ZERO PRODUCT PRESERVING MAPS OF OPERATOR VALUED FUNCTIONS

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Abstract. Let $X, Y$ be locally compact Hausdorff spaces and $\mathcal{M}, \mathcal{N}$ be Banach algebras. Let $\theta : C_0(X, \mathcal{M}) \to C_0(Y, \mathcal{N})$ be a zero-product preserving bounded linear map with dense range. We show that $\theta$ is given by a continuous field of algebra homomorphisms from $\mathcal{M}$ into $\mathcal{N}$ if $\mathcal{N}$ is irreducible. As corollaries, such a surjective $\theta$ arises from an algebra homomorphism, provided that $\mathcal{M}$ is a $W^*$-algebra and $\mathcal{N}$ is a semi-simple Banach algebra, or both $\mathcal{M}$ and $\mathcal{N}$ are C*-algebras.

1. Introduction

Let $X$ be a locally compact Hausdorff space. Denote by $X_\infty = X \cup \{\infty\}$ the one-point compactification of $X$. In case $X$ is already compact, $\infty$ is an isolated point in $X_\infty$. For a real or complex Banach algebra $\mathcal{M}$, let $C_0(X, \mathcal{M}) = \{f \in C(X, \mathcal{M}) : f(\infty) = 0\}$ be the Banach algebra of all continuous vector-valued functions from $X$ into $\mathcal{M}$ vanishing at infinity. Note that $C_0(X, \mathcal{M})$ is isometrically and algebraically isomorphic to the (projective) tensor product $C_0(X) \otimes \mathcal{M}$.

In this paper, we shall study those bounded linear maps $\theta$ from $C_0(X, \mathcal{M})$ into another such algebra $C_0(Y, \mathcal{N})$ preserving zero products. Namely, $fg = 0$ implies $\theta(f)\theta(g) = 0$. In other words,

$$f(x)g(x) = 0 \text{ in } \mathcal{M} \text{ for all } x \in X \implies \theta(f)(y)\theta(g)(y) = 0 \text{ in } \mathcal{N} \text{ for all } y \in Y.$$  

For example, let $\sigma : Y \to X$ be a continuous function, let $h$ be a uniformly bounded norm continuous function from $Y$ into the center of $\mathcal{N}$, and let $\varphi$ be a uniformly bounded SOT continuous function from $Y$ into $B(\mathcal{M}, \mathcal{N})$ such that each $\varphi_y = \varphi(y)$ is an algebra homomorphism. Then

$$(1.1) \quad \theta(f)(y) = h(y)\varphi_y(f(\sigma(y)))$$

defines a zero-product preserving bounded linear map from $C_0(X, \mathcal{M})$ into $C_0(Y, \mathcal{N})$. In particular, $\theta = h\varphi$ for a bounded central element $h$ in the algebra $C(Y, \mathcal{N})$ and an

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algebra homomorphism \( \varphi \) from \( C_0(X,M) \) into \( C_0(Y,N) \). We will investigate when zero product preserving bounded linear maps arise in this way.

For the scalar case, every zero-product preserving bounded linear map \( \theta \) from \( C_0(X) \) into \( C_0(Y) \) is of the expected form (1.1) [13, 11, 14]. Recall that a subalgebra \( S \) of the algebra \( B(E) \) of all bounded linear operators on a Banach space \( E \) is said to be standard if \( S \) contains all continuous finite rank operators. Using an interesting geometric approach, Araujo and Jarosz [2] showed that when \( X, Y \) are realcompact and \( M \) and \( N \) are standard operator algebras, every bijective linear map from \( C(X,M) \) onto \( C(Y,N) \) preserving zero products in both directions is in the form of (1.1). However, in the non-bijective case it becomes a very difficult task without assuming continuity. Even discontinuous algebra homomorphisms have complicated structure ([15, 19]). Finally, readers are referred to [1, 12, 6, 21] for problems of similar interests.

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2. Results

A linear map \( \theta \) from \( C_0(X,M) \) into \( C_0(Y,N) \) is said to be strictly separating if

\[
\|f(x)\|\|g(x)\| = 0 \quad \text{for all} \quad x \in X \quad \implies \quad \|Tf(y)\|\|Tg(y)\| = 0 \quad \text{for all} \quad y \in Y.
\]

Denote by \( \text{coz}(f) = \{ x \in X : f(x) \neq 0 \} \) the cozero set of an \( f \) in \( C_0(X,M) \). Then \( \theta \) is strictly separating if and only if it preserves the disjointness of cozeroes. We note that a subset \( U \) of \( X \) is the cozero of a continuous function in \( C_0(X,M) \) if and only if \( U \) is \( \sigma \)-compact and open. For any \( \sigma \)-compact open subset \( U \) of \( X \), denote by \( C_0(U,M) \) the subalgebra of all \( f \) in \( C_0(X,M) \) with \( \text{coz}(f) \subseteq U \).

