ORTHOGONALLY ADDITIVE AND ORTHOGONALLY MULTIPLICATIVE HOLOMORPHIC FUNCTIONS OF MATRICES

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This paper is dedicated to Professor Tsuyoshi Ando

Abstract. Let $H : M_m \to M_m$ be a holomorphic function of the algebra M_m of complex $m \times m$ matrices. Suppose that H is orthogonally additive and orthogonally multiplicative on self-adjoint elements. We show that either the range of H consists of zero trace elements, or there is a scalar sequence $\{\lambda_n\}$ and an invertible S in M_m such that

$$H(x) = \sum_{n\geq 1} \lambda_n S^{-1} x^n S, \quad \forall x \in M_m,$$

or

$$H(x) = \sum_{n\geq 1} \lambda_n S^{-1} (x^t)^n S, \quad \forall x \in M_m.$$

Here, x^t is the transpose of the matrix x. In the latter case, we always have the first representation form when H also preserves zero products. We also discuss the cases where the domain and the range carry different dimensions.

1. Introduction

Let E and F be real or complex Banach spaces, and n a positive integer. A map $P : E \to F$ is called a *bounded n-homogeneous polynomial* if there is a bounded symmetric n-linear operator $T : E \times \cdots \times E \to F$ such that

$$P(x) = T(x, \ldots, x), \quad \forall x \in E.$$

In this case, we have

$$T(x_1, \ldots, x_n) = \frac{1}{2^n n!} \sum_{\epsilon_i = \pm 1} \epsilon_1 \cdots \epsilon_n P\left(\sum_{i=1}^{n} \epsilon_i x_i\right), \quad \forall x_1, \ldots, x_n \in E.$$

A map $H : U \to F$ is said to be *holomorphic* on a nonempty open subset U of E if for each a in U there exist an open ball $B_E(a; r) \subseteq U$, centered at a with radius $r > 0$, and a unique sequence of bounded n-homogeneous polynomials $P_n : E \to F$ such that

$$H(x) = \sum_{n=0}^{\infty} P_n(x - a)$$

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uniformly for all x in $B_E(a;r)$.

To study holomorphic functions, we might assume, after translation, $a = 0$. A holomorphic function $H : B_E(0;r) \to F$ has its Taylor series at zero:

$$H(x) = \sum_{n=0}^{\infty} P_n(x) \tag{1.1}$$

uniformly for all x in $B_E(0;r)$. In the complex case, we have the Cauchy integral formulae:

$$P_n(x) = \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{H(\lambda x)}{\lambda^{n+1}}\, d\lambda, \qquad n = 0, 1, 2, \ldots. \tag{1.2}$$

For the general theory of homogeneous polynomials and holomorphic functions, we refer to [11, 18].

When E, F are Banach algebras, a function $\Phi : E \to F$ is said to be *orthogonally additive* if

$$fg = gf = 0 \quad \text{implies} \quad \Phi(f+g) = \Phi(f) + \Phi(g), \quad \forall f, g \in E,$$

and *orthogonally multiplicative* if

$$fg = gf = 0 \quad \text{implies} \quad \Phi(f)\Phi(g) = 0, \quad \forall f, g \in E.$$

The notions of orthogonally additive and orthogonally multiplicative transformations have been studied by many authors, for example, [2, 14, 15, 13, 9, 1, 16, 21, 3, 7, 19, 8, 4, 20, 17].

Our goal is to study orthogonally additive and orthogonally multiplicative holomorphic functions between C*-algebras. Every abelian C*-algebra is the algebra $C_0(X)$ of continuous functions of a locally compact Hausdorff space X vanishing at infinity. In general, a C*-algebra can be embedded into $B(H)$ as a norm closed self-adjoint subalgebra. When E, F are algebras of continuous functions, it is established in [5] the following nice representation.

