t = r - m \geq 0$. There is an invertible $S_0$ in $M_r$ such that

$$S_0^{-1}\Phi(A)S_0 = \Phi_1(A) \oplus 0_t \quad \text{for all } A \in M_n,$$

for a unital ring homomorphism $\Phi_1 : M_n \to M_m$. Replacing $\Phi$ by $\Phi_1$, we may assume that $\Phi(I_n) = I_r$.

Since $E_{ij}E_{kl} = \delta_{jk}E_{il}$, we have

$$\Phi(E_{ij})\Phi(E_{kl}) = \delta_{jk}\Phi(E_{il}), \quad i, j, k, l = 1, 2, \ldots, n. \tag{3.6}$$

Moreover,

$$I_r = \Phi(I_n) = \sum_{i=1}^{n} \Phi(E_{ii}).$$

Replacing $\Phi$ with the map $X \mapsto S_1^{-1}\Phi(X)S_1$ for some invertible $S_1$ in $M_r$, we can assume that the idempotents

$$\Phi(E_{ii}) = 0_{k_1} \oplus \cdots \oplus I_{k_i} \oplus 0_{k_{i+1}} \oplus \cdots \oplus 0_{k_n}, \quad i = 1, \ldots, n.$$

Here, $k_1 + \cdots + k_n = r$.

Let $s = r - k_1 - k_2$. It follows from (3.6) that

$$\Phi(E_{12}) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \oplus 0_s \quad \text{and} \quad \Phi(E_{21}) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \oplus 0_s,$$

where $B_{ij}, C_{ij}$ are $k_i \times k_j$ matrices for $i, j = 1, 2$. Since $E_{11}E_{12} = E_{12}$ and $E_{12}E_{11} = 0$, we have $B_{11}$, $B_{22}$ and $B_{21}$ are all zero matrices. Similarly, $C_{11}$, $C_{22}$ and $C_{12}$ are also zero matrices. Hence,

$$\Phi(E_{12}) = \begin{pmatrix} 0 & B_{12} \\ 0 & 0 \end{pmatrix} \oplus 0_s \quad \text{and} \quad \Phi(E_{21}) = \begin{pmatrix} 0 & 0 \\ C_{21} & 0 \end{pmatrix} \oplus 0_s, \tag{3.7}$$

On the other hand, $(E_{12} + E_{21})^2 = E_{11} + E_{22}$ implies

$$\begin{pmatrix} 0 & B_{12} \\ C_{21} & 0 \end{pmatrix}^2 = \begin{pmatrix} B_{12}C_{21} & 0 \\ 0 & C_{21}B_{12} \end{pmatrix} = \begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix}.$$

This ensures $k_1 = k_2$ and $B_{12} = C_{21}^{-1}$. Let $k = k_1$.

Dealing in a similar way for other pairs $i, j$ of indices, we see that

$$\Phi(E_{jj}) = E_{jj} \otimes I_k, \quad \Phi(E_{ij}) = E_{ij} \otimes B_{ij} \text{ for } i < j, \quad \Phi(E_{ij}) = E_{ij} \otimes B_{ji}^{-1} \text{ for } j < i.$$

In particular, $r/n = k$.

Replacing $\Phi$ by the map $X \mapsto S\Phi(X)S^{-1}$ with $S = I_k \oplus B_{12} \oplus B_{13} \oplus \cdots \oplus B_{1n}$, we can further assume that

$$B_{12} = \cdots = B_{1n} = I_k \qquad \text{and} \qquad B_{21} = \cdots = B_{n1} = I_k.$$

Actually, we have

$$\Phi(E_{ij}) = E_{ij} \otimes I_k \quad \text{ for all } i, j = 1, \ldots, n.$$

To see this, observe $E_{ij} = (E_{i1} + E_{1j} + E_{ij})^2$ for $1 < i < j$. We thus have

$$\Phi(E_{ij}) = (\Phi(E_{i1}) + \Phi(E_{1j}) + \Phi(E_{ij}))^2.$$

This gives

$$E_{ij} \otimes B_{ij} = (E_{i1} \otimes I_k + E_{1j} \otimes I_k + E_{ij} \otimes B_{ij})^2 = E_{ij} \otimes I_k.$$

Reordering the basic vectors, i.e., applying a permutation similarity, we can assume instead

$$\Phi(E_{ij}) = I_k \otimes E_{ij} \quad \text{ for all } i, j = 1, \ldots, n. \tag{3.8}$$

By linearity of $\Phi$, we establish (3.5).

The assumption in (b) asserts that $\Phi$ sends rank one projections to self-adjoint matrices. By (3.6), all $\Phi(E_{ii})$ are projections and orthogonal to each other. Moreover, all $\Phi(E_{ij} + E_{ji})$ are self-adjoint, since $E_{ij} + E_{ji} = P_+ - P_-$ is the difference of two rank one projections $P_\pm = (E_{ii} \pm E_{ij} \pm E_{ji} + E_{jj})/2$. Hence, we can choose a unitary matrix $S_1$ from $M_r$ such that

$$S_1^*\Phi(E_{ii})S_1 = 0_{k_1} \oplus \cdots \oplus I_{k_i} \oplus 0_{k_{i+1}} \oplus \cdots \oplus 0_{k_n}, \quad i = 1, \ldots, n.$$

In view of (3.7), the matrix

$$S_1^*\Phi(E_{12} + E_{21})S_1 = \begin{pmatrix} 0 & B_{12} \\ C_{21} & 0 \end{pmatrix} \oplus 0_s$$

is self-adjoint. Since $B_{12}^* = C_{21} = B_{12}^{-1}$, the afterward change of basis transformation $A \mapsto (B_{12}^* \oplus I_{r-k})S_1^*\Phi(A)S_1(B_{12} \oplus I_{r-k})$ is also unitary. Consequently, we can choose $S$ to be a unitary matrix in $M_r$.

The assumption in (c) implies that $\Phi$ sends rank one real symmetric idempotents to symmetric matrices (but might not be of all real entries). In particular, all $\Phi(E_{ii})$ are symmetric idempotents. Moreover, as the images of the differences of two disjoint rank one real symmetric idempotents, all $\Phi(E_{ij} + E_{ji})$ are symmetric. Consequently, $\Phi$ sends symmetric matrices to symmetric matrices.

Recall that a complex symmetric matrix $B$ is complex orthogonally diagonalizable, i.e., there exists a complex matrix $U$ such that $U^{\mathrm{t}}BU$ is diagonal and $U^{\mathrm{t}} = U^{-1}$, exactly when $B$ is diagonalizable (see, e.g., [11, Theorem 4.4.27]). We have seen that all complex symmetric idempotents $\Phi(E_{11}), \ldots, \Phi(E_{nn})$ have rank $k = r/n$, and all of them are diagonalizable. It follows that each $\Phi(E_{ii})$ has $k$ complex eigenvectors $s_{i1}, \ldots, s_{ik}$ for the eigenvalue 1 such that ${s_{ij_1}}^{\mathrm{t}} s_{ij_2} = \delta_{j_1 j_2}$ for $j_1, j_2 = 1, \ldots, k$. Let $v_i, v_j$ be eigenvectors of $\Phi(E_{ii}), \Phi(E_{jj})$ in $\mathbb{C}^r$ associated with the common eigenvalue 1, respectively. Observe that for $i \neq j$, we have

$$v_i^{\mathrm{t}} v_j = v_i^{\mathrm{t}} \Phi(E_{ii})^{\mathrm{t}} \Phi(E_{jj}) v_j = v_i^{\mathrm{t}} \Phi(E_{ii}) \Phi(E_{jj}) v_j = v_i^{\mathrm{t}} \Phi(E_{ii} E_{jj}) v_j = 0.$$

Therefore, we can find a basis $\{s_{11}, \ldots, s_{1k}, \ldots, s_{n1}, \ldots, s_{nk}\}$ of $\mathbb{C}^r$ consisting of complex eigenvectors of $\Phi(E_{ii})$'s associated with the common eigenvalue 1 such that ${s_{i_1 j_1}}^{\mathrm{t}} s_{i_2 j_2} = \delta_{i_1 i_2}\delta_{j_1 j_2}$ for $i_1, i_2 = 1, \ldots, n$ and $j_1, j_2 = 1, \ldots, k$. Using these basic vectors as column vectors, we have an orthogonal matrix $S_1$ in $M_r$ (might contain complex entries) such that

$$S_1^{\mathrm{t}} \Phi(E_{ii}) S_1 = 0_{k_1} \oplus \cdots \oplus I_{k_i} \oplus 0_{k_{i+1}} \oplus \cdots \oplus 0_{k_n}, \quad i = 1, \ldots, n.$$

In view of (3.7), as the real symmetric matrix

$$S_1^{\mathrm{t}} \Phi(E_{12} + E_{21}) S_1 = \begin{pmatrix} 0 & B_{12} \\ C_{21} & 0 \end{pmatrix} \oplus 0_s,$$

we have $B_{12}^{\mathrm{t}} = C_{21} = B_{12}^{-1}$. Thus the afterward change of basis transformation

$$A \mapsto (B_{12}^{\mathrm{t}} \oplus I_{r-k}) S_1^{\mathrm{t}} \Phi(A) S_1 (B_{12} \oplus I_{r-k})$$

is also complex orthogonal. Consequently, we can choose $S$ to be a complex orthogonal matrix in $M_r$. ∎

**Corollary 3.4.** *Let $\mathbb{F}$ be any field. Let $\Phi : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ be a linear map preserving zero products. If $\Phi(I_n)$ is an idempotent, then there is a nonsingular matrix $S$ in $M_r(\mathbb{F})$ such that*

$$\Phi(I_n)\Phi(A) = \Phi(A)\Phi(I_n) = S \begin{pmatrix} I_k \otimes A & 0 \\ 0 & 0_{r-kn} \end{pmatrix} S^{-1}.$$

*In other words, $\Psi_1 := \Phi(I_n)\Phi$ is an algebra homomorphism. Moreover, $\Psi_0 := (I_r - \Phi(I_n))\Phi$ is a linear map such that its image has trivial multiplications. Clearly,*

$$\Phi = \Psi_1 + \Psi_0.$$

*In particular, if $\Phi(I_n) = I_r$ then $\Phi$ is a unital algebra homomorphism, and has the form $A \mapsto S(I_k \otimes A)S^{-1}$.*

**Theorem 3.5.** *Let the underlying field $\mathbb{F}$ be arbitrary. Let $\Phi : M_n \to M_r$ be a linear map preserving zero products. Then $\Phi$ has the form*

$$A \mapsto S \begin{pmatrix} R_1 \otimes A & 0 \\ 0 & \Phi_0(A) \end{pmatrix} S^{-1}, \tag{3.9}$$

*for some invertible $S \in M_r$, $R_1 \in M_k$, and a zero product preserving linear map $\Phi_0$ sending $M_n(\mathbb{F})$ into nilpotent matrices.*