Recall that a representation \( \pi : N \to B(E) \) of a Banach algebra \( N \) is said to be faithful if the kernel of \( \pi \) is \( \{0\} \). We call \( \pi \) an irreducible representation of \( N \) if there is no proper linear subspace \( F \) of the Banach space \( E \) such that \( \pi(N)F \subseteq F \). It amounts to say that for each nonzero vector \( e \) in \( E \), the linear subspace \( \pi(N)e \) is the whole of \( E \). Every irreducible representation of a Banach algebra is automatically bounded [15]. A Banach algebra \( N \) is said to be irreducible if it has a faithful irreducible representation \( \pi : N \to B(E) \).

**Theorem 1.** Let \( X \) and \( Y \) be locally compact Hausdorff spaces. Let \( M \) and \( N \) be Banach algebras such that \( N \) is irreducible, and let \( \theta \) be a continuous zero-product preserving linear map from \( C_0(X,M) \) into \( C_0(Y,N) \) with dense range. Then \( \theta \) is strictly separating.

Indeed, there exists a continuous map \( \sigma : Y \to X \), and for each \( y \) in \( Y \) a bounded zero-product preserving linear map \( H_y : M \to N \) with dense range such that

\[
\theta(f)(y) = H_y(f(\sigma(y))) \quad \text{for all} \quad f \in C_0(X,M) \quad \text{and} \quad y \in Y.
\]
Moreover, the correspondence $y \mapsto H_y$ defines a uniformly bounded map $H : Y \to B(\mathcal{M}, \mathcal{N})$ continuous in the strong operator topology.

**Proof.** Let $\pi : \mathcal{N} \to B(E)$ be a faithful irreducible representation of $\mathcal{N}$. Composing $\theta$ with $\pi$, we can assume that $\mathcal{N}$ is an irreducible subalgebra of $B(E)$ and $\theta$ is again bounded and zero-product preserving with dense range.

Fix $y$ in $Y$, and denote by

$$S_y = \left\{ x \in X_\infty : \text{for all } \sigma\text{-compact open neighborhood } U \text{ of } x, \right.$$

there is an $f$ in $C_0(U, \mathcal{M})$ such that $\theta(f)(y) \neq 0$,

that is, $\theta|_{C_0(U, \mathcal{M})}$ is not trivial at $y$.

**Claim 1.** $S_y \neq \emptyset$.

Suppose not, and for each $x$ in $X_\infty$, there is a $\sigma$-compact open neighborhood $U_x$ of $x$ such that $\theta|_{C_0(U_x, \mathcal{M})}$ is trivial at $y$. Write

$$X_\infty = U_0 \cup U_1 \cup \cdots \cup U_n$$

for $x_0 = \infty$, and some $x_1, \ldots, x_n$ in $X$, with a $\sigma$-compact open neighborhood $U_i$ for $i = 0, 1, \ldots, n$, respectively. Let

$$1 = f_0 + f_1 + \cdots + f_n$$

be a continuous partition of the unity such that $\text{coz } f_i \subseteq U_i$ for $i = 0, 1, \ldots, n$. Then for all $f$ in $C_0(X, \mathcal{M})$,

$$\theta(f) = \theta(f_0 f + f_1 f + \cdots + f_n f) = 0,$$

since $\text{coz}(f_i f) \subseteq U_i$ for each $i = 0, 1, \ldots, n$. This is impossible.

**Claim 2.** $x_1, x_2 \in S_y \implies x_1 = x_2$.

Suppose $x_2 \neq x_1 \neq \infty$. Let $U_1$ and $U_2$ be disjoint $\sigma$-compact open neighborhoods of $x_1$ and $x_2$, respectively. We can assume that $\infty \notin U_1$. Since

$$f_1 f_2 = f_2 f_1 = 0 \quad \text{for all } f_i \in C_0(U_i, \mathcal{M}), \quad i = 1, 2,$$

we have

$$\theta(f_1) \theta(f_2) = \theta(f_2) \theta(f_1) = 0 \quad \text{in } C_0(Y, \mathcal{N}).$$

Let $E_1$ be the intersection of the kernels of all $\theta(f_1)(y)$ with $f_1$ in $C_0(U_1, \mathcal{M})$. Because both $\theta|_{C_0(U_1, \mathcal{M})}$ and $\theta|_{C_0(U_2, \mathcal{M})}$ are not trivial at $y$, we have $E_1$ is a proper subspace of $E$, that is, $\{0\} \neq E_1 \neq E$. 