Proposition 1.1 ([5]). *Let $H : B_{C_0(X)}(0;r) \to C_0(Y)$ be a bounded orthogonally additive and orthogonally multiplicative holomorphic function. Then there exist a sequence $\{h_n\}$ of bounded scalar continuous functions in $C(Y)$ and a map $\varphi : Y \to X$ such that*

$$H(f)(y) = \sum_{n\geq 1} h_n(y)(f(\varphi(y)))^n, \quad \forall y \in Y,$$

uniformly for all f in $B_{C_0(X)}(0;r)$. Here, φ is continuous wherever any h_n is nonvanishing.

Talking orthogonality of a pair of elements a, b in a general C*-algebra (in $B(H)$), people usually refers one of the following situations.

(i) Zero product: $ab = 0$.
(ii) Two side zero product: $ab = ba = 0$.
(iii) Range orthogonality: $a^*b = 0$.
(iv) Initial space or domain orthogonality: $ab^* = 0$.
(v) Range and domain orthogonality: $a^*b = ab^* = 0$.

In the abelian case, however, all these concepts coincide. They coincide in general when both a, b are self-adjoint.

Some partial results concerning the structures of homogeneous polynomials between general C*-algebras are also given in [5]. For example, we have

Proposition 1.2 ([5]). *Let H be a complex Hilbert space of arbitrary dimension. Let $P : B(H) \to B(H)$ be a bounded n-homogeneous polynomial, which is additive and multiplicative on pairs of orthogonal self-adjoint elements. Suppose that $P(1)$ is invertible or $P(B(H)) \supset B(H)^+$. Then there is a nonzero scalar λ and an invertible operator S in $B(H)$ such that either*

$$P(x) = \lambda S^{-1} x^n S, \ \forall x \in B(H),$$

or

$$P(x) = \lambda S^{-1} (x^t)^n S, \ \forall x \in B(H).$$

Here, x^t is the transpose of a bounded linear operator x in $B(H)$ with respect to some arbitrary but fixed orthogonal basis of the Hilbert space H. For a matrix $x = (x_{ij})$, we simply define $x^t = (x_{ji})$ to be the transpose of x.

However, results in [5] usually assume a rather strong hypothesis that $P(1)$ is invertible or $P(A) \supset B^+$. It is not very likely every summand P_n in the Taylor series (1.1) of a holomorphic function $H : B_A(0; r) \to B$ would satisfy one of these conditions. Thus, a general structure result about such holomorphic functions is still far away from reaching.

In this paper, we will establish another important case. We will give a description of orthogonally additive and orthogonally multiplicative holomorphic function $H : M_m \to M_m$ of complex matrix algebras.

In the following we say that a map H between complex matrices is *orthogonally additive* (resp. *multiplicative*) *on self-adjoint elements* if

$$H(a + b) = H(a) + H(b)$$

(resp.

$$H(a)H(b) = 0)$$

whenever a, b are self-adjoint complex matrices in its domain with $ab = 0$.

Theorem 1.3. *Let m and s be positive integers with $m \geq 2$ and $m \geq s$. Let $H : B_{M_m}(0; r) \to M_s$ be a holomorphic function between complex matrix algebras. Assume H is orthogonally additive and orthogonally multiplicative on self-adjoint elements. Then either*

(A) the range of H consists of zero trace elements (this case occurs whenever $s < m$), or

(B) there exist a scalar sequence $\{\lambda_n\}$ (some λ_n can be zero) and an invertible $m \times m$ matrix S such that

$$H(x) = \sum_{n \geq 1} \lambda_n S^{-1} x^n S, \quad \forall x \in B_{M_m}(0; r), \tag{‡}$$

or

$$H(x) = \sum_{n \geq 1} \lambda_n S^{-1} (x^t)^n S, \quad \forall x \in B_{M_m}(0; r).$$

In the case (B), we always have the representation (‡) when H also preserves zero products, i.e.,

$$ab = 0 \quad \implies H(a)H(b) = 0, \qquad \forall a, b \in B_{M_m}(0; r).$$

The proof of Theorem 1.3 will be given in the next section. The following example shows that the exception case in Theorem 1.3 can occur when $s = m$.