(a) *If $\Phi$ sends rank one idempotents to idempotents then $\Phi_0$ is the zero map, and $\Phi$ has the form*

$$A \mapsto S \begin{pmatrix} R_1 \otimes A & 0 \\ 0 & 0_{r-nk} \end{pmatrix} S^{-1}. \tag{3.10}$$

(b) *Suppose $\mathbb{F} = \mathbb{C}$ and $\Phi(A)^* = \Phi(A)$ for every rank one orthogonal projection $A$. Then $\Phi$ has the form (3.10) where $R_1 = R_1^*$ and $S$ can be chosen to be unitary, i.e., $S^{-1} = S^*$.*

(c) *Suppose $\mathbb{F} = \mathbb{R}$, and $\Phi(A) = \Phi(A)^t$ for every symmetric rank one idempotent $A$. Then $\Phi$ has the form (3.9) where $R_1 \in M_k$, and $S \in M_r$ can be chosen to be real orthogonal, $\Phi_0(A) = 0$ for symmetric matrices $A$, and $\Phi_0(X)\Phi_0(Y) = 0$ in general.*

(d) *Suppose $\mathbb{F} = \mathbb{C}$ and $\Phi(I_n)$ is diagonalizable and $\Phi(A)^{\mathrm{t}} = \Phi(A)$ for every rank one real symmetric idempotent $A$ in $M_n$. Then $\Phi$ has the form (3.9), where $R_1 = R_1^{\mathrm{t}}$ and $S$ can be chosen to be complex orthogonal, i.e., $S^{-1} = S^{\mathrm{t}}$.*

*Proof.* We use the notations in Theorems 3.1 and 3.3. In particular, $S^{-1}\Phi(A)S = R\Phi_1(A) \oplus \Phi_0(A)$, in which the unital algebra homomorphism $\Phi_1$ has the form $S_1(I_k \otimes A)S_1^{-1}$ for some invertible $S_1 \in M_{nk}$. Since $R\Phi_1(A) = \Phi_1(A)R$ for all $A \in M_n$, we have $R = S_1(R_1 \otimes I_n)S_1^{-1}$ for some invertible $R_1 \in M_k$. It follows (3.9) after resetting $S(S_1 \oplus I_{r-nk})$ to be $S$.

(a) Assume $\Phi$ sends rank one idempotents to idempotents. Then $\Phi_0(A) = \Phi_0(A)^r = 0$ for every rank one idempotent $A$ in $M_n$. Since every idempotent is a sum of rank one idempotents, $\Phi_0$ sends idempotents to zero. Since every matrix is a linear sum of three idempotents by Lemma 2.1(b), we see that $\Phi_0$ is the zero map.

(b) Because $\Phi(A)^* = \Phi(A)$ for every rank one orthogonal projection, we see that $\Phi(H)^* = \Phi(H)$ whenever $H = H^*$ by Lemma 2.1(f). In particular, $\Phi(I) = \Phi(I)^*$. We can choose an invertible $S$ with $S^* = S^{-1}$ such that $S^*\Phi(I)S = R \oplus 0_{r-nk}$. In this way, $R = R^*$. By Theorem 3.3, we can find $S_1$ satisfying $S_1^* = S_1^{-1}$ such that the unital algebra homomorphism $\Phi_1$ has the form $X \mapsto S_1(I_k \otimes X)S_1^{-1}$. As $S_1^*RS_1 = R_1 \otimes I_n$, we see that $R_1$ is self-adjoint. It follows (3.9) after resetting $S(S_1 \oplus I_{r-nk})$ to be $S$.

Since $\Phi_0(H)^r = 0$, we see that the self-adjoint matrix $\Phi_0(H) = 0$ for all $H = H^*$ in $M_n$. Because every $A$ in $M_n(\mathbb{C})$ has the form $A = H + \mathrm{i}G$ with $H^* = H = (A + A^*)/2$ and $G = (A - A^*)/(2\mathrm{i})$. Thus, $\Phi_0(A) = \Phi_0(H) + \mathrm{i}\Phi_0(G) = 0$. So, $\Phi_0$ is the zero map.

(c) The first part of the proof is similar to that of (b), and we can conclude that $\Phi_0(A) = 0$ for every symmetric matrix $A$ in $M_n(\mathbb{R})$. In particular, $\Phi_0(I_n) = 0$. Thus the range of $\Phi_0$ has trivial multiplication by (3.2).

(d) In the complex matrix case, suppose $\Phi(X) = \Phi(X)^{\mathrm{t}}$ for every rank one real symmetric idempotent $X$. As in the proof of Theorem 3.3(c), we see that $\Phi$ sends symmetric matrices to symmetric matrices. In particular, the diagonalizable matrix $\Phi(I_n)$ is symmetric, and thus complex orthogonally diagonalizable. We can thus find a complex matrix $S$ such that $S^{\mathrm{t}} = S^{-1}$ and $S^{\mathrm{t}}\Phi(I_n)S = R \oplus 0_{r-nk}$, where the invertible matrix $R$ is symmetric. Now, we can apply Theorem 3.1 and Theorem 3.3(c) to conclude that there is a complex matrix $S_1$ in $M_{nk}$ such that $S_1^{\mathrm{t}} = S_1^{-1}$ satisfying (3.1). As $S_1^{\mathrm{t}} R S_1 = R_1 \otimes I_n$, we see that $R_1$ is symmetric. Again, it follows (3.9) after resetting $S(S_1 \oplus I_{r-nk})$ to be $S$. ∎

**Example 3.6.** Consider the linear map $\Phi : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 0 & b-c \\ 0 & 0 \end{pmatrix}.$$

It is clear that $\Phi = \Phi_0$ preserves zero products and sends symmetric matrices to symmetric matrices (indeed, the zero matrix). The range of $\Phi_0$ has trivial multiplications, while $\Phi_0$ is not the zero map.

3.3. **Zero product preserving maps into nilpotents.** By Theorem 3.1, every zero product preserving additive map $\Phi : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ has the form

$$A \mapsto S(R\Phi_1(A) \oplus \Phi_0(A))S^{-1} = S(\Phi_1(A)R \oplus \Phi_0(A))S^{-1}, \tag{3.11}$$

where $R, S$ are invertible matrices, $\Phi_1 : M_n(\mathbb{F}) \to M_{nk}(\mathbb{F})$ is a unital ring homomorphism and $\Phi_0 : M_n(\mathbb{F}) \to M_{r-nk}(\mathbb{F})$ is a zero product preserving additive maps sending matrices to nilpotent matrices. By the discussion in Subsection 3.2, we have a good understanding of $\Phi_1$. In this section, we focus on $\Phi_0$.

If $\Phi_0(I_n) = 0$, Theorem 3.1 tells us that $\Phi_0(M_n)$ has trivial multiplications. The following provides us a sufficient and necessary condition for $\Phi_0(M_n)$ having trivial multiplications.

**Proposition 3.7.** *Let $\Phi : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ be an additive zero product preserver. When the underlying field $\mathbb{F}$ is an infinite field of characteristic 2, we assume in addition that $\Phi$ is $\mathbb{F}$-linear. The range of $\Phi$ has trivial multiplications exactly when $\Phi$ sends every scalar multiple of a rank one idempotent to a square zero element. In the case $\mathbb{F} = \mathbb{C}$, it is also equivalent to the condition that $\Phi$ sends every scalar multiple of a rank one projection to a square zero element.*

*Proof.* We verify the sufficiency only. Since every idempotent matrix is a sum of disjoint rank one idempotents, the assumption implies that $\Phi(\alpha E)^2 = 0$ for all idempotents $E$ in $M_n$ and $\alpha$ in $\mathbb{F}$. By Lemma 2.1(b), for every $X, Y$ in $M_n$ we can write their product as a linear sum of three idempotents, $XY = \beta_1 E_1 + \beta_2 E_2 + \beta_3 E_3$, say. In the case when 2 is invertible in $\mathbb{F}$, we see that each scalar $\beta = \left(\frac{\beta+1}{2}\right)^2 - \left(\frac{\beta-1}{2}\right)^2$. In the case when $\mathbb{F}$ is a finite field of characteristic 2, the map $\beta \mapsto \beta^2$ is an injective, and thus bijective, map from $\mathbb{F}$ onto $\mathbb{F}$. Thus in both cases we can assume that $\beta_k = \alpha_k^2 - \gamma_k^2$ for some $\alpha_k, \gamma_k$ in $\mathbb{F}$ for $k = 1, 2, 3$.

If $\Phi$ is assumed additive and $\mathbb{F}$ is not an infinite field of characteristic 2, then with (3.2) we have

$$\begin{aligned}\Phi(X)\Phi(Y) &= \Phi(I_n)\Phi((\alpha_1^2-\gamma_1^2)E_1+(\alpha_2^2-\gamma_2^2)E_2+(\alpha_3^2-\gamma_3^2)E_3)\\ &= \sum_{k=1}^{3}\Phi(I_n)\Phi((\alpha_k E_k)^2)-\sum_{k=1}^{3}\Phi(I_n)\Phi((\gamma_k E_k)^2)\\ &= \sum_{k=1}^{3}\Phi(\alpha_k E_k)^2-\sum_{k=1}^{3}\Phi(\gamma_k E_k)^2=0.\end{aligned}$$

For the exceptional case that $\mathbb{F}$ is an infinite field of characteristic 2, with the linearity of $\Phi$ it follows from (3.2) that

$$\Phi(X)\Phi(Y)=\Phi(I_n)\Phi(\beta_1E_1+\beta_2E_2+\beta_3E_3)=\sum_{k=1}^{3}\beta_k\Phi(I_n)\Phi(E_k)=\sum_{k=1}^{3}\beta_k\Phi(E_k)^2=0.$$

Finally, for the complex case, we note that every complex matrix is a linear sum of projections by Lemma 2.1(f). Since complex scalars have square roots, the above arguments bring us the desired conclusion. ■

The following theorem shows that even when $\Phi_0$ is linear and $\Phi_0(M_n(\mathbb{F}))$ has trivial multiplications, the structure of $\Phi_0$ can be quite liberal. Indeed, $\Phi_0$ can be any linear map from $M_n$ into a certain subspace $\mathbf{V}$ of $M_n$ satisfying $XY=0$ for any $X,Y$ in $\mathbf{V}$.

