Maps Preserving Schatten $p$-Norms of Convex Combinations

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Abstract. In this paper, we study maps $\phi$ of positive operators of Schatten $p$-classes $(1 < p < +\infty)$, which preserve the $p$-norms of convex combinations, that is,

$$
\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in S^+_p(H), t \in [0,1].
$$

They are exactly those carrying the form $\phi(\rho) = U\rho U^*$ for a unitary or anti-unitary $U$. In the case $p = 2$, we have the same conclusion whenever it just holds

$$
\|\rho + \sigma\|_2 = \|\phi(\rho) + \phi(\sigma)\|_2
$$

for all positive Hilbert-Schmidt class operators $\rho, \sigma$ of norm 1. Some examples are demonstrated.

Keywords: Schatten $p$-classes; norm and distance preservers; Wigner type theorems.

1. Introduction

The Mazur-Ulam theorem states that every bijective distance preserving map $\Phi$ from a Banach space onto another is affine, i.e.,

$$
\Phi(tx + (1-t)y) = t\Phi(x) + (1-t)\Phi(y), \quad \forall x, y, 0 \leq t \leq 1.
$$

After translation, we can assume $\Phi(0) = 0$ and $\Phi$ is indeed a surjective real linear isometry. Let us consider another version of this statement. Suppose that $\Phi$ is a bijective map from a Hilbert space $H$ onto $H$ and $\Phi$ preserves norm of convex combinations:

$$
\|t\Phi(x) + (1-t)\Phi(y)\| = \|tx + (1-t)y\|, \quad \forall x, y \in H, 0 \leq t \leq 1. \quad (1.1)
$$

Let us further relax the assumption that (1.1) holds for just one fixed $t$ in $(0,1)$. By letting $y = x$ in (1.1), we see that $\|\Phi(x)\| = \|x\|$ for all $x$ in $H$. Squaring both sides of (1.1), we will see that the real parts of the inner products coincide, i.e.

$$
\text{Re} \langle x, y \rangle = \text{Re} \langle \Phi(x), \Phi(y) \rangle, \quad \forall x, y \in H.
$$

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Then the classical Wigner theorem (see, e.g., [7, Theorem 3]) ensures that there is a surjective real linear isometry $U : H \to H$ such that $\Phi(x) = Ux$ for all $x$ in $H$.

Characterizing isometries, linear or not, of spaces of operators under various norms has been a fruitful area of research for a long time. See, e.g., [3,5] for good surveys. In particular, the spaces $S_p(H)$ of Schatten $p$-class operators on a (complex) Hilbert space $H$ ($1 \leq p < +\infty$) are important objects in both analysis and physics. They are widely used in operator theory and quantum mechanics, for example.

Let $S^+_p(H)$ be the set of all positive operators in $S_p(H)$, and let $S^+_p(H)_1$ be the set of all positive operators in $S^+_p(H)$ of $p$-norm one. Recall that an affine automorphism (or S-automorphism in [2], or Kadison automorphism in [11]) is a bijective affine map $\phi : S^+_1(H)_1 \to S^+_1(H)_1$, i.e.,

$$\phi(t\rho + (1-t)\sigma) = t\phi(\rho) + (1-t)\phi(\sigma), \quad \forall \rho, \sigma \in S^+_1(H)_1, t \in [0,1].$$

It is known (see, e.g., [9]) that affine automorphisms are exactly those carrying the form $\phi(\rho) = U\rho U^*$ for a unitary or anti-unitary $U$ on $H$.

Recently, Nagy [10] established a Mazur-Ulam type result for the Schatten $p$-class operators. Suppose $\phi : S^+_p(H)_1 \to S^+_p(H)_1$ ($1 < p < +\infty$) is a bijective map preserving the distance induced by the norm $\| \cdot \|_p$. Then $\phi$ is implemented by a unitary or an anti-unitary operator $U$ such that $\phi(\rho) = U\rho U^*$. In this paper, we will establish a counter part of Nagy’s result similar to the one demonstrated in the first paragraph. More precisely, we will characterize those maps $\phi : S^+_p(H)_1 \to S^+_p(H)_1$ satisfying

$$\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in S^+_p(H)_1, t \in [0,1].$$

We will show that they are implemented by a unitary or an anti-unitary operator.