Example 1.4. Let E_{ij} be the matrix unit with '1' at the (i, j)th entry and '0' elsewhere. Consider the linear map $T : M_2 \to M_2$ defined by $T(E_{11}) = E_{12}$, and $T(E_{ij}) = 0$ for all other i, j. Then T is an orthogonally multiplicative, and linear (and thus holomorphic) map. It is plain that the range of T consists of nilpotent matrices.

On the other hand, we can have other possibilities when the range have larger dimension than the domain.

Example 1.5. Consider $\theta : M_k \to M_{k+2}$ defined by

$$(a_{ij}) \mapsto \begin{pmatrix} 0 & a_{11} & a_{12} & \dots & a_{1k} & 0 \\ 0 & 0 & 0 & \dots & 0 & a_{11} \\ 0 & 0 & 0 & \dots & 0 & a_{21} \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_{k1} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then θ is linear (and thus holomorphic), and orthogonally multiplicative on self-adjiont elements. Note that the range of θ does not have trivial multiplication, since $\theta(E_{11})^2 = E_{1,k+2}$. However, θ cannot be written as the form $c\varphi$ for any fixed element c in M_{k+2} and any homomorphism or anti-homomorphism $\varphi : M_k \to M_{k+2}$. Assume on the contrary that $\theta = c\varphi$. Then we arrive at a contradiction

$$\begin{aligned} E_{1,k+2} &= \theta(E_{11})^2 = \theta(E_{11})c\varphi(E_{11}) = \theta(E_{11})c(\varphi(E_{12})\varphi(E_{21})) \\ &= \theta(E_{11})(c\varphi(E_{12}))\varphi(E_{21})) = \theta(E_{11})\theta(E_{12})\varphi(E_{21}) = 0\varphi(E_{21}) = 0. \end{aligned}$$

Example 1.6. Consider $\phi : M_k \to M_{2k+2}$ defined by

$$\left(a_{ij}\right) \mapsto \left(a_{ij}\right) \oplus \theta \left(a_{ij}\right).$$

Here, θ is the map defined in Example 1.5. Again θ is linear (and thus holomorphic), and orthogonally multiplicative on self-adjoint elements. However, θ cannot be written in any form stated in Theorem 1.3(B), although its range contains elements of nonzero trace.

The infinite dimensional case can be more complicated.

Example 1.7. Let H be a separable infinite dimensional Hilbert space with an orthonormal basis $\{e_n : n = 1, 2, \dots\}$.

(a) Consider $\theta : B(H) \to B(H)$ defined by

$$\left(a_{ij}\right) \mapsto J^2 \left(a_{ij}\right) J^{*2} + \begin{pmatrix} 0 & a_{11} \\ 0 & 0 \end{pmatrix}.$$

Here, $J : B(H) \to B(H)$ is the unilateral shift operator sending e_n to e_{n+1} for $n = 1, 2, \ldots$. Then θ is not of the standard form, while its range contains elements of nonzero trace.

(b) Let E and F be the isometries in $B(H)$ such that $E(e_n) = e_{2n}$ and $F(e_n) = e_{2n-1}$ for $n = 1, 2, \ldots$, respectively. Define a holomorphic function $H : B(H) \to B(H)$ by

$$H(a) = EaE^* + F(a^t)^2F^*, \quad \forall a \in B(H).$$

Then H is orthogonally additive and orthogonally multiplicative, but not zero product preserving. (Readers can make up one preserving zero products easily.) The range of H contains the identity $H(1) = 1$. However, it cannot be written in any form stated in Theorem 1.3(B).

2. The proofs

We begin with an observation.

Lemma 2.1. *Let $H : B_E(0; r) \to F$ be a holomorphic function between C^*-algebras with Taylor series at zero $H = \sum_{n=0}^{\infty} P_n$.*

(a) If H is orthogonally additive on self-adjoint elements then each P_n is also orthogonally additive on self-adjoint elements.