**Theorem 3.8.** *Suppose the underlying field $\mathbb{F}$ has more than $(l+2)/2$ elements. A linear map $\Phi: M_n\to M_l$ satisfies $\Phi(X)\Phi(Y)=0$ for any $X,Y$ in $M_n$ if and only if there is an invertible matrix $S_0$ in $M_l$ such that for all $A$ in $M_n$ the matrix $S_0^{-1}\Phi(A)S_0$ has the form*

$$\begin{pmatrix}0_p & Z_{12} & Z_{13}\\ 0_p & 0_p & 0\\ 0 & Z_{32} & 0_q\end{pmatrix}\quad \textit{with}\quad Z_{13}=\begin{pmatrix}\hat{Z}_{13} & 0_{u,q-v}\\ 0_{p-u,v} & 0_{p-u,q-v}\end{pmatrix}\quad \textit{and}\quad Z_{32}=\begin{pmatrix}0_{v,u} & 0_{v,p-u}\\ 0_{q-v,u} & \hat{Z}_{32}\end{pmatrix},$$

*for some nonnegative integers $p,q,u,v$.*

*Proof.* The sufficiency is clear. We consider the necessity. Suppose $X\in M_n$ such that $Y=\Phi(X)$ has the highest rank among the matrices in the range of $\Phi$. Because $Y^2=0$, we may apply a similarity transform and assume that

$$Y=\begin{pmatrix}0_p & I_p & 0\\ 0_p & 0_p & 0\\ 0 & 0 & 0_q\end{pmatrix}$$

with $2p+q=l$. Then for any $Z=\Phi(A)$, we have $ZY=YZ=0_l$. We see that

$$Z=\begin{pmatrix}0_p & Z_{12} & Z_{13}\\ 0_p & 0_p & 0\\ 0 & Z_{32} & Z_{33}\end{pmatrix}.$$

Clearly, $Z_{33} = 0_q$; for else, since $\mathbb{F}$ has more than $(l+2)/2 \geq p+1$ elements, there would be a nonzero scalar $\gamma$ such that

$$\gamma Y + Z = \begin{pmatrix} 0_p & \gamma I_p + Z_{12} & Z_{13} \\ 0_p & 0_p & 0 \\ 0 & Z_{32} & Z_{33} \end{pmatrix}$$

has rank larger than $p$, which contradicts to the choice of $Y$. Consequently, we can assume every $Z$ in $\Phi(M_n)$ carries the form

$$Z = \begin{pmatrix} 0_p & Z_{12} & Z_{13} \\ 0_p & 0_p & 0 \\ 0 & Z_{32} & 0_q \end{pmatrix}.$$

Consider the column spaces and row spaces of the matrix appearing as the $(1,3)$ block $Z_{13}$ of all $Z$ from $\Phi(M_n)$. There are invertible $P$ in $M_p$ and $Q$ in $M_q$ such that the first $u$ columns of $P$ span the sum of the column spaces of all these blocks, and the first $v$ rows of $Q$ span the sum of the row spaces of all these blocks. Thus, the $(1,3)$ blocks of all such $Z$ from $\Phi(M_n)$ always have the form

$$Z_{13} = P \begin{pmatrix} \hat{Z}_{13} & 0_{u,q-v} \\ 0_{p-u,v} & 0_{p-u,q-v} \end{pmatrix} Q.$$

Let $T = P \oplus I_p \oplus Q^{-1}$. Replace $\Phi$ with the map $A \mapsto T^{-1}\Phi(A)T$, we may assume that the $(1,3)$ block of $\Phi(A)$ always has the form

$$Z_{13} = \begin{pmatrix} \hat{Z}_{13} & 0_{u,q-v} \\ 0_{p-u,v} & 0_{p-u,q-v} \end{pmatrix}.$$

For any $B$ in $M_n$ and

$$W = \Phi(B) = \begin{pmatrix} 0_p & W_{12} & W_{13} \\ 0_p & 0_p & 0 \\ 0 & W_{32} & 0_q \end{pmatrix},$$

we have $ZW = 0_l$ for all $Z$ from $\Phi(M_n)$. Thus,

$$0_p = Z_{13}W_{32} = \begin{pmatrix} \hat{Z}_{13} & 0_{u,q-v} \\ 0_{p-u,v} & 0_{p-u,q-v} \end{pmatrix} W_{32}.$$

We see that the first $v$ rows of $W_{32}$ must be the zero row. Choose some $Z^{(j)}$ from $\Phi(M_n)$ such that the first column of the $(1,3)$ block $Z_{13}^{(j)}$ of $Z^{(j)}$ is the column vector $e_j$ with the $j$th entry 1 and other entries 0 for $j = 1, 2, \ldots, u$. Note that for all scalars $\alpha, \beta$, the matrix

$$\alpha Y + \beta Z^{(j)} + W = \begin{pmatrix} 0_p & \alpha I_p + \beta Z_{12}^{(j)} + W_{12} & \beta Z_{13}^{(j)} + W_{13} \\ 0_p & 0_p & 0 \\ 0 & \beta Z_{32}^{(j)} + W_{32} & 0_q \end{pmatrix}$$

has column rank at most $p$. Moreover, the $(1,3)$ block of the above matrix assumes the form

$$\begin{pmatrix} \beta \hat{Z}_{13}^{(j)} + \hat{W}_{13} & 0_{u,q-v} \\ 0_{p-u,v} & 0_{p-u,q-v} \end{pmatrix}.$$

Performing row operations on the first $p$ rows of the matrix $\alpha Y + \beta Z^{(j)} + W$, we will obtain another matrix such that the first column of its $(1,3)$ block is $e_j$ for all but one scalar $\beta$. Fix a choice of $\beta$ such that this happens. Note that this new matrix also has rank at most $p$. Because $\mathbb{F}$ contains more than $p+1$ elements, we can always find a scalar $\alpha$ such that the $\alpha I_p + \beta Z_{12}^{(j)} + W_{12}$ has rank $p$. This forces the $j$th column of $\beta Z_{32}^{(j)} + W_{32}$ is zero. Since we have more than one choices of $\beta$, the $j$th columns of both $Z_{32}^{(j)}$ and $W_{32}$ are zero, for $j = 1, 2, \ldots, u$. Consequently,

$$W_{32} = \begin{pmatrix} 0_{v,u} & 0_{v,p-u} \\ 0_{q-v,u} & \hat{W}_{32} \end{pmatrix}.$$

Hence, we conclude that $\Phi(A)$ has the asserted block form. ∎

The following examples show that for a zero product preserving linear map $\Phi : M_n \to M_r$, if $\Phi(I_n)$ is only a nilpotent matrix, the range of $\Phi$ might have nontrivial multiplications.

**Example 3.9.** Let the underlying field $\mathbb{F}$ be arbitrary. The linear map $\Phi : M_n \to M_{kn}$ defined by

$$A \mapsto \begin{pmatrix} 0_n & A & 0_n & \ldots & 0_n \\ 0_n & 0_n & A & \ldots & 0_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_n & 0_n & 0_n & \ldots & A \\ 0_n & 0_n & 0_n & \ldots & 0_n \end{pmatrix}$$

preserves zero products. The matrix $\Phi(I_n)$ is nilpotent such that $\Phi(I_n)^{k-1} \neq 0$ and $\Phi(I_n)^k = 0$. In particular, the range of $\Phi$ does not have trivial multiplications if $k > 2$.

In [3, Theorem 5.2], it is shown that every zero product preserving additive map $\Phi : M_n(\mathbb{D}) \to M_n(\mathbb{D})$ of matrices over a division ring $\mathbb{D}$ either has a range with trivial multiplications, or $\Phi(\cdot) = C\Psi(\cdot) = \Psi(\cdot)C$ for a ring endomorphism $\Psi$ and a matrix $C$. However, for those maps between matrices of different sizes we can have some wired examples.

**Example 3.10** (Based on [19, p. 310] and [6, Example 2.5])**.** Let the underlying field $\mathbb{F}$ be arbitrary. Consider $\Phi : M_n \to M_r$ with $r \geq n+2$ and $n \neq 1$ defined by

$$(a_{ij}) \mapsto \begin{pmatrix} 0 & a_{11} & \cdots & a_{1n} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a_{1n} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & a_{nn} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

The linear map $\Phi$ preserves zero products. Note that $\Phi(I_n)^2 = 0$. Since $\Phi(E)^2 \neq 0$ with $E = E_{11} + E_{1n}$, the image of $\Phi$ carries a nontrivial multiplication.

We claim that $\Phi$ cannot be written as the form $C\Psi$ for any $C$ in $M_r(\mathbb{F})$ and any homomorphism $\Psi : M_n(\mathbb{F}) \to M_r(\mathbb{F})$. Assume on the contrary that $\Phi = C\Psi$. Then we get a contradiction.

$$\begin{aligned}\Phi(E)^2 &= \Phi(E)C\Psi(E) = \Phi(E)C\Psi(E_{11}E) = \Phi(E)C\Psi(E_{11})\Psi(E)\\ &= \Phi(E)\Phi(E_{11})\Psi(E) = 0\Psi(E) = 0.\end{aligned}$$

When $r \le n+1$ and $n \ne 1$, we do have a good counterpart of [3, Theorem 5.2].

**Proposition 3.11.** *Suppose that $r \le n+1$ and $n \ne 1$. Let $\Phi : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ be an additive zero product preserver.*

*(a) If $\Phi(I_n)$ is not a nilpotent, then $r = n$ or $r = n+1$, and $\Phi$ has the form*

$$A \mapsto \alpha S(A_\tau \oplus 0_{r-n})S^{-1} \tag{3.12}$$

*for some nonzero scalar $\alpha$, an invertible matrix $S$ in $M_r(\mathbb{F})$, and a unital ring endomorphism $\tau$ of $\mathbb{F}$.*

*(b) If $\Phi(I_n)$ is a nilpotent, then the range of $\Phi$ always has trivial multiplications. In this case when the underlying field $\mathbb{F}$ is an infinite field of characteristic 2, we assume in addition that $\Phi$ is $\mathbb{F}$-linear.*

*Proof.* (a) We assume that $\Phi(I_n)$ is not a nilpotent. Then the 'algebraic part' $R$ given in Theorem 3.1 is an invertible matrix with rank at least $n$. It is then necessary that $r = n$ or $r = n+1$. In view of (3.11), $\Phi$ has the form

$$A \mapsto S(R\Phi_1(A) \oplus 0_{r-n})S^{-1} = S(\Phi_1(A)R \oplus 0_{r-n})S^{-1}$$

for an invertible matrix $S$ in $M_r(\mathbb{F})$, an invertible matrix $R$ in $M_n(\mathbb{F})$, and a unital ring homomorphism $\Phi_1 : M_n(\mathbb{F}) \to M_n(\mathbb{F})$. Arguing as in the proof of Theorem 3.3 we will establish (3.8) for the unital ring homomorphism $\Phi_1$. We have indeed

$$\Phi_1(E_{ij}) = E_{ij} \qquad \text{for all } i, j = 1, \ldots, n.$$

For any $a$ in $\mathbb{F}$, the matrix $\Phi_1(aI_n)$ commutes with all $\Phi_1(E_{ij}) = E_{ij}$. Thus, $\Phi_1(aI_n) = \tau(a)I_n$ for some scalar $\tau(a)$ in $\mathbb{F}$. It is easy to see that $a \mapsto \tau(a)$ is a unital ring homomorphism of $\mathbb{F}$. Observe that

$$\Phi_1(A) = \sum_{i,j=1}^{n} \Phi_1(a_{ij}E_{ij}) = \sum_{i,j=1}^{n} \Phi_1(a_{ij}I_n)\Phi_1(E_{ij}) = \sum_{i,j=1}^{n} \tau(a_{ij})E_{ij} = A_\tau,$$

where $A_\tau = (\tau(a_{ij}))$ if $A = (a_{ij})$. Since $R$ commutes with all $A_\tau$, in particular with all $E_{ij}$ since $\tau(1) = 1$, the invertible matrix $R = \alpha I_n$ for some nonzero scalar $\alpha$. Consequently, $\Phi$ has the form (3.12).

