MAPS PRESERVING SCHATTEN p-NORMS OF CONVEX COMBINATIONS

DAVID LI-WEI KUO, MING-CHENG TSAI, NGAI-CHING WONG AND JUN ZHANG[†]

Abstract. In this paper, we study maps ϕ of positive operators of Schatten p-classes ($1 < p < +\infty$), which preserve the p-norms of convex combinations, that is,

$$\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H)_1, t \in [0,1].$$

They are exactly those carrying the form $\phi(\rho) = U\rho U^*$ for a unitary or anti-unitary U. In the case $p = 2$, we have the same conclusion whenever it just holds

$$\|\rho + \sigma\|_2 = \|\phi(\rho) + \phi(\sigma)\|_2$$

for all positive Hilbert-Schmidt class operators ρ, σ of norm 1. Some examples are demonstrated.

Keywords: Schatten p-classes; norm and distance preservers; Wigner type theorems.

1. Introduction

The Mazur-Ulam theorem states that every bijective distance preserving map Φ from a Banach space onto another is affine, i.e.,

$$\Phi(tx + (1-t)y) = t\Phi(x) + (1-t)\Phi(y), \quad \forall x, y, 0 \le t \le 1.$$

After translation, we can assume $\Phi(0) = 0$ and Φ is indeed a surjective real linear isometry. Let us consider another version of this statement. Suppose that Φ is a bijective map from a Hilbert space H onto H and Φ preserves norm of convex combinations:

$$\|t\Phi(x) + (1-t)\Phi(y)\| = \|tx + (1-t)y\|, \quad \forall x, y \in H, 0 \le t \le 1. \tag{1.1}$$

Let us further relax the assumption that (1.1) holds for just one fixed t in $(0,1)$. By letting $y = x$ in (1.1), we see that $\|\Phi(x)\| = \|x\|$ for all x in H. Squaring both sides of (1.1), we will see that the real parts of the inner products coincide, i.e.

$$\mathrm{Re}\,\langle x, y\rangle = \mathrm{Re}\,\langle \Phi(x), \Phi(y)\rangle, \quad \forall x, y \in H.$$

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† corresponding author.

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Then the classical Wigner theorem (see, e.g., [7, Theorem 3]) ensures that there is a surjective real linear isometry $U : H \to H$ such that $\Phi(x) = Ux$ for all x in H.

Characterizing isometries, linear or not, of spaces of operators under various norms has been a fruitful area of research for a long time. See, e.g., [3,5] for good surveys. In particular, the spaces $\mathcal{S}_p(H)$ of Schatten p-class operators on a (complex) Hilbert space H ($1 \leq p < +\infty$) are important objects in both analysis and physics. They are widely used in operator theory and quantum mechanics, for example.

Let $\mathcal{S}_p^+(H)$ be the set of all positive operators in $\mathcal{S}_p(H)$, and let $\mathcal{S}_p^+(H)_1$ be the set of all positive operators in $\mathcal{S}_p^+(H)$ of p-norm one. Recall that an affine automorphism (or S-automorphism in [2], or Kadison automorphism in [11]) is a bijective affine map $\phi : \mathcal{S}_1^+(H)_1 \to \mathcal{S}_1^+(H)_1$, i.e.,

$$\phi(t\rho + (1-t)\sigma) = t\phi(\rho) + (1-t)\phi(\sigma), \quad \forall \rho, \sigma \in \mathcal{S}_1^+(H)_1, t \in [0,1].$$

It is known (see, e.g., [9]) that affine automorphisms are exactly those carrying the form $\phi(\rho) = U\rho U^*$ for a unitary or anti-unitary U on H.

Recently, Nagy [10] established a Mazur-Ulam type result for the Schatten p-class operators. Suppose $\phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ ($1 < p < +\infty$) is a bijective map preserving the distance induced by the norm $\|\cdot\|_p$. Then ϕ is implemented by a unitary or an anti-unitary operator U such that $\phi(\rho) = U\rho U^*$. In this paper, we will establish a counter part of Nagy's result similar to the one demonstrated in the first paragraph. More precisely, we will characterize those maps $\phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ satisfying

$$\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H)_1, t \in [0,1].$$

We will show that they are implemented by a unitary or an anti-unitary operator.

Our main theorem follows.