(b) If H is orthogonally multiplicative on self-adjoint elements then each P_n is also orthogonally multiplicative on self-adjoint elements. Indeed, for orthogonal self-adjoint elements x, y in $B_E(0; r)$ we have

$$xy = 0 \quad \Longrightarrow \quad P_m(x)P_n(y) = 0, \quad m, n = 0, 1, 2, \ldots.$$

Proof. Let $\{x, y\}$ be an orthogonal pair of self-adjoint elements in $B_E(0; r)$. Suppose first that H is orthogonally additive. For sufficiently small scalar α, we have

$$\begin{aligned} H(\alpha x + \alpha y) &= \sum_n P_n(\alpha x + \alpha y) = \sum_n \alpha^n P_n(x + y) \\ = H(\alpha x) + H(\alpha y) &= \sum_n (P_n(\alpha x) + P_n(\alpha y)) = \sum_n \alpha^n (P_n(x) + P_n(y)). \end{aligned}$$

As α can be arbitrary (but small), we see that

$$P_n(x + y) = P_n(x) + P_n(y), \quad n = 0, 1, 2, \ldots.$$

Suppose then H is orthogonally multiplicative. For sufficiently small scalars α, β, we have

$$0 = H(\alpha x)H(\beta y) = \sum_{m,n} P_m(\alpha x)P_n(\beta y) = \sum_{m,n} \alpha^m \beta^n P_m(x)P_n(y).$$

As α, β can be arbitrary (but small), we see that

$$P_n(x)P_m(y) = 0, \quad n, m = 0, 1, 2, \ldots.$$

$\square$

It follows from Lemma 2.1 that if H is orthogonally additive then $P_0 = 0$.

The following linearization of orthogonally additive n-homogeneous polynomials of matrix algebras is an important tool of us. The result for general C*-algebras is given by C. Palazuelos, A. M. Peralta and I. Villanueva [19], and M. Burgosy, F. J. Fernández-Poloz, J. J. Garcésx and A. M. Peralta [6], which extend the commutative version of D. Perez-Garcia and I. Villanueva [21] (see also [19]).

Lemma 2.2. *Let F be a complex Banach space, and $P : M_m \to F$ an n-homogeneous polynomial. If P is orthogonally additive on self-adjoint elements then there exists a linear operator $T : M_m \to F$ such that*

$$P(x) = T(x^n), \quad \forall x \in M_m.$$

Recall that we say a map θ between rings preserving zero products if $\theta(x)\theta(y) = 0$ whenever $xy = 0$. We say that a set Z of a ring has trivial products, if $xy = 0$ for all x, y in Z.

Lemma 2.3 ([9, Corollary 2.4])**.** *Let m and s be positive integers with $m \geq 2$ and $m \geq s$. Let $\mathbb{F}$ be an algebraically closed field of characteristic 0 and $\theta : M_m(\mathbb{F}) \to M_s(\mathbb{F})$ a linear map preserving zero products. Then either the range of θ has trivial multiplication, or $m = s$ and there exist an invertible matrix S in $M_m(\mathbb{F})$ and a nonzero scalar c such that*

$$\theta(x) = cS^{-1}xS \quad \forall x \in M_m(\mathbb{F}).$$

Note that the orthogonal multiplicity of an orthogonal additive polynomial P does not guarantee its linearization T preserving zero products. So we cannot apply Lemma 2.3 directly. But when x, y are idempotents with $xy = 0$, we have $T(x)T(y) = P(x)P(y) = 0$. This suggests we establish the following two lemmas. Fortunately, they are sufficient for our proof of Theorem 1.3.