(b) Let $\Phi(I_n)$ be a nilpotent. Suppose on contrary that $\Phi(M_n)$ does not have trivial multiplications. By Proposition 3.7, $\Phi(\alpha E)^2 \ne 0$ for a rank one idempotent $E$ in $M_n$ and $\alpha \ne 0$ in $\mathbb{F}$. Let $\{e_1, \ldots, e_n\}$ be a basis of $\mathbb{F}^n$ consisting of eigenvectors of $E$ such that $Ee_1 = e_1$ and $Ee_j = 0$ for $j = 2, \ldots, n$. In this setting, we can write $E = E_{11}$, where $E_{ij}$ is the matrix unit of $M_n$ with respect to the basis $\{e_1, \ldots, e_n\}$.

Observe that

$$(\alpha E_{11} + \alpha E_{1j})(\alpha E_{11} - \alpha E_{j1}) = 0$$

implies

$$\Phi(\alpha E_{1j})\Phi(\alpha E_{j1}) = \Phi(\alpha E_{11})^2 \neq 0, \qquad j = 1, \dots, n, \tag{3.13}$$

and

$$\Phi(\alpha E_{ij})\Phi(\alpha E_{kl}) = 0, \quad \text{whenever } j \neq k, \text{ and } i, j, k, l = 1, \dots, n. \tag{3.14}$$

Since $\Phi(I_n)$ is a nilpotent, $\Phi(\alpha E_{11})$ is a nilpotent as well by Lemma 3.2(b). After a similarity transformation, we can assume that $\Phi(\alpha E_{11}) = J_1 \oplus \cdots \oplus J_m$ is a direct sum of its zero Jordan blocks. Since $\Phi(\alpha E_{11})^2 \neq 0$, we can further assume that $J_1$ is of size at least 3; namely,

$$J_1 = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Since $E_{1j}E_{11} = E_{11}E_{j1} = 0$, we see that the first and the second columns of $\Phi(\alpha E_{1j})$ are zero columns, and the second and the third rows of $\Phi(\alpha E_{j1})$ are zero rows for $j = 2, \dots, n$.

Denote by $R_j$ the first row of $\Phi(\alpha E_{1j})$, and by $C_j$ the third column of $\Phi(\alpha E_{j1})$ for $j = 2, 3, \dots, n$. Let

$$R = \begin{pmatrix} R_2 \\ R_3 \\ \vdots \\ R_n \end{pmatrix}_{(n-1)\times r} \qquad \text{and} \quad C = \begin{pmatrix} C_2 & C_3 & \cdots & C_n \end{pmatrix}_{r\times(n-1)}.$$

The conditions (3.13) and (3.14) tell us that $R_iC_j = 1$ whenever $i = j$, and 0 whenever $i \neq j$. In other words, $RC = I_{n-1}$. Note that the first and second columns of $R$ are both the zero columns. On the other hand, since the third row of $C$ is the zero row, we can replace the third column of $R$ by the zero column to get a new $(n-1)\times r$ matrix $R'$ such that $R'C = RC = I_{n-1}$. Therefore, $R'$ has rank $n-1$. Since the first three columns of $R'$ are zero, we have $r - 3 \geq n - 1$. This contradiction establishes the assertion. ∎

If $r > n + 1$ or $n = 1$, even a ring homomorphism from $M_n(\mathbb{F})$ into $M_r(\mathbb{F})$ can carry a far more complicated form. The following example tells us that Proposition 3.11 does not hold when $r = 2n = 2$.

**Example 3.12.** Let $\mathbb{F}$ be a purely transcendental extension over another field $\mathbb{K}$, for example $\mathbb{R}/\mathbb{Q}$. According to [25, Corollary 1' in p. 124], there is a nonzero additive derivation $x \mapsto x'$ of $\mathbb{F}$. Consider the unital ring homomorphism $\Phi : M_1(\mathbb{F}) \to M_2(\mathbb{F})$ defined by

$$(a) \mapsto \begin{pmatrix} a & a' \\ 0 & a \end{pmatrix}.$$

This does not carry the form as stated in (3.11).

## 4. Zero product preserving maps for other types of products

In this section, we will use the results and techniques in the last section to study zero product preservers and homomorphisms for other types of products.

### 4.1. **Jordan homomorphisms and *-homomorphisms.**

The following lemmas can be known, and we include them with short proofs here for completeness.

**Lemma 4.1.** *Let $\mathbb{F}$ be a field of characteristic not 2. Let $\Phi : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ be an additive map. If $\Phi$ is a Jordan homomorphism then $\Phi$ preserves double zero products; when $\mathbb{F}$ is the real or complex, $\Phi$ also preserves zero products of self-adjoint or symmetric elements, i.e.,*

$$\Phi(A)\Phi(B) = 0 \quad \textit{whenever} \quad AB = 0 \textit{ and both } A, B \textit{ are self-adjoint or symmetric.}$$

*Proof.* Note that a Jordan homomorphism between matrices preserves commutativity ( [9]). If $AB = BA = 0$ then $\Phi(AB + BA) = 0$ and thus $\Phi(A)\Phi(B) = \pm\Phi(B)\Phi(A) = 0$. On the other hand, if both $A, B$ are self-adjoint and $AB = 0$ then $BA = (AB)^* = 0$, and we have $\Phi(A)\Phi(B) = 0$ from above arguments. The case for symmetric matrices is similar. ■

**Lemma 4.2.** *Let $\mathbb{F}$ be any field. Let $\theta : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ be an additive Jordan homomorphism.*

(a) *If $\theta$ is not the zero map, then it is injective.*
(b) *There are two disjoint idempotents $P, Q$ in $M_r(\mathbb{F})$ such that*
   i. *$P\theta(A) = \theta(A)P$ and $Q\theta(A) = \theta(A)Q$ for all $A \in M_n(\mathbb{F})$,*
   ii. *the maps $\theta_1, \theta_2 : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ defined by $\theta_1(A) = \theta(A)P$ and $\theta_2(A) = \theta(A)Q$ are ring homomorphism and ring anti-homomorphism, respectively, such that $\theta = \theta_1 + \theta_2$.*
(c) *Suppose the underlying field is the complex $\mathbb{C}$. If $\theta$ is a linear Jordan *-homomorphism then we can choose $\theta_1, \theta_2$ above to be an algebra *-homomorphism and an algebra *-anti-homomorphism, respectively.*

*Proof.* We note that Jordan ideals of a matrix ring are two-sided ideals [9, Theorem 11]. Thus the kernel $\theta^{-1}(0)$ of $\theta$ is a two-sided ideal of the simple ring $M_n(\mathbb{F})$. If $\theta$ is not zero, then we see that its kernel is zero, and thus $\theta$ is injective. Moreover, it follows from [9, Theorem 7] that any additive Jordan homomorphism $\theta : M_n(\mathbb{F}) \to \mathcal{B}$ from the matrix ring $M_n(\mathbb{F})$ into another ring $\mathcal{B}$ is a sum of a ring homomorphism and a ring anti-homomorphism as stated.

On the other hand, any linear Jordan *-homomorphism $\theta : \mathcal{A} \to \mathcal{B}$ between C*-algebras such that $\theta(\mathcal{A})$ generates $\mathcal{B}$ is a sum of *-algebra homomorphism $A \mapsto \theta(A)P$ and a *-algebra anti-homomorphism $A \mapsto \theta(A)Q$ for an orthogonal pair of central projections $P, Q$ in $\mathcal{B}^{**}$ with $P + Q = 1$ [22, Theorem 3.3]. Thus the last assertion follows. ■

**Lemma 4.3.** *Let $\mathbb{F}$ be any field. Let $\theta : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ be an additive Jordan homomorphism. If $\theta$ preserves zero products, then $\theta$ is a ring homomorphism.*

*Proof.* By Lemma 4.2, we write $\theta = \theta_1 + \theta_2$ where $\theta_1(\cdot) = \theta(\cdot)P = P\theta(\cdot)$ is a ring homomorphism and $\theta_2(\cdot) = \theta(\cdot)Q = Q\theta(\cdot)$ is a ring anti-homomorphism with idempotents $P, Q$ such that

$PQ = QP = 0$. The goal is to assert that $\theta_2 = 0$. If it is not, choose any $A, B$ from $M_n$ such that $AB = 0$ but $BA \neq 0$. Since both $\theta$ and $\theta_1$ preserve zero products, we see

$$0 = \theta(A)\theta(B) = \theta_1(A)\theta_1(B) + \theta_2(A)\theta_2(B) = \theta_2(A)\theta_2(B) = \theta_2(BA).$$

Hence $\theta_2$ is not injective. By Lemma 4.2(a), $\theta_2 = 0$. ∎

**Theorem 4.4.** *Let $\mathbb{F}$ be a field. Suppose $\Phi : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ is a linear Jordan homomorphism.*

(a) *There exist nonnegative integers $k_1, k_2$ such that $t = r - nk_1 - nk_2 \geq 0$, and an invertible matrix $S$ in $M_r(\mathbb{F})$ such that $\Phi$ has the form*

$$A \mapsto S \begin{pmatrix} I_{k_1} \otimes A & & \\ & I_{k_2} \otimes A^{\mathrm{t}} & \\ & & 0_t \end{pmatrix} S^{-1}. \tag{4.1}$$

(b) *Assume $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. If $\Phi(A)^* = \Phi(A)$ for every rank one projection $A$, then $S$ can be chosen such that $S^{-1} = S^*$.*

(c) *Assume $\mathbb{F} = \mathbb{C}$. If $\Phi(A)$ is symmetric for every rank one real symmetric idempotent $A$, then $S$ can be chosen to be complex orthogonal.*

*Proof.* It suffices to verify the case when $n \geq 2$. It follows from Lemma 4.2 that there are idempotents $P, Q$ in $M_r$ such that $P + Q = I_r$, $PQ = QP = 0$ and $\Phi = \Phi_1 + \Phi_2$ is a direct sum of the algebra homomorphism $\Phi_1 = P\Phi$ and the algebra anti-homomorphism $\Phi_2 = Q\Phi$. Considering the algebra homomorphism $\Phi_2(\cdot)^{\mathrm{t}}$, we arrive at the conclusions with Theorem 3.3. ∎

4.2. **Idempotents and disjointness linear preservers.** A special case of the following result when $n = r$ has been obtained in [5, Theorem 8], while a more general case for matrices over a unital commutative ring can be found in [2, Theorem 2.1]. We provide here an elementary proof with a more detailed description of the map involved.

