INNER PRODUCTS AND MODULE MAPS OF HILBERT C*-MODULES

MING-HSIU HSU AND NGAI-CHING WONG

ABSTRACT. Let $E$ and $F$ be two Hilbert $C^*$-modules over $C^*$-algebras $A$ and $B$, respectively. Let $T$ be a surjective linear isometry from $E$ onto $F$ and $\varphi$ a map from $A$ into $B$. We will prove in this paper that if the $C^*$-algebras $A$ and $B$ are commutative, then $T$ preserves the inner products and $T$ is a module map, i.e., there exists a $*$-isomorphism $\varphi$ between the $C^*$-algebras such that $\langle Tx, Ty \rangle = \varphi(\langle x, y \rangle)$, and $T(ax) = T(x)\varphi(a)$.

In case $A$ or $B$ is noncommutative $C^*$-algebra, $T$ may not satisfy the equations above in general. We will also give some condition such that $T$ preserves the inner products and $T$ is a module map.

1. Introduction

A (right) Hilbert $C^*$-module over a $C^*$-algebra $A$ is a right $A$-module $E$ equipped with $A$-valued inner product $\langle \cdot, \cdot \rangle$ which is conjugate $A$-linear in the first variable and $A$-linear in the second variable such that $E$ is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$.

Let $X$ be a locally compact Hausdorff space and $H$ a Hilbert space, the Banach-Stone theorem states that every surjective linear isometry between $C_0(X, H_1)$ and $C_0(Y, H_2)$ is also of the form (1)

Let $X$ and $Y$ be two locally compact Hausdorff spaces, the Banach-Stone theorem states that every surjective linear isometry between $C_0(X)$ and $C_0(Y)$ is a weighted composition operator. More precisely, let $T$ be a surjective linear isometry from $C_0(X)$ onto $C_0(Y)$, then there exists a continuous function $h \in C_0(Y)$ with $|h(y)| = 1$, for all $y$ in $Y$, and a homeomorphism $\varphi$ from $Y$ onto $X$ such that $T$ is of the form:

(1) $Tf(y) = h(y)f(\varphi(y)), \forall f \in C_0(X), \forall y \in Y.$

Let $H_1$ and $H_2$ be two Hilbert spaces. In [7], Jerison characterizes surjective linear isometries between $C_0(X, H_1)$ and $C_0(Y, H_2)$, see also [12, 6]. It is said that every surjective linear isometry $T$ from $C_0(X, H_1)$ onto $C_0(Y, H_2)$ is also of the form (1)

2000 Mathematics Subject Classification. 46L08, 46E40, 46B04.

Key words and phrases. Hilbert $C^*$-modules, TROs, complete isometries, triple products, Banach-Stone type theorems.

This work is jointly supported by a Taiwan NSC Grant ( ).
in which \( h(y) \) is a unitary operator from \( H_1 \) onto \( H_2 \) and \( h \) is continuous from \( Y \) into \( (B(H_1, H_2), SOT) \), the space of all bounded linear operators with the strong operator topology. In this case, we can find a relationship of inner products of \( C_0(X, H_1) \) and \( C_0(Y, H_2) \) by a simple calculation:

\[
\langle Tf, Tg \rangle(y) = \langle Tf(y), Tg(y) \rangle = \langle h(y)(f(\varphi(y))), h(y)(f(\varphi(y))) \rangle = \langle f(\varphi(y)), f(\varphi(y)) \rangle = \langle f, g \rangle \circ \varphi(y).
\]

i.e.

\[
\langle Tf, Tg \rangle = \langle f, g \rangle \circ \varphi.
\]

Let \( R_\varphi : C_0(X) \to C_0(Y) \) be the \(*\)-isomorphism defined by \( R_\varphi(\phi) = \phi \circ \varphi \). Then \( T \) preserves the inner products with respect to \( R_\varphi \), i.e.,

\[
\langle Tf, Tg \rangle = R_\varphi(f, g).
\]