Let $V$ be a nonempty open set in $Y$ such that $\overline{V} \subseteq U_1$. Let $g$ be in $C_0(X)$ such that $\mathrm{coz} \ g \subseteq U_1$ and $g|_V = 1$. For each $f$ in $C_0(X, \mathcal{M})$, write
\[ f = fg + f(1 - g). \]
Since $\mathrm{coz}(fg) \subseteq U_1$, we have
\[ \theta(fg)(y)|_{E_1} = 0. \]
Hence
\[ \theta(f)(y)|_{E_1} = \theta(f(1 - g))(y)|_{E_1}. \]
For any $k$ in $C_0(X, \mathcal{M})$ with $\mathrm{coz} \ k \subseteq V$, we have $k(f(1 - g)) = 0$. This implies
\[ \theta(k)(y)\theta(f)(y)|_{E_1} = \theta(k)(y)\theta(f(1 - g))(y)|_{E_1} = 0 \quad \text{for all } f \in C_0(X, \mathcal{M}). \]
However, $\{\theta(f)(y) : f \in C_0(X, \mathcal{M})\}$ is dense in $\mathcal{N}$, which is irreducible on $E$. Therefore,
\[ \theta(k)(y) = 0 \quad \text{for all } k \in C_0(X, \mathcal{M}) \text{ with } \mathrm{coz} \ k \subseteq V. \]
Since $V$ is an arbitrary nonempty open set with closure contained in $U_1$, we have
\[ \theta(k)(y) = 0 \quad \text{for all } k \in C_0(U_1, \mathcal{M}). \]
This conflict establishes Claim 2.

By Claims 1 and 2, $S_y$ is a singleton.

**Claim 3.** If $S_y = \{x\}$ then
\[ f(x) = 0 \quad \implies \quad \theta(f)(y) = 0. \]

By Urysohn’s Lemma, we can assume $f$ vanishes in a neighborhood of $x$. Now $x \notin \overline{\mathrm{coz} f}$, which is compact in $X_\infty$. For each $x'$ in $\overline{\mathrm{coz} f}$, there is a \( \sigma \)-compact open neighborhood $U'$ of $x'$ such that $\theta|_{C_0(U', \mathcal{M})}$ is trivial at $y$. By a compactness argument as the one proving Claim 1, we see that $\theta(f)(y) = 0$.

It follows from Claim 3 that $S_y \neq \{\infty\}$ for all $y$ in $Y$ since $\theta$ has dense range. Denote by $\sigma(y) = x$ if $S_y = \{x\}$. Then there is a linear map $H_y : \mathcal{M} \to \mathcal{N}$ such that
\[ \theta(f)(y) = H_y (f(\sigma(y))) \quad \text{for all } f \in C_0(X, \mathcal{M}) \text{ and } y \in Y. \]
In particular, $\theta$ is strictly separating.

The rest of the proof follows in a straightforward manner, or one can quote the standard results about strictly separating maps in [6, 12].

The following lemma might be known, although we do not find a proof from the literature. Remark that it is shown in [17] every non-zero Banach algebra homomorphism from $B(H)$ into $B(K)$ is injective if both $H$ and $K$ are separable Hilbert spaces. However, there is an example in [18] of a non-zero homomorphism from $B(H)$ into $B(H)$ with compact operators as its kernel, where $H$ is inseparable. Moreover, it
is known that every irreducible representation of a Banach algebra is norm continuous [15] and every algebra isomorphism between C*-algebras is a *-isomorphism [20, Theorem 4.1.20].

Lemma 2. Let $H, K$ be real or complex Hilbert spaces of arbitrary dimension. Let $B(H)$ and $B(K)$ be the algebras of all bounded linear operators on $H$ and $K$, respectively. Then every surjective algebra homomorphism from $B(H)$ onto $B(K)$ is an isomorphism.