**Theorem 4.5.** *Assume that the underlying field $\mathbb{F}$ has characteristic not two. Let $\Phi : M_n \to M_r$ be a linear map preserving idempotents. Then $\Phi$ is a Jordan homomorphism, and there exist an invertible $S$ in $M_r$, and nonnegative integers $k_1, k_2$ such that $t = r - nk_1 - nk_2 \geq 0$, and*

$$\Phi(A) = S \begin{pmatrix} I_{k_1} \otimes A & & \\ & I_{k_2} \otimes A^{\mathrm{t}} & \\ & & 0_t \end{pmatrix} S^{-1}.$$

*Proof.* We first note that $\Phi(I_n)$ is an idempotent. Observe that for idempotents $P, Q$, they are orthogonal to each other exactly when $P + Q$ is again an idempotent. Therefore, $\Phi$ sends disjoint idempotents $P, Q$ in $M_n$ to disjoint idempotents $\Phi(P), \Phi(Q)$ in $M_r$. In particular, for every idempotent $P$ in $M_n$ we have

$$\Phi(I_n)\Phi(P) = (\Phi(P) + \Phi(I_n - P))\Phi(P) = \Phi(P) = \Phi(P)\Phi(I_n).$$

By Lemma 2.1(b), we see that

$$\Phi(A) = \Phi(I_n)\Phi(A) = \Phi(A)\Phi(I_n), \qquad \text{for all } A \in M_n.$$

Therefore, if we change $\Phi(\cdot)$ to $S_0^{-1}\Phi(\cdot)S_0$ for some suitable invertible $S_0$ in $M_r$, we can assume $\Phi(I_n) = I_s \oplus 0_{r-s}$ and $\Phi(A) = A' \oplus 0_{r-s}$ for some $A'$ in $M_s$, where $s$ is the rank of $\Phi(I_n)$.

In the following, we assume further that $s = r$, and in particular, $\Phi(I_n) = I_r$. Since $\Phi$ sends disjoint idempotents in $M_n$ to such in $M_r$, it follows from Lemma 2.1(e) that

$$\Phi(A^2) = \Phi(A)^2, \quad \text{for all real symmetric matrix } A \in M_n. \tag{4.2}$$

In particular,

$$\Phi(E_{ii})\Phi(E_{jj}) = \delta_{ij}\Phi(E_{ii}), \quad i, j = 1, 2, \ldots, n. \tag{4.3}$$

Moreover,

$$I_r = \Phi(I_n) = \sum_{i=1}^{n} \Phi(E_{ii}).$$

Replacing $\Phi$ by the map $X \mapsto S_1^{-1}\Phi(X)S_1$ for some invertible $S_1$ in $M_r$, we can assume that the idempotents

$$\Phi(E_{ii}) = 0_{k_1} \oplus \cdots \oplus I_{k_i} \oplus 0_{k_{i+1}} \oplus \cdots \oplus 0_{k_n}, \quad i = 1, \ldots, n.$$

Here, $k_1 + \cdots + k_n = r$.

Since $E_{11} + E_{12}$, $E_{22} + E_{12}$, $E_{11} + E_{21}$ and $E_{22} + E_{21}$ are all idempotents and have pairwise zero products with $E_{ii}$ for $i = 3, \ldots, n$, we see that $\Phi(E_{11}) + \Phi(E_{12})$, $\Phi(E_{22}) + \Phi(E_{12})$, $\Phi(E_{11}) + \Phi(E_{21})$ and $\Phi(E_{22}) + \Phi(E_{21})$ are all idempotents and have zero products with $\Phi(E_{ii})$ for $i = 3, \ldots, n$. This forces

$$\Phi(E_{12}) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \oplus 0_s \quad \text{and} \quad \Phi(E_{21}) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \oplus 0_s,$$

where $s = r - k_1 - k_2$, $B_{ij}, C_{ij}$ are $k_i \times k_j$ matrices for $i, j = 1, 2$. Moreover,

$$\begin{aligned} \begin{pmatrix} I_{k_1} + B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} &= \begin{pmatrix} I_{k_1} + B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}^2 \\ &= \begin{pmatrix} I_{k_1} + 2B_{11} + B_{11}^2 + B_{12}B_{21} & B_{12} + B_{11}B_{12} + B_{12}B_{22} \\ B_{21} + B_{21}B_{11} + B_{22}B_{21} & B_{21}B_{12} + B_{22}^2 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & I_{k_2} + B_{22} \end{pmatrix} &= \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & I_{k_2} + B_{22} \end{pmatrix}^2 \\ &= \begin{pmatrix} B_{11}^2 + B_{12}B_{21} & B_{11}B_{12} + B_{12} + B_{12}B_{22} \\ B_{21}B_{11} + B_{21} + B_{22}B_{21} & B_{21}B_{12} + I_{k_2} + 2B_{22} + B_{22}^2 \end{pmatrix}. \end{aligned}$$

It follows

$$B_{11} = 0_{k_1}, \quad B_{22} = 0_{k_2}, \quad B_{21}B_{12} = 0_{k_2}, \quad \text{and} \quad B_{12}B_{21} = 0_{k_1}.$$

Similarly, we have

$$C_{11} = 0_{k_1}, \quad C_{22} = 0_{k_2}, \quad C_{21}C_{12} = 0_{k_2}, \quad \text{and} \quad C_{12}C_{21} = 0_{k_1}.$$

Thus we can write

$$\Phi(E_{ij}) = \begin{pmatrix} 0 & X_{ij} \\ Y_{ij} & 0 \end{pmatrix} \oplus 0_s, \quad \text{with } X_{ij}Y_{ij} = 0_{k_1},\ Y_{ij}X_{ij} = 0_{k_2} \quad \text{for all } i \neq j,\ i,j = 1,2. \tag{4.4}$$

Because $E_{12} + E_{21}$ is real symmetric and $(E_{12} + E_{21})^2 = E_{11} + E_{22}$, by (4.2), we have

$$\begin{pmatrix} 0 & X_{12} + X_{21} \\ Y_{12} + Y_{21} & 0 \end{pmatrix}^2 = \begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix}.$$

Consequently,

$$\begin{aligned} X_{ij}Y_{ji} + X_{ji}Y_{ij} &= I_{k_i}, \\ Y_{ij}X_{ji} + Y_{ji}X_{ij} &= I_{k_j}, \quad i < j,\ i,j = 1,2. \end{aligned} \tag{4.5}$$

By comparing traces, we see that $k_1 = k_2$.

The above discussions hold for all pairs $i, j$ of distinct indices. We thus conclude that

$$k_1 = k_2 = \cdots = k_n = k$$

for a common value $k$ such that $nk = r$. Consequently, $\Phi(E_{ii}) = E_{ii} \otimes I_k$ for $i = 1, \ldots, n$. Moreover, (4.4) and (4.5) hold for all distinct indices $i, j$ from $1, 2, \ldots, n$. It follows for $A = (a_{ij})$ in $M_n$ that

$$\Phi(A) = \sum_i a_{ii}E_{ii} \otimes I_k + \sum_{i<j} a_{ij}E_{ij} \otimes X_{ij} + \sum_{i<j} a_{ij}E_{ji} \otimes Y_{ij} + \sum_{i>j} a_{ij}E_{ji} \otimes X_{ij} + \sum_{i>j} a_{ij}E_{ij} \otimes Y_{ij}.$$

By (4.5), we then see that

$$\Phi(A^2) = \Phi(A)^2.$$

Therefore, $\Phi$ is a Jordan homomorphism from $M_n$ into $M_r$. The desired assertion follows from Theorem 4.4. ■

**Theorem 4.6.** *Assume that the underlying field $\mathbb{F}$ has characteristic not two. Let $\Phi : M_n \to M_r$ be a linear map such that $\Phi$ preserves double zero products, i.e.,*

$$\Phi(A)\Phi(B) = \Phi(B)\Phi(A) = 0 \quad \textit{whenever} \quad A, B \in M_n \textit{ satisfies } AB = BA = 0.$$

*Then there exist nonnegative integers $k_1, k_2$ such that $t = r - nk_1 - nk_2 \geq 0$, and invertible matrices $S$ in $M_r$, $R_1$ in $M_{k_1}$ and $R_2$ in $M_{k_2}$ such that $\Phi$ has the form*

$$A \mapsto S \begin{pmatrix} R_1 \otimes A & 0 & 0 \\ 0 & R_2 \otimes A^{\mathrm{t}} & 0 \\ 0 & 0 & \Phi_0(A) \end{pmatrix} S^{-1}.$$

*If $\Phi(I_n)$ has nil index $\nu$, then the double zero product preserving linear map $\Phi_0 : M_n \to M_t$ satisfying that $\Phi_0(P)^{\nu+1} = 0$ for every idempotent $P$ in $M_n$.*

*In the complex (resp. real) case, $\Phi_0 : M_n \to M_t$ is a linear map preserving Jordan zero products (resp. Jordan zero products of symmetric matrices). If $\nu = 0$, especially when $\Phi(I_n)$ is diagonalizable, then*

$$\Phi_0(X)\Phi_0(Y) + \Phi_0(Y)\Phi_0(X) = 0_t \quad \textit{for all} \quad \textit{(resp. symmetric) } X, Y \in M_n.$$

*Proof.* Observe that for any idempotent $P$ in $M_n$, we have

$$P(I_n - P) = (I_n - P)P = 0.$$

Thus

$$\Phi(P)(\Phi(I_n) - \Phi(P)) = (\Phi(I_n) - \Phi(P))\Phi(P) = 0.$$

This gives

$$\Phi(P)\Phi(I_n) = \Phi(P)^2 = \Phi(I_n)\Phi(P). \tag{4.6}$$

Since every $A$ in $M_n$ is a linear sum of three idempotents (Lemma 2.1(b)),

$$\Phi(I_n)\Phi(A) = \Phi(A)\Phi(I_n) \quad \text{for all } A \in M_n.$$

As argued in the proof of Theorem 3.1, we write $\Phi = \Phi_1 \oplus \Phi_0$, and define $\Psi(\cdot) = \Phi_1(\cdot)R^{-1}$. By (4.6) we see that $\Psi$ is a linear map from $M_n$ into $M_s$ preserving idempotents. By Theorem 4.5, $\Psi$ is a unital Jordan homomorphism, and thus $\Phi$ is given in the stated form. Moreover, it follows again from (4.6) that $\Phi_0(P)^{\nu+1} = \Phi_0(I_n)^\nu\Phi_0(P) = 0$ for all idempotents $P$ in $M_n$.