By (1), it is easy to see that \( T \) is a module map with respect to \( R_\varphi \) in the sense

\[
T(f \circ \phi) = T(f)R_\varphi(\phi), \text{ for all } f \in C_0(X, H_1) \text{ and } \phi \in C_0(X).
\]

It is natural to ask if these properties are true for surjective linear isometries between Hilbert \( C^* \)-modules over \( C^* \)-algebras. We will show in this paper that the answer is yes if the \( C^* \)-algebras are commutative. Unfortunately, if one of the \( C^* \)-algebras is noncommutative, the answer is more complicated. We will give an example (see Example 3) to explain this is not true in general. And we will give a condition on \( T \) (see Theorem 9) such that \( T \) is a module map and preserves the inner products.

2. Preliminaries

Let \( E \) be a Hilbert \( C^* \)-module over \( C^* \)-algebra \( A \). We set \( \langle E, E \rangle \) to be the linear span of elements of the form \( \langle x, y \rangle, x, y \in E \). \( E \) is said to be full if the closed two-sided ideal \( \overline{\langle E, E \rangle} \) equal \( A \).

A \( JB^* \)-triple is a complex vector space \( V \) with a continuous mapping \( V^3 \to V, (x, y, z) \to \{x, y, z\} \), called a Jordan triple product, which is symmetric and linear in \( x \) and conjugate linear in \( y \) such that for \( x, y, z, u, v \) in \( V \), we have

1. \( \{x, y, \{z, u, v\}\} = \{\{x, y, z\}, u, v\} - \{z, \{y, x, u\}, v\} + \{z, u, \{x, y, v\}\}; \)
2. the mapping \( z \to \{x, x, z\} \) is hermitian and has non-negative spectrum;
3. \( \|\{x, x, x\}\| = \|x\|^3. \)

In [5], J. M. Isidro shows that every Hilbert \( C^* \)-module is a \( JB^* \)-triple with the Jordan triple product

\[
\{x, y, z\} = \frac{1}{2}(x\langle y, z \rangle + z\langle y, x \rangle).
\]

A well-known theorem of Kaup [10] (see also [1]) states that every surjective linear isometry between \( JB^* \)-triples is a Jordan triple homomorphism, i.e., it preserves the Jordan triple product

\[
T\{x, y, z\} = \{Tx, Ty, Tz\}, \forall x, y, z \in E.
\]

Hence, if \( T \) is a surjective linear isometry between Hilbert \( C^* \)-modules, then

(2) \( T(x\langle y, z \rangle + z\langle y, x \rangle) = Tx\langle Ty, Tz \rangle + Tz\langle Ty, Tx \rangle, \forall x, y, z \in E. \)
The equation (2) holds if and only if
\[ T(x\langle x, x \rangle) = Tx\langle Tx, Tx \rangle, \forall x \in E \]
by triple polarization
\[ 2\{x, y, z\} = \frac{1}{8} \sum_{\alpha^4=\beta^2=1} \alpha \beta \langle x + \alpha y + \beta z, x + \alpha y + \beta z \rangle (x + \alpha y + \beta z). \]

A ternary ring of operators (TRO) between two Hilbert spaces \( H \) and \( K \) is a linear subspace \( R \) of \( B(H, K) \), the space of all bounded linear operators from \( H \) into \( K \), satisfying \( AB^*C \in R \). Zettl shows in [17] that every Hilbert \( C^* \)-module is isomorphic to a norm closed TRO. In this case, Hilbert \( C^* \)-modules have another triple product, i.e.,
\[ \{x, y, z\} := x\langle y, z \rangle. \]

A map between TROs is said to be a triple homomorphism if it preserves the triple products. In the case of Hilbert \( C^* \)-modules, a map \( T \) is a triple homomorphism if it satisfies
\[ T(x\langle y, z \rangle) = Tx\langle Ty, Tz \rangle, \forall x, y, z. \]

We have known every surjective linear isometry is a Jordan triple homomorphism, but it could not be a triple homomorphism, see Example 3.