Theorem 1.1. *Let H be a separable complex Hilbert space of finite or infinite dimension. Let $1 < p < +\infty$. Suppose that ϕ is a map from $\mathcal{S}_p^+(H)_1$ into $\mathcal{S}_p^+(H)_1$, which will be assumed to be surjective when* $\dim\ H = +\infty$. *The following conditions are equivalent.*

(1) *ϕ preserves the Schatten p-norms of convex combinations, i.e.,*

$$\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H)_1, t \in [0,1]. \tag{1.2}$$

(2) *ϕ preserves the pairings, i.e., for all $\rho, \sigma \in \mathcal{S}_p^+(H)_1$, we have $\sigma^{p-1}\rho \in \mathcal{S}_1(H)$, and*

$$\operatorname{tr}(\sigma^{p-1}\rho) = \operatorname{tr}(\phi(\sigma)^{p-1}\phi(\rho)). \tag{1.3}$$

(3) *There exists a unitary or anti-unitary operator U on H such that*

$$\phi(\rho) = U\rho U^*, \quad \forall \rho \in \mathcal{S}_p^+(H)_1.$$

We note that the condition (1.2) becomes a tautology when $p = 1$. On the other hand, the conclusion of Theorem 1.1 holds again if we replace $\mathcal{S}_p^+(H)_1$ by $\mathcal{S}_p^+(H)$ everywhere. In this case, setting $\sigma = \rho$ in (1.2) we see that ϕ does map $\mathcal{S}_p^+(H)_1$ into $\mathcal{S}_p^+(H)_1$.

The proof of Theorem 1.1 is given in Section 2. When $p = 2$, we see in Section 3 that for ϕ carrying the expected form stated in Theorem 1.1(3) it suffices condition (1.2) held for only one fixed t in $(0, 1)$. Finally, we demonstrate some examples in Section 4.

2. Proof of the main theorem

In what follows, we fix some notation and definitions used throughout the paper. Let H stand for a separable complex Hilbert space of finite dimension or infinite dimension. Let $B(H)$ denote the algebra of all bounded linear operators on H. For a compact operator T in $B(H)$, let $s_1(T) \geq s_2(T) \geq \cdots \geq 0$ denote the singular values of T, i.e., the eigenvalues of $|T| = (TT^*)^{\frac{1}{2}}$ arranged in their decreasing order (repeating according to multiplicity). A compact operator T belongs to the Schatten p-classes $\mathcal{S}_p(H)$ $(1 \leq p < +\infty)$ if

$$\|T\|_p := \left(\sum_{i=1}^{\infty} s_i(T)^p\right)^{1/p} = (\operatorname{tr}|T|^p)^{1/p} < +\infty, \tag{2.1}$$

where tr denotes the trace functional. We call $\|T\|_p$ the Schatten p-norm of T. In particular, $\mathcal{S}_1(H)$ is the trace class and $\mathcal{S}_2(H)$ is the Hilbert-Schmidt class. The cone of positive operators in $\mathcal{S}_p(H)$ is denoted by $\mathcal{S}_p^+(H)$, and the set of rank one projections in $\mathcal{S}_p^+(H)$ is denoted by $P_1(H)$.

Recall that the norm of a normed space is Fréchet differentiable at $x \neq 0$ if $\lim_{t\to 0} \dfrac{\|x + ty\| - \|x\|}{t}$ exists and uniform for all norm one vectors y.

Lemma 2.1 ([1, Theorem 2.3]). *Let $1 < p < +\infty$ and ρ in $\mathcal{S}_p^+(H)$ be nonzero. The norm of $\mathcal{S}_p^+(H)$ is Fréchet differentiable at ρ. For any σ in $\mathcal{S}_p^+(H)$ we have*

$$\frac{d\|\rho + t\sigma\|_p}{dt}\Big|_{t=0} = \operatorname{tr}\left(\frac{\rho^{p-1}\sigma}{\|\rho\|_p^{p-1}}\right). \tag{2.2}$$

Lemma 2.2. *Suppose $\rho, \sigma \in \mathcal{S}_p^+(H)$ $(1 < p < +\infty)$. The following conditions are equivalent.*

(1) $\rho = \sigma$.
(2) $\|t\rho + (1-t)P\|_p = \|t\sigma + (1-t)P\|_p$ *for all P in $P_1(H)$ and all t in* $[0, 1]$.
(3) $\operatorname{tr}(P\rho) = \operatorname{tr}(P\sigma)$ *for all P in $P_1(H)$.*

Proof. (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (3): Differentiating both sides of $\|t\rho + (1-t)P\|_p = \|t\sigma + (1-t)P\|_p$ at $t = 0^+$, we have $\operatorname{tr} P\rho = \operatorname{tr} P^{p-1}\rho = \operatorname{tr} P^{p-1}\sigma = \operatorname{tr} P\sigma$ by Lemma 2.1.