Assume now that the underlying field is $\mathbb{C}$ (resp. $\mathbb{R}$). It follows from Lemma 2.1(f) (resp. (e)) and (4.6) that

$$\Phi(I_n)\Phi(A^2) = \Phi(A)^2 \quad \text{for all self-adjoint (resp. symmetric) } A \text{ in } M_n. \tag{4.7}$$

Note that $A + B$ is self-adjoint (resp. symmetric) whenever both $A, B$ are. We have

$$\Phi(I_n)\Phi((A + B)^2) = \Phi(A + B)^2,$$

and thus

$$\Phi(I_n)\Phi(AB + BA) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A), \tag{4.8}$$

for all self-adjoint (resp. symmetric) $A, B$ in $M_n$. Because $(A + \mathrm{i}B)^2 = A^2 + \mathrm{i}(AB + BA) + B^2$ for any self-adjoint matrices $A, B$, both (4.7) and (4.8) are true for all matrices $A, B$ in $M_n$ in the complex case. With (4.8), we see that $\Phi$, and thus also $\Phi_0$, sends pairs of (resp. symmetric) matrices with zero Jordan products to pairs with zero Jordan products. Finally, if $\nu = 0$, namely, $\Phi_0(I_n) = 0$, then $\Phi_0(A)\Phi_0(B) + \Phi_0(B)\Phi_0(A) = 0$ for any (resp. symmetric) matrices $A, B$ in $M_n$ by (4.8). ∎

The general case of the following result is known to $C^*$-algebraists (see, e.g., [17, Theorem 3.6]). We include an easy proof for the special case of complex matrices for completeness.

**Theorem 4.7.** *A complex linear map $\Phi : M_n \to M_r$ preserves range orthogonality, i.e.,*

$$\Phi(A)^*\Phi(B) = 0 \quad \textit{whenever} \quad A, B \in M_n \textit{ satisfies } A^*B = 0,$$

*if and only if there are matrices $S, T$ in $M_r$ such that $S^*S = I_r$ and*

$$\Phi(A) = S\begin{pmatrix} I_k \otimes A & 0 \\ 0 & 0_{r-nk} \end{pmatrix}T, \quad \textit{for all} \quad A \in M_n.$$

*Proof.* We first claim that

$$\Phi(I_n)^*\Phi(A^*B) = \Phi(A)^*\Phi(B) \quad \text{for all } A, B \in M_n. \tag{4.9}$$

Indeed, for any orthogonal projections $P, Q$ in $M_n$, since $P^*(QB) = Q^*(PB) = 0$, we have

$$\Phi(P)^*\Phi(QB) = \Phi(Q)^*\Phi(PB) = 0 \quad \text{for all } \ B \in M_n.$$

It follows

$$\Phi(P+Q)^*\Phi(QB) = \Phi(Q)^*\Phi(QB) = \Phi(Q)^*\Phi((P+Q)B) \quad \text{for all } \ B \in M_n.$$

In particular,

$$\Phi(I_n)^*\Phi(QB) = \Phi(Q)^*\Phi(B) \quad \text{for all } B \in M_n.$$

Since every complex matrix $A$ can be written as a linear sum of projections, we establish (4.9). In particular, if $\Phi(I_n) = 0$ then $\Phi$ is a zero map, and the assertions hold trivially. So assume the rank $s$ of $\Phi(I_n)$ is positive below.

Let $H$ be the column space of $\Phi(I_n)$, i.e., the linear span of all column vectors in $\Phi(I_n)$ in $\mathbb{C}^r$. It follows from (4.9) that

$$\Phi(I_n)^*\Phi(A^*A) = \Phi(A)^*\Phi(A) = \Phi(A^*A)^*\Phi(I_n) \quad \text{for all } \ A \in M_n.$$

In particular, $\Phi(A)x = 0$ whenever $\Phi(I_n)x = 0$ for all $x$ in $\mathbb{C}^r$. Therefore, we can define an $\pi(A)$ in $M_r$ by setting

$$\pi(A)\Phi(I_n)x = \Phi(A)x \quad \text{for all } x \in \mathbb{C}^r,$$

and $\pi(A)y = 0$ for any $y$ in the orthogonal complement of $H$ in $\mathbb{C}^r$.

Observe the $\mathbb{C}^r$ inner products

$$\begin{aligned}
\langle \pi(A^*B)\Phi(I_n)x, \Phi(I_n)y\rangle &= \langle \Phi(A^*B)x, \Phi(I_n)y\rangle = \langle \Phi(I_n)^*\Phi(A^*B)x, y\rangle \\
&= \langle \Phi(A)^*\Phi(B)x, y\rangle = \langle \Phi(B)x, \Phi(A)y\rangle \qquad (\text{by } (4.9)) \\
&= \langle \pi(B)\Phi(I_n)x, \pi(A)\Phi(I_n)y\rangle = \langle \pi(A)^*\pi(B)\Phi(I_n)x, \Phi(I_n)y\rangle,
\end{aligned}$$

for all $x, y$ in $\mathbb{C}^r$ and $A, B$ in $M_n$. Hence,

$$\pi(A^*B) = \pi(A)^*\pi(B) \quad \text{for all } \ A, B \in M_n.$$

Therefore, $\pi : M_n \to M_r$ is an algebra $*$-homomorphism. It then follows from Theorem 3.3 that there exist an integer $k$ and a unitary matrix $S$ in $M_r$ such that

$$\pi(A) = S \begin{pmatrix} I_k \otimes A & 0 \\ 0 & 0_{r-kn} \end{pmatrix} S^*.$$

By construction,

$$\Phi(A) = \pi(A)\Phi(I_n) = S \begin{pmatrix} I_k \otimes A & 0 \\ 0 & 0_{r-kn} \end{pmatrix} S^*\Phi(I_n) \quad \text{for all } \ A \in M_n.$$

Setting $T = S^*\Phi(I_n)$, we complete the proof. ■

The following can be considered as an enhanced version of a special case of the general results about orthogonality preserving linear maps of JB*-triples discussed in, e.g., [4]. When $\Phi$ is surjective, it is also discussed in [23] (see also [12, Theorem 2.2]), which applies indeed for general C*-algebras.

**Theorem 4.8.** *Let $\Phi : M_n \to M_r$ be a complex linear map preserving double orthogonality, i.e.,*

$$\Phi(A)\Phi(B)^* = \Phi(B)^*\Phi(A) = 0 \quad \textit{whenever} \quad A, B \in M_n \textit{ satisfy } AB^* = B^*A = 0.$$

*Suppose that $\Phi(I_n)$ is a self-adjoint matrix in $M_r$ of rank $s$. Then there are nonnegative integers $k_1, k_2$ such that $s = nk$ where $k = k_1 + k_2$, and there is a unitary $S$ in $M_r$, and invertible self-adjoint matrices $R_1$ in $M_{k_1}$, $R_2$ in $M_{k_2}$ such that*

$$\Phi(A) = S \begin{pmatrix} R_1 \otimes A & 0 & 0 \\ 0 & R_2 \otimes A^{\mathrm{t}} & 0 \\ 0 & 0 & 0_{r-nk} \end{pmatrix} S^* \quad \textit{for all } A \in M_n. \tag{4.10}$$

*Proof.* Let $P, Q$ be orthogonal projections in $M_n$. By the double orthogonality preserving property of $\Phi$, we have

$$\Phi(P)^*\Phi(Q) = \Phi(Q)\Phi(P)^* = 0.$$

Putting $P = I_n - Q$, we have

$$(\Phi(I_n)^* - \Phi(Q)^*)\Phi(Q) = \Phi(Q)(\Phi(I_n)^* - \Phi(Q)^*) = 0.$$

Since $\Phi(I_n)^* = \Phi(I_n)$, we have

$$\begin{aligned} \Phi(I_n)\Phi(Q) &= \Phi(Q)^*\Phi(Q) = \Phi(Q)^*\Phi(I_n), \\ \Phi(Q)\Phi(I_n) &= \Phi(Q)\Phi(Q)^* = \Phi(I_n)\Phi(Q)^*. \end{aligned} \tag{4.11}$$

Hence

$$\Phi(I_n)^2\Phi(Q) = \Phi(I_n)\Phi(Q)^*\Phi(I_n) = \Phi(Q)\Phi(I_n)^2.$$

Write the self-adjoint matrix $\Phi(I_n) = \Phi(I_n)_+ - \Phi(I_n)_-$ as the orthogonal difference of its positive and negative parts. The fact that $\Phi(Q)$ commutes with $\Phi(I_n)^2 = \Phi(I_n)_+^2 + \Phi(I_n)_-^2$ implies that $\Phi(Q)$ commutes with both $\Phi(I_n)_+$ and $\Phi(I_n)_-$, and thus also with $\Phi(I_n)$. By Lemma 2.1(f), $\Phi(I_n)$ commutes with $\Phi(A)$ for any $A$ in $M_n$. It follows from (4.11) that both the left and right support projections of $\Phi(Q)$ is dominated by the support projection of the self-adjoint $\Phi(I_n)$. Thus, it is also true for $\Phi(A)$ for any $A$ in $M_n$.

Choose a unitary matrix $S_1$ from $M_r$ such that $S_1^*\Phi(I_n)S_1 = D_1 \oplus 0_{r-s}$, where $D_1$ is an $s \times s$ invertible diagonal matrix with all nonzero (real) eigenvalues on the diagonal. From above we see that $S_1^*\Phi(A)S_1 = \Phi_1(A) \oplus 0_{r-s}$ for a linear map $\Phi_1 : M_n \to M_s$. Clearly, $\Phi_1$ also preserves double orthogonality and $\Phi_1(I_n) = D_1$. Moreover,

$$D_1\Phi_1(A) = \Phi_1(A)D_1 \quad \text{for all} \quad A \in M_n.$$

Let $\pi : M_n \to M_s$ be defined by

$$\pi(A) = D_1^{-1}\Phi_1(A) \quad \text{for all} \quad A \in M_n.$$

It is clear that $\pi$ preserves double orthogonality. Therefore, $\pi$ satisfies (4.11) as $\Phi$ does. Since $\pi(I_n) = I_s$, we have

$$\pi(Q) = \pi(Q)^*\pi(Q) = \pi(Q)^*$$

for every projection $Q$ in $M_n$. Therefore, $\pi$ sends projections to projections. By Lemma 2.2(b), $\pi$ is a unital Jordan *-homomorphism from $M_n$ into $M_s$.