Let \( R \) be a TRO. Then \( M_n(R) \), the space of all \( n \times n \) matrices whose entries are in \( R \), has a TRO-structure. Let \( T \) be a map between TROs \( R_1 \) and \( R_2 \). For all positive integer \( n \), define maps \( T^{(n)} : M_n(R_1) \to M_n(R_2) \) by \( T^{(n)}((x_{ij})_{ij}) = (T(x_{ij}))_{ij} \). We call \( T \) an isometry if \( T^{(n)} \) is isometric and complete isometry if each \( T^{(n)} \) is isometric for all \( n \). It has been shown that a surjective linear isometry between TROs is a triple homomorphism if and only if it is completely isometric. More details about TROs mentioned above, we refer to [17], see also [14, 3]. In fact, Solel shows in [16] that every surjective 2-isometry between two full Hilbert \( C^* \)-modules is necessarily completely isometric.

3. Results

Note that in the case of a commutative \( C^* \)-algebra \( A = C_0(X) \), for some locally compact Hausdorff space \( X \), Hilbert \( C^* \)-modules over \( C_0(X) \) are the same as Hilbert bundles, or equivalently, continuous fields of Hilbert spaces, over \( X \).

We showed the following theorem in [4].

**Theorem 1.** Let \( E \) and \( F \) be two Hilbert \( C^* \)-modules over commutative \( C^* \)-algebras \( C_0(X) \) and \( C_0(Y) \), respectively. Then every surjective linear isometry from \( E \) onto \( F \) is a weighted composition operator
\[ Tf(y) = h(y)(f(\varphi(y))), \forall f \in E, \forall y \in Y \]
Here, \( \varphi \) is a homeomorphism from \( Y \) onto \( X \), \( h(y) \) is a unitary operator between the corresponding fibers of \( E \) and \( F \), for all \( y \) in \( Y \).

By the similar argument discussed in the introduction, we have
Corollary 2. Every surjective linear isometry between Hilbert $C^*$-modules over commutative $C^*$-algebras preserves the inner products and is a module map.

Now we discuss the case of noncommutative $C^*$-algebras. From equation (4), it seems that a surjective linear isometry $T$ indicates that $T$ preserves inner products and that $T$ is a module map. We explain this could be not true in general by an example.

Example 3. Given a positive integer $n$. The Hilbert column space $H^n_c$ is the subspace of $M_n(\mathbb{C})$ consisting of all matrices whose non-zero entries are only in the first column. Similarly, the Hilbert row space is the subspace consisting of matrices whose non-zero entries are only in the first row. Clearly, $H_c$ and $H_r$ are right Hilbert $C^*$-modules over $C^*$-algebras $\mathbb{C}$ and $M_n(\mathbb{C})$, respectively, with the inner product $\langle A, B \rangle := A^t B$. Define a surjective linear isometry $T : H^n_c \to H^n_r$ by $T(A) = A^t$, the transpose of $A$. Then $\langle T(A), T(B) \rangle = tr(A, B)$, the trace of $\langle A, B \rangle$, but $T$ is not a module map with respect to the trace. For the surjective linear isometry $T : H^n_c \to H^n_r$, $T(A) = A^t$. Let $\varphi : \mathbb{C} \to M_n(\mathbb{C})$ be defined by $\varphi(\alpha) = \alpha I$. Then $T$ is a module map with respect to $\varphi$, but the equation $\langle TA, TB \rangle = \varphi(\langle A, B \rangle)$ does not hold. It is clear that $T$ does not satisfy the equation (4).

Remark 4. In fact, the corollary above says that there exists a $\ast$-isomorphism $\varphi$ between the $C^*$-algebras such that

$$\langle Tx, Ty \rangle = \varphi(\langle x, y \rangle)$$

and

$$T(xa) = T(x)\varphi(a).$$

We have seen in the Example 3 that even if $T$ is a module map or preserves the inner products, the map $\varphi$ might be just a linear map.