Our main theorem follows.

**Theorem 1.1.** Let $H$ be a separable complex Hilbert space of finite or infinite dimension. Let $1 < p < +\infty$. Suppose that $\phi$ is a map from $S^+_p(H)_1$ into $S^+_p(H)_1$, which will be assumed to be surjective when dim $H = +\infty$. The following conditions are equivalent.

1. $\phi$ preserves the Schatten $p$-norms of convex combinations, i.e.,

$$\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in S^+_p(H)_1, t \in [0,1]. \quad (1.2)$$

2. $\phi$ preserves the pairings, i.e., for all $\rho, \sigma \in S^+_p(H)_1$, we have $\sigma^{p-1}\rho \in S_1(H)$, and

$$\text{tr}(\sigma^{p-1}\rho) = \text{tr}(\phi(\sigma)^{p-1}\phi(\rho)). \quad (1.3)$$

3. There exists a unitary or anti-unitary operator $U$ on $H$ such that

$$\phi(\rho) = U\rho U^*, \quad \forall \rho \in S^+_p(H)_1.$$
We note that the condition (1.2) becomes a tautology when \( p = 1 \). On the other hand, the conclusion of Theorem 1.1 holds again if we replace \( S_p^+(H)_1 \) by \( S_p^+(H) \) everywhere. In this case, setting \( \sigma = \rho \) in (1.2) we see that \( \phi \) does map \( S_p^+(H)_1 \) into \( S_p^+(H) \).

The proof of Theorem 1.1 is given in Section 2. When \( p = 2 \), we see in Section 3 that for \( \phi \) carrying the expected form stated in Theorem 1.1(3) it suffices condition (1.2) held for only one fixed \( t \) in \((0, 1)\). Finally, we demonstrate some examples in Section 4.

2. Proof of the main theorem

In what follows, we fix some notation and definitions used throughout the paper. Let \( H \) stand for a separable complex Hilbert space of finite dimension or infinite dimension. Let \( B(H) \) denote the algebra of all bounded linear operators on \( H \). For a compact operator \( T \) in \( B(H) \), let \( s_1(T) \geq s_2(T) \geq \cdots \geq 0 \) denote the singular values of \( T \), i.e., the eigenvalues of \( \sqrt{|T|} = (TT^*)^{1/2} \) arranged in their decreasing order (repeating according to multiplicity). A compact operator \( T \) belongs to the Schatten \( p \)-classes \( S_p(H) \) \((1 \leq p < +\infty)\) if

\[
\|T\|_p := \left( \sum_{i=1}^{\infty} s_i(T)^p \right)^{1/p} = (\text{tr}|T|^p)^{1/p} < +\infty,
\]

where \( \text{tr} \) denotes the trace functional. We call \( \|T\|_p \) the Schatten \( p \)-norm of \( T \). In particular, \( S_1(H) \) is the trace class and \( S_2(H) \) is the Hilbert-Schmidt class. The cone of positive operators in \( S_p(H) \) is denoted by \( S_p^+(H) \), and the set of rank one projections in \( S_p^+(H) \) is denoted by \( P_1(H) \).

Recall that the norm of a normed space is Fréchet differentiable at \( x \neq 0 \) if \( \lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \) exists and uniform for all norm one vectors \( y \).

**Lemma 2.1** ([1, Theorem 2.3]). Let \( 1 < p < +\infty \) and \( \rho \) in \( S_p^+(H) \) be nonzero. The norm of \( S_p^+(H) \) is Fréchet differentiable at \( \rho \). For any \( \sigma \) in \( S_p^+(H) \) we have

\[
\frac{d\|\rho + t\sigma\|_p}{dt}\bigg|_{t=0} = \text{tr}\left( \frac{\rho^{p-1}\sigma}{\|\rho\|^{p-1}} \right).
\]

**Lemma 2.2.** Suppose \( \rho, \sigma \in S_p^+(H) \) \((1 < p < +\infty)\). The following conditions are equivalent.

1. \( \rho = \sigma \).
2. \( \|t\rho + (1 - t)P\|_p = \|t\sigma + (1 - t)P\|_p \) for all \( P \) in \( P_1(H) \) and all \( t \) in \([0, 1]\).
3. \( \text{tr}(P\rho) = \text{tr}(P\sigma) \) for all \( P \) in \( P_1(H) \).