(3) $\Rightarrow$ (1): Since ρ and σ are positive, $\rho - \sigma$ is hermitian. There exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of H such that $\rho - \sigma = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$. Choosing $P_i = e_i \otimes e_i$, we have $\lambda_i = \operatorname{tr}(P_i(\rho - \sigma)) = 0$ for all $i = 1, 2, \ldots$. It follows $\rho - \sigma = 0$. □

We say that two self-adjoint operators ρ, σ in $B(H)$ are orthogonal if $\rho\sigma = 0$, which is equivalent to the property that they have mutually orthogonal ranges.

Lemma 2.3. *Suppose that $\rho, \sigma \in \mathcal{S}_p^+(H)$ for $1 < p < +\infty$. The following conditions are equivalent.*

(1) *ρ, σ are orthogonal, i.e., $\rho\sigma = 0$.*
(2) *$\|\alpha\rho + (1-\alpha)\sigma\|_p^p = \alpha^p \|\rho\|_p^p + (1-\alpha)^p \|\sigma\|_p^p$ for any (and thus all) α in $(0, 1)$.*
(3) *$\operatorname{tr}(\rho\sigma) = 0$.*
(4) *$\|\rho + t\sigma\|_p \geq \|\rho\|_p$ for all t in $\mathbb{R}$, that is, $\rho \perp \sigma$ in Birkhoff's sense.*
(5) *$\operatorname{tr}(\rho^{p-1}\sigma) = 0$.*

Proof. (1) $\Leftrightarrow$ (2): From [6, Lemma 2.6], we know that for any two positive operators A, B in $\mathcal{S}_p^+(H)$, we have

$$\operatorname{tr}(A + B)^p \geq \operatorname{tr} A^p + \operatorname{tr} B^p.$$

Here the equality holds if and only if $AB = 0$. Setting $A = \alpha\rho$ and $B = (1-\alpha)\sigma$, we get

$$\begin{aligned} \rho\sigma = 0 &\Leftrightarrow (\alpha\rho)((1-\alpha)\sigma) = 0 \\ &\Leftrightarrow \operatorname{tr}(\alpha\rho + (1-\alpha)\sigma)^p = \operatorname{tr}(\alpha\rho)^p + \operatorname{tr}((1-\alpha)\sigma)^p \\ &\Leftrightarrow \|\alpha\rho + (1-\alpha)\sigma\|_p^p = \alpha^p \|\rho\|_p^p + (1-\alpha)^p \|\sigma\|_p^p. \end{aligned}$$

(1) $\Leftrightarrow$ (3): One direction is obvious. For the other, because ρ, σ are positive,

$$\operatorname{tr}[(\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}})(\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}})^*] = \operatorname{tr}(\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}\sigma^{\frac{1}{2}}\rho^{\frac{1}{2}}) = \operatorname{tr}(\rho\sigma) = 0.$$

This forces $\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}} = 0$, and thus $\rho\sigma = \rho^{\frac{1}{2}}(\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}})\sigma^{\frac{1}{2}} = 0$.

(1) $\Rightarrow$ (4): Since $\rho\sigma = 0$, there exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of H such that $\rho = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$, $\sigma = \sum_{i=1}^{\infty} \mu_i e_i \otimes e_i$, $\lambda_i \geq 0$, $\mu_i \geq 0$, and $\lambda_i\mu_i = 0$ for all $i = 1, 2, \cdots$. Hence,

$$\|\rho + t\sigma\|_p^p = \operatorname{tr}|\rho + t\sigma|^p = \sum_{i=1}^{\infty} (\lambda_i + |t|\mu_i)^p \geq \sum_{i=1}^{\infty} \lambda_i^p = \|\rho\|_p^p.$$

(4) $\Rightarrow$ (5): Without loss of generality, we can assume that $\rho \neq 0$. Define $f(t) = \|\rho + t\sigma\|_p \geq \|\rho\|_p$. Then $f(t)$ is differentiable and attains its minimum at $t = 0$. From Lemma 2.1,

$$0 = \frac{d\|\rho + t\sigma\|_p}{dt}\Big|_{t=0} = \operatorname{tr}\left(\frac{\rho^{p-1}\sigma}{\|\rho\|_p^{p-1}}\right),$$

and assertion (5) follows.