In the following, $E$ and $F$ stand for two Hilbert $C^*$-modules over $C^*$-algebras $A$ and $B$, respectively. $T$ is a map from $E$ into $F$ and $\varphi$ is a map from $A$ into $B$. The following lemmas explain the relations of $T$, $\varphi$, when $T$ preserves the inner products and when $T$ is a module map, see also [8].

Lemma 5. If $\varphi$ is linear, every map $T$ from $E$ into $F$ which preserves the inner products with respect to $\varphi$ is linear.

Proof. Since $T$ preserves the inner products with respect to $\varphi$. Then for all $x, y$ and $z$ in $E$, $\alpha$ in $\mathbb{C}$,

$$\langle T(\alpha x + y), Tz \rangle = \varphi(\langle \alpha x + y, z \rangle) = \alpha \varphi(\langle x, z \rangle) + \varphi(\langle y, z \rangle) = \langle \alpha Tx + Ty, Tz \rangle.$$ 

Similarly, we have

$$\langle Tx, T(\alpha y + z) \rangle = \langle Tx, \alpha Ty + Tz \rangle.$$

It is easy to show that

$$\langle T(\alpha x + y) - (\alpha Tx + Ty), T(\alpha x + y) - (\alpha Tx + Ty) \rangle = 0.$$

This proves $T(\alpha x + y) = \alpha Tx + Ty$ and hence $T$ is linear. 

Lemma 6 ([8]). Let $T$ be a surjective linear map which preserves the inner products and is a module map w.r.t. $\varphi$. If $F$ is full, then $\varphi$ is a $\ast$-homomorphism.
Proof. Let $a_1, a_2$ in $A$ and $\alpha$ in $\mathbb{C}$. It is easy to show that
\[
T(x)(\varphi(\alpha a_1 + a_2) - \alpha \varphi(a_1) - \varphi(a_2)) = T(x)\varphi(\alpha a_1 + a_2) - \alpha T(x)\varphi(a_1) - T(x)\varphi(a_2) = T(\alpha xa_1 + xa_2)\]
and
\[
T(x)(\varphi(a_1a_2) - \varphi(a_1)\varphi(a_2)) = T(x)\varphi(a_1a_2) - T(x)\varphi(a_1)\varphi(a_2) = T(xa_1a_2) - T(xa_1a_2) = 0.
\]
Since $T$ is surjective and $F$ is full, we have $\varphi(\alpha a_1 + a_2) = \alpha \varphi(a_1) + \varphi(a_2)$ and $\varphi(a_1a_2) = \varphi(a_1)\varphi(a_2)$.

Let $x, y$ in $A$, we have
\[
\varphi((x, y)^*) = \varphi((y, x)) = (Ty, Tx) = (Tx, Ty)^* = \varphi((x, y))^*.
\]
For $a$ in $A$,
\[
\langle T(x)(\varphi(a^*) - \varphi(a)^*), T(x)(\varphi(a^*) - \varphi(a)^*) \rangle = \varphi((x, y)^*)\varphi((x, y))\varphi(a) - \varphi(a)\varphi((x, y))\varphi(a^*) + \varphi(a)\varphi((x, y))\varphi(a^*) - \varphi((x, y)^*)\varphi((x, y))\varphi(a^*) - \varphi((x, y)^*)\varphi((x, y))^* + \varphi((x, y)^*)^* - \varphi((x, y)^*)^* = 0.
\]
Hence, $T(x)(\varphi(a^*) - \varphi(a)^*) = 0$ for all $x$ in $E$. Since $T$ is surjective and $F$ is full, we have $\varphi(a^*) = \varphi(a)^*$.

Lemma 7. If $\varphi$ is a $*$-homomorphism, then every map $T$ which preserves the inner products w.r.t. $\varphi$ is a module map w.r.t. $\varphi$.

Proof. Let $x$ and $y$ in $E$ and $a$ in $A$. Then
\[
\langle T(xa), Ty \rangle = \varphi((xa, y)) = (T(xa)\varphi(a), Ty) = (T(x)\varphi(a), Ty).
\]
Similarly, we have
\[
\langle T(x), Tya \rangle = (T(x), T(y)a) = \langle T(xa), T(y)\varphi(a) \rangle.
\]
It is easy to show that
\[
\langle T(xa) - T(x)\varphi(a), T(xa) - T(x)\varphi(a) \rangle = 0.
\]
Hence, $T(xa) = T(x)\varphi(a)$.