**Proof.** (1) \( \Rightarrow \) (2) is obvious.

(2) \( \Rightarrow \) (3): Differentiating both sides of \( \|t\rho + (1 - t)P\|_p = \|t\sigma + (1 - t)P\|_p \) at \( t = 0^+ \), we have \( \text{tr}P\rho = \text{tr}P^{p-1}\rho = \text{tr}P^{p-1}\sigma = \text{tr}P\sigma \) by Lemma 2.1.
(3) $\Rightarrow$ (1): Since $\rho$ and $\sigma$ are positive, $\rho - \sigma$ is hermitian. There exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $H$ such that $\rho - \sigma = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$. Choosing $P_i = e_i \otimes e_i$, we have $\lambda_i = \text{tr}(P_i(\rho - \sigma)) = 0$ for all $i = 1, 2, \ldots$. It follows $\rho - \sigma = 0$. 

We say that two self-adjoint operators $\rho, \sigma$ in $B(H)$ are orthogonal if $\rho \sigma = 0$, which is equivalent to the property that they have mutually orthogonal ranges.

**Lemma 2.3.** Suppose that $\rho, \sigma \in S_p^+(H)$ for $1 < p < +\infty$. The following conditions are equivalent.

1. $\rho, \sigma$ are orthogonal, i.e., $\rho \sigma = 0$.
2. $\|\alpha \rho + (1 - \alpha)\sigma\|_p^p = \alpha^p \|\rho\|_p^p + (1 - \alpha)^p \|\sigma\|_p^p$ for any (and thus all) $\alpha$ in $(0, 1)$.
3. $\text{tr}(\rho \sigma) = 0$.
4. $\|\rho + t\sigma\|_p \geq \|\rho\|_p$ for all $t$ in $\mathbb{R}$, that is, $\rho \perp \sigma$ in Birkhoff’s sense.
5. $\text{tr}(\rho^{p-1} \sigma) = 0$.

**Proof.** (1) $\Leftrightarrow$ (2): From [6, Lemma 2.6], we know that for any two positive operators $A, B$ in $S_p^+(H)$, we have

$$\text{tr}(A + B)^p \geq \text{tr} A^p + \text{tr} B^p.$$ 

Here the equality holds if and only if $AB = 0$. Setting $A = \alpha \rho$ and $B = (1 - \alpha)\sigma$, we get

$$\rho \sigma = 0 \Leftrightarrow (\alpha \rho)((1 - \alpha)\sigma) = 0$$

$$\Leftrightarrow \text{tr}(\alpha \rho + (1 - \alpha)\sigma)^p = \text{tr}(\alpha \rho)^p + \text{tr}((1 - \alpha)\sigma)^p$$

$$\Leftrightarrow \|\alpha \rho + (1 - \alpha)\sigma\|_p^p = \alpha^p \|\rho\|_p^p + (1 - \alpha)^p \|\sigma\|_p^p.$$

(1) $\Leftrightarrow$ (3): One direction is obvious. For the other, because $\rho, \sigma$ are positive,

$$\text{tr}[(\rho^{\frac{1}{2}} \sigma^{\frac{1}{2}})(\rho^{\frac{1}{2}} \sigma^{\frac{1}{2}})^*] = \text{tr}(\rho^{\frac{1}{2}} \sigma^{\frac{1}{2}} \sigma^{\frac{1}{2}} \rho^{\frac{1}{2}}) = \text{tr}(\rho \sigma) = 0.$$ 

This forces $\rho^{\frac{1}{2}} \sigma^{\frac{1}{2}} = 0$, and thus $\rho \sigma = \rho^{\frac{1}{2}}(\rho^{\frac{1}{2}} \sigma^{\frac{1}{2}})\sigma^{\frac{1}{2}} = 0$.