Proof. The case is trivial when $H$ is of finite dimension since $B(H)$ is then a simple algebra. Suppose the (Hilbert space) dimension of $H$ is an infinite cardinal number $\aleph$. For each infinite cardinal number $\aleph \leq \aleph$, let $I_{\aleph}$ be the closed two-sided ideal of $B(H)$ consisting of operators $T$ such that all closed subspaces contained in the range of $T$ is of dimension less than $\aleph$. In case $H$ is separable, $I_{\aleph} = \mathcal{K}(H)$, the ideal of compact operators on $H$. In general, as indicated in [5] that $I_{\aleph}$ is the largest two-sided ideal of $B(H)$. In fact, every closed two-sided ideal of $B(H)$ is in the form of $I_{\aleph}$ for some $\aleph \leq \aleph[9, \text{Section 17}].$

Let $\theta$ be an algebra homomorphism from $B(H)$ onto $B(K)$. Then the kernel $I$ of $\theta$ is a closed two-sided ideal of $B(H)$. Since the quotient algebra $B(H)/I$ is isomorphic to $B(K)$, there is an $e$ in $B(H)$ such that $(e + I)B(H)(e + I) = eB(H)e + I$ is of one dimension modulo $I$. Assume $I$ is nonzero. Let $\aleph$ be the infinite cardinal number such that $I = I_{\aleph}$. Then the range of $e$ contains a closed subspace of dimension $\aleph$. By halving this subspace into two each of dimension $\aleph$, we see that $eB(H)e$ contains two elements linear independent modulo $I_{\aleph}$, a contradiction. This completes our proof. \qed

Corollary 3. Let $X, Y$ be locally compact Hausdorff spaces. Let $\mathcal{M}, \mathcal{N}$ be either the Banach algebras $B(H), B(K)$ of all bounded operators or $\mathcal{K}(H), \mathcal{K}(K)$ of compact operators on real or complex Hilbert spaces $H, K$, respectively. Let $\theta : C_0(X, \mathcal{M}) \to C_0(Y, \mathcal{N})$ be a continuous surjective zero-product preserving linear map. Then there exist a continuous function $\sigma$ from $Y$ into $X$, a continuous scalar function $h$ on $Y$, and a SOT continuous map $y \mapsto S_y$ from $Y$ into $B(K, H)$ such that $S_y$ is invertible and

\begin{equation}
\theta(f)(y) = h(y)S_y^{-1}f(\sigma(y))S_y, \quad \forall f \in C(X, \mathcal{M}), \forall y \in Y.
\end{equation}

Proof. It follows from Theorem 1 that for each fixed $y$ in $Y$, $\theta$ induces a bounded zero-product preserving linear map $H(y)$ from $\mathcal{M}$ onto $\mathcal{N}$. By either [10, Theorem 2.1] or [7, Corollary 3.2], $H(y)$ is a scalar multiple of a bounded algebra homomorphism from $\mathcal{M}$ onto $\mathcal{N}$. Since $\mathcal{K}(H)$ is simple, this algebra homomorphism is indeed an isomorphism if $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{K}(H)$ and $\mathcal{K}(K)$, respectively. On the other hand, by Lemma 2 the algebra homomorphism above is again an isomorphism in case $\mathcal{M}$ and $\mathcal{N}$ are $B(H)$ and $B(K)$, respectively. Thus, by either [3, Theorem 4] or [8, Corollary 3.2], there exist a scalar $h(y)$ and a bounded invertible operator $S_y$ on $K$ to implement (2.1). It is then routine to check the continuity of $h$ and the map $y \mapsto S_y$. \qed
The following corollary holds, for example, when $\mathcal{M}$ is a $W^*$-algebra, or a unital $C^*$-algebra of real rank zero [4].

**Corollary 4.** Let $X$ and $Y$ be locally compact Hausdorff spaces such that $X$ is compact. Let $\mathcal{M}$ be a unital Banach algebra such that the subalgebra of $\mathcal{M}$ generated by its idempotents is norm dense in $\mathcal{M}$, and let $\mathcal{N}$ be a semi-simple Banach algebra. Let $\theta$ be a continuous zero-product preserving linear map from $C(X, \mathcal{M})$ into $C_0(Y, \mathcal{N})$ with dense range. Then $\theta(1)$ is in the center of $C_0(Y, \mathcal{N})$, and

\[
\theta(1)(fg) = \theta(f)\theta(g) \quad \text{for all } f, g \in C(X, \mathcal{M}).
\]

Suppose, in addition, that $Y$ is compact and $\mathcal{N}$ is unital. If $\theta(1)$ is invertible or $\theta$ is surjective, then $\theta = \theta(1)\varphi$ for an algebra homomorphism $\varphi$.