It follows from Theorem 4.4 that there are nonnegative integers $k_1, k_2$ such that $s = nk$ where $k = k_1 + k_2$, and there is a unitary $U_1$ in $M_{nk}$ such that

$$\pi(A) = U_1 \begin{pmatrix} I_{k_1} \otimes A & 0 \\ 0 & I_{k_2} \otimes A^{\mathrm{t}} \end{pmatrix} U_1^* \quad \text{for all } A \in M_n.$$

Let

$$S = S_1 \begin{pmatrix} U_1 & 0 \\ 0 & I_{r-nk} \end{pmatrix} \quad \text{and} \quad R = U_1^* D_1 U_1.$$

Then

$$\begin{aligned}\Phi(A) &= S_1 \begin{pmatrix} \Phi_1(A) & 0 \\ 0 & 0 \end{pmatrix} S_1^* = S_1 \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \pi(A) & 0 \\ 0 & 0 \end{pmatrix} S_1^* \\ &= S_1 \begin{pmatrix} D_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ 0 & I_{r-nk} \end{pmatrix} \begin{pmatrix} I_{k_1} \otimes A & 0 & 0 \\ 0 & I_{k_2} \otimes A^{\mathrm{t}} & 0 \\ 0 & 0 & 0_{r-nk} \end{pmatrix} \begin{pmatrix} U_1 & 0 \\ 0 & I_{r-nk} \end{pmatrix}^* S_1^* \\ &= S \begin{pmatrix} R & 0 \\ 0 & 0_{r-nk} \end{pmatrix} \begin{pmatrix} I_{k_1} \otimes A & 0 & 0 \\ 0 & I_{k_2} \otimes A^{\mathrm{t}} & 0 \\ 0 & 0 & 0_{r-nk} \end{pmatrix} S^*.\end{aligned}$$

Moreover, by construction the self-adjoint matrix $R$ satisfies

$$R \begin{pmatrix} I_{k_1} \otimes A & 0 \\ 0 & I_{k_2} \otimes A^{\mathrm{t}} \end{pmatrix} = \begin{pmatrix} I_{k_1} \otimes A & 0 \\ 0 & I_{k_2} \otimes A^{\mathrm{t}} \end{pmatrix} R \quad \text{for all } A \in M_n.$$

Thus

$$R = \begin{pmatrix} R_1 \otimes I_n & 0 \\ 0 & R_2 \otimes I_n \end{pmatrix}$$

for some self-adjoint invertible matrices $R_1$ in $M_{k_1}$ and $R_2$ in $M_{k_2}$. This establishes the assertion (4.10). ■

Denote by $H_n$ the real linear space of self-adjoint matrices in $M_n(\mathbb{C})$.

**Theorem 4.9.** *Let $\Phi : H_n \to M_r(\mathbb{C})$ be a real linear map preserving zero products. Then*

- *there are nonnegative integers $k_1, k_2$ such that $s = n(k_1 + k_2)$ is the rank of $\Phi(I_n)$,*
- *there are invertible matrices $S$ in $M_r(\mathbb{C})$, $R_1$ in $M_{k_1}(\mathbb{C})$ and $R_2$ in $M_{k_2}(\mathbb{C})$, and*
- *there is a real linear map $\Phi_0 : H_n \to M_{r-s}$ preserving zero Jordan products,*

*such that $\Phi$ carries the form*

$$A \mapsto S \begin{pmatrix} R_1 \otimes A & 0 & 0 \\ 0 & R_2 \otimes A^{\mathrm{t}} & 0 \\ 0 & 0 & \Phi_0(A) \end{pmatrix} S^{-1}.$$

*If the nilpotent part of $\Phi(I_n)$ is zero, then*

$$\Phi_0(X)\Phi_0(Y) + \Phi_0(Y)\Phi_0(X) = 0 \quad \text{for all } X, Y \in H_n. \tag{4.12}$$

*Furthermore, if $\Phi(H_n) \subseteq H_r$, i.e., $\Phi(A)^* = \Phi(A)$ for all $A$ in $H_n$, then $S$ can be chosen to be unitary, $R_1 = R_1^*$, $R_2 = R_2^*$, and $\Phi$ carries the form*

$$A \mapsto S \begin{pmatrix} R_1 \otimes A & 0 & 0 \\ 0 & R_2 \otimes A^{\mathrm{t}} & 0 \\ 0 & 0 & 0 \end{pmatrix} S^*.$$

*Proof.* For any projection $P$ in $M_n$, we have $P(I_n - P) = (I_n - P)P = 0$, which implies $\Phi(P)\Phi(I_n - P) = \Phi(I_n - P)\Phi(P) = 0$. Hence,

$$\Phi(I_n)\Phi(P) = \Phi(P)^2 = \Phi(P)\Phi(I_n).$$

By Lemma 2.1(f),

$$\Phi(I_n)\Phi(A^2) = \Phi(A)^2 = \Phi(A^2)\Phi(I_n) \quad \text{for all } A \in H_n. \tag{4.13}$$

We choose an invertible $S_1$ from $M_r$ such that $S_1^{-1}\Phi(I_n)S_1 = R \oplus N$, where $R$ is an $s \times s$ invertible matrix and $N$ is an $(r-s)\times(r-s)$ nilpotent matrix. Replacing $\Phi(\cdot)$ with $S_1^{-1}\Phi(\cdot)S_1$, we can assume that $\Phi(I_n) = R \oplus N$. Note that $N = 0_{r-s}$ and we can choose $S_1$ to be unitary and $R = R^*$ to be a real diagonal invertible matrix if $\Phi(I_n)$ is self-adjoint. Because every self-adjoint matrix is the difference of two positive matrices, and positive matrices have positive square roots, it follows from (4.13) that $\Phi(I_n)\Phi(A) = \Phi(A)\Phi(I_n)$ for all $A$ in $H_n$.

Arguing as in the proof of Theorem 3.1, we can write $\Phi(A) = \Phi_1(A) \oplus \Phi_0(A)$, where $\Phi_1 : H_n \to M_s$ is a zero product preserving real linear map such that $R\Phi_1(A) = \Phi_1(A)R$ for all $A$ in $H_n$, and $\Phi_0 : H_n \to M_{r-s}$ is a zero product real linear map into nilpotent matrices. When $\Phi(H_n) \subseteq H_r$, we see that $\Phi_0$ is the zero map. In general, if $N = 0$ then by (4.13) we establish (4.12).

Consider the map $\Phi_2(\cdot) = R^{-1}\Phi_1(\cdot) = \Phi_1(\cdot)R^{-1}$. We see that $\Phi_2$ is a unital zero product preserving real linear map from $H_n$ into $M_s$. It follows from (4.13) again that $\Phi_2$ is a unital real linear Jordan homomorphism. Extend $\Phi_2$ to a complex linear map $\Phi_3 : M_n \to M_s$ by setting

$$\Phi_3(A + iB) = \Phi_2(A) + i\Phi_2(B) \quad \text{for all } A, B \in H_n.$$

Then $\Phi_3$ is a unital complex linear Jordan homomorphism. It follows from Theorem 4.4 that $\Phi_3$ assumes the form

$$A \mapsto S_2[(I_{k_1} \otimes A) \oplus (I_{k_2} \otimes A^{\mathrm{t}})]S_2^{-1}$$

for some invertible $S_2$ in $M_s$ and some nonnegative integers $k_1, k_2$ such that $s = n(k_1 + k_2)$. When $\Phi$, and thus $\Phi_2$, sends into self-adjoint matrices, $\Phi_3$ is a Jordan *-homomorphism, and we can assume $S_2$ is unitary.

Consequently,

$$\begin{aligned}
\Phi(A) &= S_1(R \oplus I_{r-s})(S_2 \oplus I_{r-s}) \begin{pmatrix} I_{k_1} \otimes A & 0 & 0 \\ 0 & I_{k_2} \otimes A^{\mathrm{t}} & 0 \\ 0 & 0 & \Phi_0(A) \end{pmatrix} (S_2^{-1} \oplus I_{r-s}) S_1^{-1} \\
&= S_1(S_2 \oplus I_{r-s})(S_2^{-1} R S_2 \oplus I_{r-s}) \begin{pmatrix} I_{k_1} \otimes A & 0 & 0 \\ 0 & I_{k_2} \otimes A^{\mathrm{t}} & 0 \\ 0 & 0 & \Phi_0(A) \end{pmatrix} (S_2^{-1} \oplus I_{r-s}) S_1^{-1} \\
&= S(S_2^{-1} R S_2 \oplus I_{r-s}) \begin{pmatrix} I_{k_1} \otimes A & 0 & 0 \\ 0 & I_{k_2} \otimes A^{\mathrm{t}} & 0 \\ 0 & 0 & \Phi_0(A) \end{pmatrix} S^{-1} \\
&= S(R' \oplus I_{r-s}) \begin{pmatrix} I_{k_1} \otimes A & 0 & 0 \\ 0 & I_{k_2} \otimes A^{\mathrm{t}} & 0 \\ 0 & 0 & \Phi_0(A) \end{pmatrix} S^{-1}.
\end{aligned}$$

Here, $S = S_1(S_2 \oplus I_{r-s})$ is invertible in $M_r$ and $R' = S_2^{-1} R S_2$ in $M_s$ satisfies

$$R' \begin{pmatrix} I_{k_1} \otimes A & 0 \\ 0 & I_{k_2} \otimes A^{\mathrm{t}} \end{pmatrix} = \begin{pmatrix} I_{k_1} \otimes A & 0 \\ 0 & I_{k_2} \otimes A^{\mathrm{t}} \end{pmatrix} R' \quad \text{for all } A \in H_n.$$

Hence,

$$R' = \begin{pmatrix} R_1 \otimes I_n & 0 \\ 0 & R_2 \otimes I_n \end{pmatrix}$$

for some invertible matrices $R_1$ in $M_{k_1}$ and $R_2$ in $M_{k_2}$. When $\Phi$ has self-adjoint images, we can assume that $S$ is unitary and $R_1, R_2$ are self-adjoint. The assertions follow. ■

Denote by $S_n(\mathbb{F})$ the set of $n \times n$ symmetric matrices in $M_n(\mathbb{F})$.

**Theorem 4.10.** *Let $\Phi : S_n(\mathbb{C}) \to M_r(\mathbb{C})$ be a zero product preserving complex linear map. The following are equivalent.*

(a) *$\Phi(A)$ is a (resp. symmetric) idempotent whenever $A$ is a rank one idempotent in $S_n(\mathbb{R})$.*

(b) *There is a nonnegative integer $k$ and an invertible (resp. complex orthogonal) matrix $S$ in $M_r$ such that $\Phi$ has the form*

$$A \mapsto S^{-1} \begin{pmatrix} I_k \otimes A & 0 \\ 0 & 0_{r-kn} \end{pmatrix} S.$$

*Proof.* We verify the implication (a) $\implies$ (b) only. By (a), $\Phi(E_{ii})$ are all idempotents and $\Phi(E_{ii})\Phi(E_{jj}) = 0$ if $i \neq j$. Hence, $\Phi(I_n) = \sum_{i=1}^n \Phi(E_{ii})$ is an idempotent. Assume $S$ in $M_r(\mathbb{C})$ is invertible and that $S^{-1}\Phi(I_n)S = I_s \oplus 0_{r-s}$. As in the proof of Theorem 3.3, replacing $\Phi(\cdot)$ with $S^{-1}\Phi(\cdot)S$, we can assume that $r = s$, $\Phi(I_n) = I_s$ and

$$\Phi(E_{ii}) = 0_{k_1} \oplus \cdots \oplus I_{k_i} \oplus 0_{k_{i+1}} \oplus \cdots \oplus 0_{k_n}, \quad i = 1, \ldots, n.$$

Here, $k_1 + k_2 + \cdots + k_n = s$.