(1) $\Rightarrow$ (4): Since $\rho \sigma = 0$, there exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $H$ such that $\rho = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i$, $\sigma = \sum_{i=1}^{\infty} \mu_i e_i \otimes e_i$, $\lambda_i \geq 0$, $\mu_i \geq 0$, and $\lambda_i \mu_i = 0$ for all $i = 1, 2, \ldots$. Hence,

$$\|\rho + t\sigma\|_p^p = \text{tr} |\rho + t\sigma|^p = \sum_{i=1}^{\infty} (\lambda_i + |t|\mu_i)^p \geq \sum_{i=1}^{\infty} \lambda_i^p = \|\rho\|_p^p.$$ 

(4) $\Rightarrow$ (5): Without loss of generality, we can assume that $\rho \neq 0$. Define $f(t) = \|\rho + t\sigma\|_p \geq \|\rho\|_p$. Then $f(t)$ is differentiable and attains its minimum at $t = 0$. From Lemma 2.1,

$$0 = \frac{d\|\rho + t\sigma\|_p}{dt} \bigg|_{t=0} = \text{tr} \left( \frac{\rho^{p-1} \sigma}{\|\rho\|_p^{p-1}} \right),$$

and assertion (5) follows.
As in proving (3) ⇒ (1), we have $\rho^{p-1} = 0$. Then there exists an orthonormal basis $\{e'_i\}_{i=1}^{\infty}$ of $H$ such that $\rho^{p-1} = \sum_{i=1}^{\infty} \xi_i e'_i \otimes e'_i$, $\sigma = \sum_{i=1}^{\infty} \eta_i e'_i \otimes e'_i$, with $\xi_i \geq 0$, $\eta_i \geq 0$, and $\xi_i \mu_i = 0$ for all $i = 1, 2, \cdots$. Thus, $\text{tr}(\rho \sigma) = \sum_{i=1}^{\infty} \frac{1}{\xi_i^{p-1}} \eta_i = 0$. □

Lemma 2.4. Let $1 < p < +\infty$. Suppose that $\phi$ is a map from $S_p^+(H)_1$ into $S_p^+(H)_1$ preserving the Schatten $p$-norms of convex combinations, i.e., equation (1.2) holds. Then we have

$$\text{tr}(\sigma^{p-1} \rho) = \text{tr}(\phi(\sigma)^{p-1} \phi(\rho)).$$

Proof. Differentiating both sides of (1.2) with respect to $t$ and evaluating at $t = 0$, we have

$$\frac{d}{dt} \|t \rho + (1-t) \sigma\|_p \bigg|_{t=0} = \frac{d}{dt} \|\sigma + t(\rho - \sigma)\|_p \bigg|_{t=0} = \text{tr} \left( \frac{\sigma^{p-1}(\rho - \sigma)}{\|\sigma\|_p^{p-1}} \right) = \text{tr}(\sigma^{p-1} \rho) \|\sigma\|_p^{p-1} - \|\sigma\|_p = \text{tr}(\sigma^{p-1} \rho) - 1,$$

and

$$\frac{d}{dt} \|t \phi(\rho) + (1-t) \phi(\sigma)\|_p \bigg|_{t=0} = \frac{d}{dt} \|\phi(\sigma)\|_p^{p-1} - \|\phi(\sigma)\|_p = \text{tr}(\phi(\sigma)^{p-1} \phi(\rho)) - 1.$$

Since (1.2) holds for $t$ in $(0, 1]$, these derivatives agree. Therefore, $\text{tr}(\sigma^{p-1} \rho) = \text{tr}(\phi(\sigma)^{p-1} \phi(\rho))$. □

Proposition 2.5. Suppose $\phi : S_p^+(H)_1 \to S_p^+(H)_1$ satisfies that

$$\text{tr}(\sigma^{p-1} \rho) = \text{tr}(\phi(\sigma)^{p-1} \phi(\rho)), \quad \forall \rho, \sigma \in S_p^+(H)_1.$$

Then the following assertions hold.

1. $\phi$ preserves orthogonality in both directions, that is

   $\rho\sigma = 0 \Leftrightarrow \phi(\rho)\phi(\sigma) = 0$, \quad $\forall \rho, \sigma \in S_p^+(H)_1$.

2. When $\text{dim } H < +\infty$, $\phi$ maps rank one projections to rank one projections. This also holds when $\text{dim } H = +\infty$ and $\phi$ is surjective.

3. When $\text{dim } H < +\infty$, we have

   $$\text{tr } PQ = \text{tr } \phi(P)\phi(Q), \quad \forall P, Q \in P_1(H).$$

   This also holds when $\text{dim } H = +\infty$ and $\phi$ is surjective.
Proof. (1) follows from Lemma 2.3.