(5) $\Rightarrow$ (1): As in proving (3) $\Rightarrow$ (1), we have $\rho^{p-1}\sigma = 0$. Then there exists an orthonormal basis $\{e_i'\}_{i=1}^{\infty}$ of H such that $\rho^{p-1} = \sum_{i=1}^{\infty} \xi_i e_i' \otimes e_i'$, $\sigma = \sum_{i=1}^{\infty} \eta_i e_i' \otimes e_i'$, with $\xi_i \geq 0, \eta_i \geq 0$, and $\xi_i \mu_i = 0$ for all $i = 1, 2, \cdots$. Thus, $\operatorname{tr}(\rho\sigma) = \sum_{i=1}^{\infty} \xi_i^{\frac{1}{p-1}} \eta_i = 0$. □

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Lemma 2.4. *Let $1 < p < +\infty$. Suppose that ϕ is a map from $\mathcal{S}_p^+(H)_1$ into $\mathcal{S}_p^+(H)_1$ preserving the Schatten p-norms of convex combinations, i.e., equation (1.2) holds. Then we have*

$$\operatorname{tr}(\sigma^{p-1}\rho) = \operatorname{tr}(\phi(\sigma)^{p-1}\phi(\rho)). \tag{2.3}$$

Proof. Differentiating both sides of (1.2) with respect to t and evaluating at $t = 0$, we have

$$\begin{aligned} \frac{d\|t\rho + (1-t)\sigma\|_p}{dt}\Big|_{t=0} &= \frac{d\|\sigma + t(\rho - \sigma)\|_p}{dt}\Big|_{t=0} \\ &= \operatorname{tr}\left(\frac{\sigma^{p-1}(\rho - \sigma)}{\|\sigma\|_p^{p-1}}\right) \\ &= \frac{\operatorname{tr}(\sigma^{p-1}\rho)}{\|\sigma\|_p^{p-1}} - \|\sigma\|_p \\ &= \operatorname{tr}(\sigma^{p-1}\rho) - 1, \end{aligned}$$

and

$$\begin{aligned} \frac{d\|t\phi(\rho) + (1-t)\phi(\sigma)\|_p}{dt}\Big|_{t=0} &= \frac{\operatorname{tr}(\phi(\sigma)^{p-1}\rho)}{\|\phi(\sigma)\|_p^{p-1}} - \|\phi(\sigma)\|_p \\ &= \operatorname{tr}(\phi(\sigma)^{p-1}\rho) - 1. \end{aligned}$$

Since (1.2) holds for t in $(0, 1]$, these derivatives agree. Therefore, $\operatorname{tr}(\sigma^{p-1}\rho) = \operatorname{tr}(\phi(\sigma)^{p-1}\phi(\rho))$. □

Proposition 2.5. *Suppose $\phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ satisfies that*

$$\operatorname{tr}(\sigma^{p-1}\rho) = \operatorname{tr}(\phi(\sigma)^{p-1}\phi(\rho)), \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H)_1. \tag{2.4}$$

Then the following assertions hold.

(1) *ϕ preserves orthogonality in both directions, that is*

$$\rho\sigma = 0 \Leftrightarrow \phi(\rho)\phi(\sigma) = 0, \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H)_1.$$

(2) *When $\dim H < +\infty$, ϕ maps rank one projections to rank one projections. This also holds when $\dim H = +\infty$ and ϕ is surjective.*

(3) *When $\dim H < +\infty$, we have*

$$\operatorname{tr} PQ = \operatorname{tr} \phi(P)\phi(Q), \quad \forall P, Q \in P_1(H).$$

This also holds when $\dim H = +\infty$ and ϕ is surjective.