**Proof.** Let $\pi : \mathcal{N} \to B(E)$ be an irreducible representation of $\mathcal{N}$. Then $\theta_\pi = \pi \circ \theta$ is again a continuous zero-product preserving linear map from $C(X, \mathcal{M})$ into $C_0(Y, \pi(\mathcal{N}))$ with dense range. By Theorem 1, we find that $\theta_\pi$ carries a weighted composition operator form

\[
\theta_\pi(f)(y) = H_y(f(\sigma(y))) \quad \text{for all } f \in C(X, \mathcal{M}) \text{ and } y \in Y.
\]

In particular, each $H_y$ is a continuous zero-product preserving linear map from $\mathcal{M}$ into $\pi(\mathcal{N})$ with dense range.

By results in [10] (see also [7]), for each $y$ in $Y$ we have $\theta_\pi(1)(y) = H_y(1)$ is in the center of $\mathcal{N}$ and

\[
H_y(1)H_y(ab) = H_y(a)H_y(b) \quad \text{for all } a, b \in \mathcal{M}.
\]

Hence

\[
\pi \left( \theta(1)(f) - \theta(f)(1) \right) = 0
\]

and

\[
\pi \left( \theta(1)(fg) - \theta(f)(g) \right) = 0
\]

for all $f, g$ in $C(X, \mathcal{M})$. Being semi-simple, $\mathcal{N}$ has a faithful family of irreducible representations. Thus $\theta(1)$ is in the center of $C_0(Y, \mathcal{N})$ and (2.2) holds.

Now, we assume that $Y$ is compact and $\mathcal{N}$ is unital. If $\theta$ is surjective, $1 = \theta(f)$ for some $f$ in $C(X, \mathcal{M})$. It follows from $\theta(1)\theta(f^2) = \theta(f)^2 = 1$ that $\theta(1)$ is invertible. Assume $\theta(1)$ is invertible. Then $\theta(1)^{-1}\theta$ is again a bounded zero-product preserving linear map with dense range, and sends 1 to 1. Suppose now $\theta(1) = 1$. Then (2.2) ensures that $\theta$ is an algebra homomorphism. □

A recent result in [7] states that every surjective zero-product preserving bounded linear map $\theta$ between unital C*-algebras is a product $\theta = \theta(1)\varphi$ of the invertible central element $\theta(1)$ and an algebra homomorphism $\varphi$. Since $C(X, \mathcal{A})$ (resp. $C(Y, \mathcal{B})$) is *-isomorphic to the (projective) tensor product $C(X) \otimes \mathcal{A}$ (resp. $C(Y) \otimes \mathcal{B}$) as C*-algebras (see, e.g., [16]), we have the following
Corollary 5. Let $X$ and $Y$ be compact Hausdorff spaces, and $A, B$ be unital $C^*$-algebras. Let $\theta$ be a continuous zero-product preserving linear map from $C(X, A)$ onto $C(Y, B)$. Then $\theta(1)$ is an invertible element in the center of $C(Y, B)$, and $\theta = \theta(1)\varphi$ for an algebra homomorphism $\varphi$.

The following example shows that the irreducibility condition on $\mathcal{N}$ cannot be dropped in Theorem 1, and the map $\theta$ in the Corollaries 4 and 5 cannot be written as a weighted composition operator in the form of (1.1) in general.

Example 6. Let $X = \{0\}$ and $\mathcal{M} = \mathbb{C} \oplus \mathbb{C}$ be the two-dimensional $C^*$-algebra, and let $Y = \{1, 2\}$ and $\mathcal{N} = \mathbb{C}$ be the one-dimensional $C^*$-algebra. Define $\theta : C(X, \mathcal{M}) \to C(Y, \mathcal{N})$ by $\theta(a \oplus b) = g$ with $g(1) = a$ and $g(2) = b$. Then $\theta$ is bijective and preserves zero products in both directions.

Remark that $\theta : C(X, \mathcal{M}) \to C(Y, \mathcal{N})$ satisfies the condition stated in Theorem 1. In fact, let $h_1(a \oplus b) = a$ and $h_2(a \oplus b) = b$ be the canonical projection of $\mathbb{C} \oplus \mathbb{C}$ onto its summands, and set $\sigma(1) = \sigma(2) = 0$. Then

$$\theta(f)(y) = h_y(f(\sigma(y))), \quad \forall f \in C(X, \mathcal{M}), \forall y \in Y.$$ 

However, $\mathcal{M}$ is not irreducible and $T^{-1} : C(Y, \mathcal{N}) \to C(X, \mathcal{M})$ does not carry a weighted composition operator form. Note also that $X$ and $Y$ are not homeomorphic although both $C(X, \mathcal{M})$ and $C(Y, \mathcal{N})$ are isomorphic to $\mathbb{C} \oplus \mathbb{C}$ as $C^*$-algebras and $\theta$ implements an algebra isomorphism between them.

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