Lemma 8 ([13]). Let $T$ be a surjective linear isometry and $\varphi$ a $*$-isomorphism. If $T$ is a module map w.r.t. $\varphi$, then $T$ preserves the inner products with respect to $\varphi$.

Proof. It suffices to prove that $\langle Tx, Tx \rangle = \varphi((x, x))$ for all $x$ in $E$. Note that $|a| := (a^*a)^{1/2}$. For all $b$ in $B$, let $\varphi(a) = b$, then
\[
||Tx||^2 = ||T(x)b||^2 = ||\langle T(x)\varphi(a), T(x)\varphi(a) \rangle|| = ||\langle T(xa), T(xa) \rangle|| = ||\langle xa, xa \rangle|| = ||xa||^2 = ||\varphi(|x|)b||^2 = ||\varphi(|x|)|b|^2.
\]
By Lemma 3.5 in [11], we get $|Tx| = (\varphi(|x|)$ and hence $\langle Tx, Tx \rangle = \varphi((x, x))$. \qed
Theorem 9. Let $T$ be a surjective linear 2-isometry from $E$ onto $F$. Then there exists a $*$-isomorphism $\varphi$ from $\langle E, E \rangle$ onto $\langle F, F \rangle$ such that, for all $x, y$ in $E$, and $a$ in $A$,

$$\langle Tx, Ty \rangle = \varphi(\langle x, y \rangle)$$

and

$$T(ax) = T(x)\varphi(a).$$

Proof. We can regard $E$ and $F$ as full modules over $\langle E, E \rangle$ and $\langle F, F \rangle$, respectively. In this case, as we mentioned above, $T$ is completely isometric and hence it preserves the triple products

$$T(z\langle x, y \rangle) = Tz\langle Tx, Ty \rangle, \forall x, y, z \in E.$$ 

Define $\varphi : \langle E, E \rangle \to \langle F, F \rangle$ by

$$\varphi(\sum_{i=1}^{n} \alpha_i \langle x_i, y_i \rangle) := \sum_{i=1}^{n} \alpha_i \langle Tx_i, Ty_i \rangle, \quad x_i, y_i \in E, \quad \alpha_i \in \mathbb{C}, \quad i = 1, \cdots, n.$$ 

Let $x_i, y_i$ and $z \in E$, $\alpha_i \in \mathbb{C}$, $i = 1, \cdots, n$. Then $\sum_{i=1}^{n} \alpha_i \langle x_i, y_i \rangle = 0$ if and only if $z(\sum_{i=1}^{n} \alpha_i \langle x_i, y_i \rangle) = 0$ for all $z$ if and only if $T(z)(\sum_{i=1}^{n} \alpha_i \langle Tx_i, Ty_i \rangle) = \sum_{i=1}^{n} \alpha_i Tz\langle Tx_i, Ty_i \rangle = T(z)(\sum_{i=1}^{n} \alpha_i \langle x_i, y_i \rangle)$ if and only if $\sum_{i=1}^{n} \alpha_i \langle Tx_i, Ty_i \rangle = 0$. This shows that $\varphi$ is well-defined and injective. From the definition of $\varphi$, since $T$ is surjective, it is clear that $\varphi$ is a surjective $*$-homomorphism and $T$ preserves the inner products w.r.t. $\varphi$. By lemma 7, $T$ is a module map w.r.t $\varphi$. \hfill $\square$

Corollary 10. Every surjective linear 2-isometry between two full Hilbert $C^*$-modules preserves the inner products and is a module map with respect to some $*$-isomorphism of underlying $C^*$-algebras.

References


Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan.

*E-mail address*, Ming-Hsiu Hsu: hsumh@math.nsysu.edu.tw

*E-mail address*, Ngai-Ching Wong: wong@math.nsysu.edu.tw