Proof. (1) follows from Lemma 2.3.

(2) First, we assume dim $H = n < +\infty$. Suppose ρ is a rank one projection. We can find $n-1$ pairwise orthogonal rank one projections $\rho_1, \cdots, \rho_{n-1}$ such that $\rho\rho_i = 0$ for $1 \leq i \leq n-1$. From (1), we know that $\phi(\rho), \phi(\rho_1), \cdots, \phi(\rho_{n-1})$ are nonzero and pairwise orthogonal. This forces $\phi(\rho)$ has rank one since $\dim H = n$. By (2.4), taking $\sigma = \rho$, we see that $\operatorname{tr} \phi(\rho)^p = \operatorname{tr} \rho^p = \operatorname{tr} \rho = 1$. Therefore, the rank one positive operator $\phi(\rho)$ is a projection.

Next, we consider the case dim $H = +\infty$ and ϕ is surjective. Suppose that there exists a rank one projection ρ in $\mathcal{S}_p^+(H)$ such that $\phi(\rho)$ has rank greater than one. Then there are two nonzero orthogonal operators T_1 and T_2 in $\mathcal{S}_p^+(H)$ such that $\phi(\rho) = T_1 + T_2$. Since ϕ is surjective and preserves orthogonality in both directions, there are two nonzero orthogonal operators ρ_1 and ρ_2 in $\mathcal{S}_p^+(H)_1$ such that $\phi(\rho_1) = T_1/\|T_1\|_p$ and $\phi(\rho_2) = T_1/\|T_2\|_p$. For any σ in $\mathcal{S}_p^+(H)$ with $\sigma\rho = 0$, we have

$$\phi(\sigma)(\|T_1\|_p \phi(\rho_1) + \|T_2\|_p \phi(\rho_2)) = \phi(\sigma)(T_1 + T_2) = \phi(\sigma)\phi(\rho) = 0.$$

It forces

$$\|T_1\|_p \phi(\sigma)\phi(\rho_1)\phi(\sigma) = -\|T_2\|_p \phi(\sigma)\phi(\rho_2)\phi(\sigma) = 0,$$

and hence $\phi(\sigma)\phi(\rho_1) = \phi(\sigma)\phi(\rho_2) = 0$, because $\phi(\sigma)$, $\phi(\rho_1)$ and $\phi(\rho_2)$ are all positive. This implies $\sigma\rho_1 = \sigma\rho_2 = 0$. Therefore, $\rho_1 = \lambda_1\rho$ and $\rho_2 = \lambda_2\rho$ for some nonzero λ_1, λ_2. This contradicts to that $\rho_1\rho_2 = 0$.

(3) From (2), we know that $\phi(P), \phi(Q)$ are rank one projections in $P_1(H)$. Therefore, $P^{p-1} = P, \phi(P)^{p-1} = \phi(P)$. Using Equation (2.4) with $\sigma = P, \rho = Q$ we have

$$\operatorname{tr} PQ = \operatorname{tr}(P^{p-1}Q) = \operatorname{tr}(\phi(P)^{p-1}\phi(Q)) = \operatorname{tr} \phi(P)\phi(Q).$$

□

Proof of Theorem 1.1. (1) ⇒ (2) follows from Lemma 2.4.

(3) ⇒ (1) is obvious.

(2) ⇒ (3): From Proposition 2.5, we obtain that $\phi|_{P_1(H)} : P_1(H) \to P_1(H)$ satisfies $\operatorname{tr} PQ = \operatorname{tr} \phi(P)\phi(Q)$ for all rank one projections P, Q in $P_1(H)$. From a nonsurjective version of Wigner's theorem, c.f. [9, Theorem 2.1.4], there exists an isometry or anti-isometry U on H such that

$$\phi(P) = UPU^*, \qquad \forall P \in P_1(H).$$

Note that U is indeed surjective even when H is of infinite dimension, since ϕ is assumed to be surjective in this case.