Let

$$B = \Phi(E_{12} + E_{21}) = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \oplus 0_{s'},$$

where $B_{ij}$ are $k_i \times k_j$ complex matrices for $i, j = 1, 2$, and $s' = s - k_1 - k_2$. For any nonzero real $\gamma$, consider

$$X_1 = \begin{pmatrix} \gamma & 1 \\ 1 & 1/\gamma \end{pmatrix} \oplus 0_{n-2} \quad \text{and} \quad X_2 = \begin{pmatrix} 1/\gamma & -1 \\ -1 & \gamma \end{pmatrix} \oplus 0_{n-2}.$$

Because $X_1 X_2 = 0_n$, we see that

$$\begin{aligned} 0_s &= \Phi(X_1)\Phi(X_2) = (\Phi(\gamma E_{11} + E_{22}/\gamma) + B)(\Phi(E_{11}/\gamma + \gamma E_{22}) - B) \\ &= \left[\begin{pmatrix} \gamma I_{k_1} & 0 \\ 0 & I_{k_2}/\gamma \end{pmatrix} \oplus 0_{s'} + B\right]\left[\begin{pmatrix} I_{k_1}/\gamma & 0 \\ 0 & \gamma I_{k_2} \end{pmatrix} \oplus 0_{s'} - B\right] \\ &= \begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'} - B^2 - \begin{pmatrix} \gamma B_{11} & \gamma B_{12} \\ B_{21}/\gamma & B_{22}/\gamma \end{pmatrix} \oplus 0_{s'} + \begin{pmatrix} B_{11}/\gamma & \gamma B_{12} \\ B_{21}/\gamma & \gamma B_{22} \end{pmatrix} \oplus 0_{s'} \\ &= \begin{pmatrix} I_{k_1} & 0 \\ 0 & I_{k_2} \end{pmatrix} \oplus 0_{s'} - B^2 - \begin{pmatrix} (\gamma - 1/\gamma)B_{11} & 0 \\ 0 & (1/\gamma - \gamma)B_{22} \end{pmatrix} \oplus 0_{s'}. \end{aligned}$$

Since this is true for all nonzero real $\gamma$, we see that $B_{11}$ and $B_{22}$ are zero blocks. Because the $(1,1)$ and $(2,2)$ blocks of $B$ are zero, we get

$$B_{12}B_{21} = I_{k_1} \quad \text{and} \quad B_{21}B_{12} = I_{k_2}.$$

Hence, $k_1 = k_2$ and $B_{21} = B_{12}^{-1}$. Similarly, we get all $k_1 = k_2 = \cdots = k_n$, and we set this common value to be $k$. It follows $s = nk$.

Now, we may replace $\Phi$ with the map $(B_{12}^{-1} \oplus I_k \oplus I_{s-2k})\Phi(X)(B_{12} \oplus I_k \oplus I_{s-2k})$ so that $B_{12}$ is changed to $I_k$. Consequently, we can assume

$$B = \Phi(E_{12} + E_{21}) = \begin{pmatrix} 0 & I_k \\ I_k & 0 \end{pmatrix} \oplus 0_{s-2k}.$$

In a similar manner, we can assume, up to similarity,

$$\Phi(E_{1j} + E_{j1}) = (E_{1j} + E_{j1}) \otimes I_k \quad \text{for all } j = 1, \ldots, n.$$

Notice that all $E_{ij} + E_{ji}$ with $i, j = 2, \ldots, n$, are disjoint from $E_{11}$. It follows that all $\Phi(E_{ij}) + \Phi(E_{ji})$ are disjoint from $\Phi(E_{11}) = I_k \oplus 0_{s-k}$ for $i, j = 2, \ldots, n$. Consequently, all $\Phi(E_{ij}) + \Phi(E_{ji})$ are contained in $(0_k \oplus I_{s-k})M_s(0_k \oplus I_{s-k})$. Therefore, $\Phi$ induces a zero product preserving complex linear map $\Phi' : S_{n-1}(\mathbb{C}) \to M_{s-k}(\mathbb{C})$ such that condition (a) is satisfied. Therefore, with an induction argument we can show that

$$\Phi(E_{ij} + E_{ji}) = (E_{ij} + E_{ji}) \otimes I_k \quad \text{for all } i, j = 1, \ldots, n.$$

After a permutation similarity, we can assume instead

$$\Phi(E_{ij} + E_{ji}) = I_k \otimes (E_{ij} + E_{ji}) \quad \text{for all } i, j = 1, \ldots, n.$$

Since $\{E_{ij} + E_{ji} : i, j = 1, \ldots, n\}$ is a basis for $S_n(\mathbb{C})$, we arrive at the asserted representation.

Finally, if $\Phi$ sends rank one real symmetric idempotents to symmetric idempotents, the matrix $S$ above can be chosen to be complex orthogonal as in the proof of Theorem 3.3(c). ∎

## 5. Future projects

One can consider additive or linear maps $\Phi : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ satisfying

$$\Phi(A) \cdot \Phi(B) = 0_r \quad \text{whenever} \quad A, B \in M_n(\mathbb{F}) \text{ satisfy } A \cdot B = 0_n$$

for different kinds of products (binary operations). For instance, one may consider the Jordan product $A \cdot B = AB + BA$, the Jordan triple product $A \cdot B = ABA$, the Lie product $A \cdot B = AB - BA$, the skew product $A \cdot B = AB^*$, etc. We note that although some results about these problems are known for the case when $r \leq n$, they are indeed challenging in general.

In particular, we are interested in characterizing the following additive/linear maps $\Phi : M_n(\mathbb{F}) \to M_r(\mathbb{F})$ between matrix algebras. We hope their characterizations will be done in a future project.

(1) $\Phi(A)\Phi(B) + \Phi(B)\Phi(A) = 0$ whenever $A, B \in M_n(\mathbb{F})$ satisfy $AB + BA = 0$.
(2) $\Phi(ABA) = 0$ whenever $A, B \in M_n(\mathbb{F})$ satisfy $ABA = 0$.

## Acknowledgment

C.-K. Li is an affiliate member of the Institute for Quantum Computing, University of Waterloo. His research was supported by USA NSF grant DMS 1331021 and Simons Foundation Grant 351047. This research started during his academic visit to Taiwan in 2018, which was supported by grants from Taiwan MOST. He would like to express his gratitude to the hospitality of several institutions there, including the Academia Sinica, National Chung Hsing University, National Sun Yat-sen University, and National Taipei University of Science and Technology.

M.-C. Tsai, Y.-S. Wang and N.-C. Wong are supported by Taiwan MOST grants 108-2115-M-027-003, 108-2115-M-005-001 and 108-2115-M-110-004-MY2, respectively.

## References

[1] M. Brešar, Multiplication algebra and maps determined by zero products, *Linear Alg. Appl.*, **60**(7) (2012), 763–768.

[2] M. Brešar and P. Šemrl, Mappings which preserve idempotents, local automorphisms, and local derivations, *Canad. J. Math.*, **45** (1993), 483–496.

[3] M. Brešar and P. Šemrl, Zero product preserving maps on matrix rings over division rings, *Contemp. Math.*, to appear.

[4] M. Burgos, F. J. Fernández-Polo, J. J. Garcés, J. Martínez Moreno and A. M. Peralta, Orthogonality preservers in $C^*$-algebras, JB$^*$-algebras and JB$^*$-triples, *J. Math. Anal. Appl.*, **348** (2008), 220-233.

[5] G.-H. Chan, M.-H. Lim and K.-K. Tan, Linear preservers on matrices, *Linear Alg. Appl.*, **93** (1987), 67–80.

[6] M. A. Chebotar, W.-F. Ke, P.-H. Lee and N.-C. Wong, Mappings preserving zero products, *Studia Math.*, **155**(1) (2003), 77–94.

[7] W.-S. Cheung, C.-K. Li, and Y.-T. Poon, Isometries between matrix algebras, *J. Aust. Math. Soc.* **77** (2004), 1-16.

[8] J. A. Erdos, On products of idempotent matrices, *Glasgow Math. J.*, **8** (1967) 118–122.

[9] N. Jacobson and C. E. Rickart, Jordan homomorphisms of rings, *Trans. Amer. Math. Soc.*, **69**(3) (1950), 479–502.

[10] I. N. Herstein, Jordan homomorphisms, *Trans. Amer. Math. Soc.*, **81**(2) (1956), 331–341.

[11] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge, UK, 2013.

[12] A. T.-M. Lau and N.-C. Wong, Orthogonality and disjointness preserving linear maps between Fourier and Fourier-Stieltjes algebras of locally compact groups, *J. Funct. Anal.*, **265** (2013), no. 4, 562–593.

[13] C.-K. Li and S. Pierce, Linear preserver problems, *Amer. Math. Monthly* **108** (2001), 591–605.

[14] C.-K. Li, L. Rodman, and P. Šemrl, Linear transformations between matrix spaces that map one rank specific set into another, *Linear Alg. Appl.*, **357** (2002), 197-208.

[15] C.-K. Li, M.-C. Tsai, Y.-S. Wang, and N.-C. Wong, Nonsurjective maps between rectangular matrix spaces preserving disjointness, triple products, or norms, *J. Operator Th.*, **83** (2020), 27–53.

[16] C.-K. Li and N.-K. Tsing, Linear preserver problems: A brief introduction and some special techniques, *Linear Alg. Appl.*, **162–164** (1992), 217–235.

[17] J.-H. Liu, C.-Y. Chou, C.-J. Liao and N.-C. Wong, Linear disjointness preservers of operator algebras and related structures, *Acta Sci. Math. (Szeged)*, **84** (2018), 277–307.

[18] L. Monlar, *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces*, Springer-Verlag, Berlin, 2007.

[19] M. Omladič, H. Radjavi, P. Šemrl, Preserving commutativity, *J. Pure Appl. Algebra* **156** (2001), 309–328.

[20] C. S. Pazzis, On decomposing any matrix as a linear combination of three idempotents, *Linear Alg. Appl.*, **433** (2010), 843–855.

[21] P. Šemrl, Maps on matrix spaces, *Linear Alg. Appl.*, **413** (2006), 364–393.

[22] E. Størmer, On the Jordan structure of C*-algebras, *Trans. Amer. Math. Soc.*, **120**(3), 1965, 438–447.

[23] N.-C. Wong, Triple homomorphisms of operators algebras, *Southeast Asian Bulletin of Math.* **29** (2005), 401–407.

[24] N.-C. Wong, Zero product preservers of $C^*$-algebras, *Contemporary Math.*, **435** (2007), 377–380. (arXiv:0708.3718)

[25] O. Zariski and P. Samuel, *Commutative algebra, I*, Graduate Texts in Mathematics 28, Springer-Verlag, Berlin, New York, 1975.

(Li) Department of Mathematics, The College of William & Mary, Williamsburg, VA 13185, USA.
*Email address*: `ckli@math.wm.edu`

(Tsai) General Education Center, National Taipei University of Science and Technology, Taipei 10608, Taiwan
*Email address*: `mctsai2@mail.ntut.edu.tw`

(Wang) Department of Applied Mathematics, National Chung Hsing University, Taichung 40227, Taiwan
*Email address*: `yashu@nchu.edu.tw`

(Wong) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan.
*Email address*: `wong@math.nsysu.edu.tw`