(2) First, we assume dim $H = n < +\infty$. Suppose $\rho$ is a rank one projection. We can find $n - 1$ pairwise orthogonal rank one projections $\rho_1, \cdots, \rho_{n-1}$ such that $\rho \rho_i = 0$ for $1 \leq i \leq n - 1$. From (1), we know that $\phi(\rho), \phi(\rho_1), \cdots, \phi(\rho_{n-1})$ are nonzero and pairwise orthogonal. This forces $\phi(\rho)$ has rank one since dim $H = n$. By (2.4), taking $\sigma = \rho$, we see that $\text{tr} \phi(\rho)^p = \text{tr} \rho^p = \text{tr} \rho = 1$. Therefore, the rank one positive operator $\phi(\rho)$ is a projection.

Next, we consider the case dim $H = +\infty$ and $\phi$ is surjective. Suppose that there exists a rank one projection $\rho$ in $S_p^+(H)$ such that $\phi(\rho)$ has rank greater than one. Then there are two nonzero orthogonal operators $T_1$ and $T_2$ in $S_p^+(H)$ such that $\phi(\rho) = T_1 + T_2$. Since $\phi$ is surjective and preserves orthogonality in both directions, there are two nonzero orthogonal operators $\rho_1$ and $\rho_2$ in $S_p^+(H)$ such that $\phi(\rho_1) = T_1/\|T_1\|_p$ and $\phi(\rho_2) = T_1/\|T_2\|_p$. For any $\sigma$ in $S_p^+(H)$ with $\sigma \rho = 0$, we have

$$\phi(\sigma)(\|T_1\|_p \phi(\rho_1) + \|T_2\|_p \phi(\rho_2)) = \phi(\sigma)(T_1 + T_2) = \phi(\sigma)\phi(\rho) = 0.$$  

It forces

$$\|T_1\|_p \phi(\sigma) \phi(\rho_1)\phi(\sigma) = -\|T_2\|_p \phi(\sigma) \phi(\rho_2)\phi(\sigma) = 0,$$

and hence $\phi(\sigma) \phi(\rho_1) = \phi(\sigma) \phi(\rho_2) = 0$, because $\phi(\sigma)$, $\phi(\rho_1)$ and $\phi(\rho_2)$ are all positive. This implies $\sigma \rho_1 = \sigma \rho_2 = 0$. Therefore, $\rho_1 = \lambda_1 \rho$ and $\rho_2 = \lambda_2 \rho$ for some nonzero $\lambda_1, \lambda_2$. This contradicts to that $\rho_1 \rho_2 = 0$.

(3) From (2), we know that $\phi(P), \phi(Q)$ are rank one projections in $P_1(H)$. Therefore, $P^{p-1} = P, \phi(P)^{p-1} = \phi(P)$. Using Equation (2.4) with $\sigma = P, \rho = Q$ we have

$$\text{tr} \ P \ Q = \text{tr} (P^{p-1} Q) = \text{tr}(\phi(P)^{p-1} \phi(Q)) = \text{tr} \phi(P) \phi(Q).$$

$\square$

Proof of Theorem 1.1. (1) $\Rightarrow$ (2) follows from Lemma 2.4.

(3) $\Rightarrow$ (1) is obvious.

(2) $\Rightarrow$ (3): From Proposition 2.5, we obtain that $\phi|_{P_1(H)} : P_1(H) \rightarrow P_1(H)$ satisfies $\text{tr} \ P \ Q = \text{tr} \phi(P) \phi(Q)$ for all rank one projections $P, Q$ in $P_1(H)$. From a nonsurjective version of Wigner’s theorem, c.f. [9, Theorem 2.1.4], there exists an isometry or anti-isometry $U$ on $H$ such that

$$\phi(P) = UPU^*, \quad \forall P \in P_1(H).$$

Note that $U$ is indeed surjective even when $H$ is of infinite dimension, since $\phi$ is assumed to be surjective in this case.