For any rank one projection P in $P_1(H)$, setting $\sigma = P$ in (1.3) we have

$$\begin{aligned}\operatorname{tr}(P\rho) &= \operatorname{tr}(P^{p-1}\rho) = \operatorname{tr}(\phi(P)^{p-1}\phi(\rho)) = \operatorname{tr}(\phi(P)\phi(\rho)) \\ &= \operatorname{tr}(UPU^*\phi(\rho)U) = \operatorname{tr}(PU^*\phi(\rho)U)\end{aligned}$$

We have $U^*\phi(\rho)U = \rho$ by Lemma 2.2. This gives $\phi(\rho) = U\rho U^*$. □

3. Maps preserving norms of just a special convex combination

A careful look at the proof of Lemma 2.4 tells us that the condition $\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p$ suffices to hold for the members of any sequence in $(0, 1]$ converging to 0 rather than for any point t in $[0, 1]$. Indeed, in order to get some good properties of ϕ stated in the previous section, we only need to assume that ϕ preserves the Schatten p-norm of convex combination with a given system of coefficients.

Proposition 3.1. *Let $\phi : \mathcal{S}_p^+(H)_1 \to \mathcal{S}_p^+(H)_1$ $(1 < p < +\infty)$. Let α in $(0, 1)$ be arbitrary but fixed. Suppose*

$$\|\alpha\rho + (1-\alpha)\sigma\|_p = \|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H)_1, \tag{3.1}$$

The following properties are satisfied.

(1) *ϕ is injective.*
(2) *ϕ preserves orthogonality in both directions.*
(3) *When $\dim\ H < +\infty$, ϕ maps rank one projections to rank one projections. This also holds when $\dim H = +\infty$ and ϕ is surjective.*

Proof. (1) Assume $\phi(\rho) = \phi(\sigma)$. We have $\|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_p = 1$. From (3.1) we get $\|\alpha\rho + (1-\alpha)\sigma\|_p = 1$. Hence,

$$\|\alpha\rho + (1-\alpha)\sigma\|_p = \alpha\|\rho\|_p + (1-\alpha)\|\sigma\|_p.$$

This forces $\rho = \sigma$ since the norm $\|\cdot\|_p$ is strictly convex for $1 < p < +\infty$.

(2) Assume $\rho\sigma = 0$. From Lemma 2.3, we have

$$\begin{aligned}\|\alpha\rho + (1-\alpha)\sigma\|_p^p &= \alpha^p\|\rho\|^p + (1-\alpha)^p\|\sigma\|^p \\ &= \alpha^p\|\phi(\rho)\|^p + (1-\alpha)^p\|\phi(\sigma)\|^p.\end{aligned}$$

Together with (3.1), we have

$$\|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_p^p = \alpha^p\|\phi(\rho)\|^p + (1-\alpha)^p\|\phi(\sigma)\|^p.$$

Hence, we have $\phi(\rho)\phi(\sigma) = 0$ from Lemma 2.3 again. The other implication follows similarly.

(3) The proof is similar to that of Proposition 2.5(2). □

When $p = 2$, we get an improvement of Theorem 1.1.

Theorem 3.2. *Let H be a separable complex Hilbert space. Suppose that $\phi : \mathcal{S}_2^+(H)_1 \to \mathcal{S}_2^+(H)_1$, which needs to be surjective when $\dim\ H = +\infty$. The following conditions are equivalent.*

(1) *ϕ preserves the Hilbert-Schmidt norms of all convex combinations, i.e.,*

$$\|t\rho + (1-t)\sigma\|_2 = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_2, \quad \forall \rho, \sigma \in \mathcal{S}_2^+(H)_1, t \in [0,1].$$

(2) *For any (and thus all) α in $(0,1)$ we have*

$$\|\alpha\rho + (1-\alpha)\sigma\|_2 = \|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_2, \quad \forall \rho, \sigma \in \mathcal{S}_2^+(H)_1.$$

A special case states

$$\|\rho + \sigma\|_2 = \|\phi(\rho) + \phi(\sigma)\|_2, \quad \forall \rho, \sigma \in \mathcal{S}_2^+(H)_1.$$

(3) $\operatorname{tr}(\rho\sigma) = \operatorname{tr}(\phi(\rho)\phi(\sigma))$ *for all ρ, σ in $\mathcal{S}_2^+(H)_1$.*