For any rank one projection $P$ in $P_1(H)$, setting $\sigma = P$ in (1.3) we have

$$\text{tr}(P \rho) = \text{tr}(P^{p-1} \rho) = \text{tr}(\phi(P)^{p-1} \phi(\rho)) = \text{tr}(\phi(P) \phi(\rho))$$

$$= \text{tr}(UPU^* \phi(\rho) U) = \text{tr}(PU^* \phi(\rho) U)$$
We have $U^*\phi(\rho)U = \rho$ by Lemma 2.2. This gives $\phi(\rho) = U\rho U^*$.

3. Maps preserving norms of just a special convex combination

A careful look at the proof of Lemma 2.4 tells us that the condition $\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p$ suffices to hold for the members of any sequence in $(0,1]$ converging to 0 rather than for any point $t$ in $[0,1]$. Indeed, in order to get some good properties of $\phi$ stated in the previous section, we only need to assume that $\phi$ preserves the Schatten $p$-norm of convex combination with a given system of coefficients.

**Proposition 3.1.** Let $\phi : S_p^+(H)_1 \to S_p^+(H)_1$ $(1 < p < +\infty)$. Let $\alpha$ in $(0,1)$ be arbitrary but fixed. Suppose

$$\|\alpha\rho + (1-\alpha)\sigma\|_p = \|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in S_p^+(H)_1, \quad (3.1)$$

The following properties are satisfied.

1. $\phi$ is injective.
2. $\phi$ preserves orthogonality in both directions.
3. When $\dim H < +\infty$, $\phi$ maps rank one projections to rank one projections. This also holds when $\dim H = +\infty$ and $\phi$ is surjective.

**Proof.** (1) Assume $\phi(\rho) = \phi(\sigma)$. We have $\|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_p = 1$. From (3.1) we get $\|\alpha\rho + (1-\alpha)\sigma\|_p = 1$. Hence,

$$\|\alpha\rho + (1-\alpha)\sigma\|_p = \alpha\|\rho\|_p + (1-\alpha)\|\sigma\|_p.$$ 

This forces $\rho = \sigma$ since the norm $\| \cdot \|_p$ is strictly convex for $1 < p < +\infty$.

(2) Assume $\rho\sigma = 0$. From Lemma 2.3, we have

$$\|\alpha\rho + (1-\alpha)\sigma\|_p^p = \alpha^p\|\rho\|_p^p + (1-\alpha)^p\|\sigma\|_p^p = \alpha^p\|\phi(\rho)\|_p^p + (1-\alpha)^p\|\phi(\sigma)\|_p^p.$$ 

Together with (3.1), we have

$$\|\alpha\phi(\rho) + (1-\alpha)\phi(\sigma)\|_p^p = \alpha^p\|\phi(\rho)\|_p^p + (1-\alpha)^p\|\phi(\sigma)\|_p^p.$$ 

Hence, we have $\phi(\rho)\phi(\sigma) = 0$ from Lemma 2.3 again. The other implication follows similarly.

(3) The proof is similar to that of Proposition 2.5(2). \qed

When $p = 2$, we get an improvement of Theorem 1.1.

**Theorem 3.2.** Let $H$ be a separable complex Hilbert space. Suppose that $\phi : S_2^+(H)_1 \to S_2^+(H)_1$, which needs to be surjective when $\dim H = +\infty$. The following conditions are equivalent.
(1) $\phi$ preserves the Hilbert-Schmidt norms of all convex combinations, i.e.,
$$\|t\rho + (1 - t)\sigma\|_2 = \|t\phi(\rho) + (1 - t)\phi(\sigma)\|_2, \quad \forall \rho, \sigma \in S^+_2(H), t \in [0, 1].$$

(2) For any (and thus all) $\alpha$ in $(0, 1)$ we have
$$\|\alpha\rho + (1 - \alpha)\sigma\|_2 = \|\alpha\phi(\rho) + (1 - \alpha)\phi(\sigma)\|_2, \quad \forall \rho, \sigma \in S^+_2(H).$$

A special case states
$$\|\rho + \sigma\|_2 = \|\phi(\rho) + \phi(\sigma)\|_2, \quad \forall \rho, \sigma \in S^+_2(H).$$

(3) $\text{tr}(\rho\sigma) = \text{tr}(\phi(\rho)\phi(\sigma))$ for all $\rho, \sigma$ in $S^+_2(H)$.

(4) There exists a unitary or anti-unitary operator $U$ such that
$$\phi(\rho) = U\rho U^*, \quad \forall \rho \in S^+_2(H).$$

**Proof.** We prove (2) $\Rightarrow$ (3) only. Observe
$$\|\alpha\rho + (1 - \alpha)\sigma\|_2^2 = \text{tr}(\alpha\rho + (1 - \alpha)\sigma)^2 = \alpha^2 \text{tr}\rho^2 + 2\alpha(1 - \alpha)\text{tr}(\rho\sigma) + (1 - \alpha)^2 \text{tr}\sigma^2,$$
and
$$\|\alpha\phi(\rho) + (1 - \alpha)\phi(\sigma)\|_2^2 = \alpha^2 \text{tr}(\phi(\rho))^2 + 2\alpha(1 - \alpha)\text{tr}(\phi(\rho)\phi(\sigma)) + (1 - \alpha)^2 \text{tr}(\phi(\sigma))^2.$$

We have, $\text{tr}(\rho\sigma) = \text{tr}(\phi(\rho)\phi(\sigma))$. 

4. **Examples**

We remark that all results in previous sections hold for a map $\phi : S^+_p(H) \rightarrow S^+_p(H)$ which satisfies instead of (1.2), the condition
$$\|t\rho + (1 - t)\sigma\|_p = \|t\phi(\rho) + (1 - t)\phi(\sigma)\|_p, \quad \forall \rho, \sigma \in S^+_p(H), t \in [0, 1].$$

The proofs go in exactly the same ways.

The following example shows that a norm preserver of $S^+_p(H)$ might not be affine.

**Example 4.1.** Let $H$ be a finite dimensional Hilbert space with an orthonormal basis $\{e_i\}_{i=1}^n$. Let $1 < p < +\infty$. Define a map $\phi$ from $S^+_p(H)$ into itself by

$$\phi(\rho) = \begin{cases} 0, & \text{if } \rho = 0, \\ \frac{\|\rho\|_p}{\sum_{i=1}^n P_i\rho P_i\|_p} \sum_{i=1}^n P_i\rho P_i, & \text{if } \rho \neq 0, \end{cases} \quad (4.1)$$

where $P_i = e_i \otimes e_i$ is a rank one projection for $i = 1, \ldots, n$. Obviously, $\phi(\rho)$ is positive and $\|\phi(\rho)\|_p = \|\rho\|_p$ for all $\rho$ in $S^+_p(H)$. However, $\phi$ does not preserve the Schatten $p$-norms of convex combinations, as the eigenvalues of $\rho$ and $\phi(\rho)$ can be different from each other.
Our theorems are about Schatten $p$-norms for $1 < p < +\infty$. Here is an example of a map of $S^+_1(H)$ which preserves trace norms of convex combinations. However, it is not implemented by a unitary or anti-unitary.

**Example 4.2.** Consider Example 4.1 in the case where $p = 1$. In this case,

$$\phi(\rho) = \sum_{i=1}^{n} P_i \rho P_i.$$  \hspace{1cm} (4.2)

It is easy to see that the map $\phi$ satisfies the condition

$$\|t \rho + (1-t) \sigma\|_1 = \|t \phi(\rho) + (1-t) \phi(\sigma)\|_1, \quad \forall \rho, \sigma \in S^+_1(H), t \in [0, 1].$$

But there does not exist a unitary or anti-unitary $U$ such that $\phi(\rho) = U \rho U^*$ for all $\rho$ in $S^+_1(H)$.

**Example 4.3.** Let $H$ be a separable Hilbert space of infinite dimension, and $\{e_n : n = 1, 2, \ldots\}$ be a basis of $H$. Let $S$ be the unilateral shift on $H$ defined by $Se_n = e_{n+1}$ for $n = 1, 2, \ldots$. Let $\phi$ be defined by $\phi(\rho) = S \rho S^*$ for $\rho$ in $S^+_p(H)$. The map $\phi$ is not surjective, as $e_1 \otimes e_1$ is not in its range. It is easy to see that $\|t \rho + (1-t) \sigma\|_p = \|t \phi(\rho) + (1-t) \phi(\sigma)\|_p$ holds for all $\rho, \sigma$ in $S^+_p(H)$ and $t$ in $[0, 1]$. However, $\phi$ is not implemented by a unitary or anti-unitary.

5. **Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this article.

**References**


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