(4) *There exists a unitary or anti-unitary operator U such that*

$$\phi(\rho) = U\rho U^*, \quad \forall \rho \in \mathcal{S}_2^+(H)_1.$$

Proof. We prove (2) $\Rightarrow$ (3) only. Observe

$$\|\alpha\rho + (1-\alpha)\sigma\|_2^2 = \operatorname{tr}(\alpha\rho + (1-\alpha)\sigma)^2 = \alpha^2 \operatorname{tr}\rho^2 + 2\alpha(1-\alpha)\operatorname{tr}(\rho\sigma) + (1-\alpha)^2 \operatorname{tr}\sigma^2,$$

and

$$\|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_2^2 = \alpha^2 \operatorname{tr}\phi(\rho)^2 + 2\alpha(1-\alpha)\operatorname{tr}(\phi(\rho)\phi(\sigma)) + (1-\alpha)^2 \operatorname{tr}\phi(\sigma)^2.$$

We have, $\operatorname{tr}(\rho\sigma) = \operatorname{tr}(\phi(\rho)\phi(\sigma))$.

□

4. Examples

We remark that all results in previous sections hold for a map $\phi : \mathcal{S}_p^+(H) \to \mathcal{S}_p^+(H)$ which satisfies instead of (1.2), the condition

$$\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H), t \in [0,1].$$

The proofs go in exactly the same ways.

The following example shows that a norm preserver of $\mathcal{S}_p^+(H)$ might not be affine.

Example 4.1. *Let H be a finite dimensional Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^n$. Let $1 < p < +\infty$. Define a map ϕ from $\mathcal{S}_p^+(H)$ into itself by*

$$\phi(\rho) = \begin{cases} 0, & \text{if } \rho = 0, \\ \dfrac{\|\rho\|_p}{\|\sum_{i=1}^n P_i\rho P_i\|_p} \displaystyle\sum_{i=1}^n P_i\rho P_i, & \text{if } \rho \neq 0, \end{cases} \tag{4.1}$$

where $P_i = e_i \otimes e_i$ is a rank one projection for $i = 1, \ldots, n$. Obviously, $\phi(\rho)$ is positive and $\|\phi(\rho)\|_p = \|\rho\|_p$ for all ρ in $\mathcal{S}_p^+(H)$. However, ϕ does not preserve the Schatten p-norms of convex combinations, as the eigenvalues of ρ and $\phi(\rho)$ can be different from each other.

Our theorems are about Schatten p-norms for $1 < p < +\infty$. Here is an example of a map of $\mathcal{S}_1^+(H)$ which preserves trace norms of convex combinations. However, it is not implemented by a unitary or anti-unitary.

Example 4.2. *Consider Example 4.1 in the case where* $p = 1$. *In this case,*

$$\phi(\rho) = \sum_{i=1}^{n} P_i \rho P_i. \tag{4.2}$$

It is easy to see that the map ϕ *satisfies the condition*

$$\|t\rho + (1-t)\sigma\|_1 = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_1, \quad \forall \rho, \sigma \in \mathcal{S}_1^+(H), t \in [0,1].$$

But there does not exist a unitary or anti-unitary U *such that* $\phi(\rho) = U\rho U^*$ *for all* ρ *in* $\mathcal{S}_1^+(H)$.

Example 4.3. *Let* H *be a separable Hilbert space of infinite dimension, and* $\{e_n : n = 1, 2, \ldots\}$ *be a basis of* H. *Let* S *be the unilateral shift on* H *defined by* $Se_n = e_{n+1}$ *for* $n = 1, 2, \ldots$. *Let* ϕ *be defined by* $\phi(\rho) = S\rho S^*$ *for* ρ *in* $\mathcal{S}_p^+(H)$. *The map* ϕ *is not surjective, as* $e_1 \otimes e_1$ *is not in its range. It is easy to see that* $\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p$ *holds for all* ρ, σ *in* $\mathcal{S}_p^+(H)$ *and* t *in* $[0,1]$. *However,* ϕ *is not implemented by a unitary or anti-unitary.*

5. Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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(Kuo, Tsai and Wong) Department of Applied Mathematics, National Sun Yat-Sen University, Kaohsiung 80424, Taiwan.

E-mail address, Kuo: mpu.verilog@gmail.com

E-mail address, Tsai: mctsai2@gmail.com

E-mail address, Wong: wong@math.nsysu.edu.tw

(Zhang) School of Mathematics and Statistics, Central China Normal University, Wuhan, Hubei 430079, China.

E-mail address, Zhang: zhjun@mail.ccnu.edu.cn