Lemma 2.4. *Let m and s be positive integers with $m \geq 2$ and $m \geq s$. Let $\theta : M_m \to M_s$ be a complex linear map. Assume that*

$$\theta(p)\theta(q) = 0 \quad \textit{whenever} \quad p, q \textit{ are orthogonal rank one projections.} \tag{2.1}$$

Then either

(A) the range of θ consists of nilpotent elements (this happens whenever $s < m$), or

(B) $m = s$ and there exist an invertible matrix S in M_m and a nonzero scalar λ such that

$$\theta(x) = \lambda S^{-1}xS, \quad \forall x \in M_m,$$

or

$$\theta(x) = \lambda S^{-1}x^tS, \quad \forall x \in M_m.$$

Proof. Note that (2.1) holds indeed for all orthogonal pairs of self-adjoint matrices (through spectral decompositions). For any projection p in M_m, we have

$$(1 - p)p = p(1 - p) = 0.$$

The assumption implies that

$$\theta(1 - p)\theta(p) = \theta(p)\theta(1 - p) = 0.$$

Hence

$$\theta(1)\theta(p) = \theta(p)\theta(1) = \theta(p)^2$$

holds for all projections p in M_m. By the spectral theory, we have

$$\theta(1)\theta(a) = \theta(a)\theta(1) \tag{2.2}$$

for all self-adjoint, and thus for all, a in M_m. Moreover,

$$\theta(1)\theta(a^2) = \theta(a^2)\theta(1) = \theta(a)^2$$

holds for all self-adjoint elements a in M_m. Considering $(a+b)^2$ for two self-adjoint elements a and b, we have

$$\theta(1)\theta(ab + ba) = \theta(a)\theta(b) + \theta(b)\theta(a).$$

Since very complex matrix a in M_m can be written as $a = b + \sqrt{-1}c$ for two self-adjoint matrices b and c, we see that

$$\theta(1)\theta(a^2) = \theta(a^2)\theta(1) = \theta(a)^2, \quad \forall a \in M_m. \tag{2.3}$$

It follows further that

$$\theta(1)\theta(ab + ba) = \theta(a)\theta(b) + \theta(b)\theta(a), \quad \forall a, b \in M_m. \tag{2.4}$$

Suppose that there is an x in M_m such that $\theta(x)$ is not nilpotent. It follows from (2.2) and (2.3) that

$$\theta(1)^s\theta(x^2)^s = (\theta(1)\theta(x^2))^s = \theta(x)^{2s} \neq 0.$$

Consequently, $\theta(1)$ is not nilpotent, and thus its spectrum contains a complex number $\lambda \neq 0$. Using the Riesz functional calculus, we have an idempotent e in M_s such that $e\theta(1) = \theta(1)e = \lambda e \neq 0$ and e commutes with every matrix commuting with $\theta(1)$ (see, e.g., [10, Prop. 4.11]). Since $\theta(1)$ commutes with all $\theta(a)$'s, so does e. Define $\Psi : M_m \to M_s$ by $\Psi(a) = e\theta(a)/\lambda$. It follows from (2.4) that

$$\Psi(ab + ba) = \Psi(a)\Psi(b) + \Psi(b)\Psi(a), \quad \forall a, b \in M_m.$$

By the well-known theorem of Herstein, we see that either $\Psi = 0$, or Ψ is an injective homomorphism or anti-homomorphism. But the first case implies the contradiction $e = \Psi(1) = 0$. Hence, the latter case occurs, and we must have $s = m$. This forces $e = 1$ and thus $\theta(1) = \lambda$. It follows from the Noether-Skolem theorem that we have one of the expected representations of θ. This completes the proof. $\square$

Lemma 2.5. *Let m and s be positive integers with $m \geq 2$ and $m \geq s$. Let $\theta : M_m \to M_s$ be a complex linear map. Assume that*

$$\theta(e)\theta(f) = 0 \quad \textit{whenever} \quad e, f \textit{ are rank one idempotents with } ef = 0.$$

If the range of θ does not consist of nilpotent elements then $s = m$, and there exist an invertible matrix S in M_m and a nonzero scalar λ such that

$$\theta(x) = \lambda S^{-1}xS, \quad \forall x \in M_n.$$

Proof. Since projections are idempotents, it follows from Lemma 2.4 that $m = s$ and there is an invertible matrix S and a nonzero scalar λ such that $\frac{1}{\lambda}S\theta(x)S^{-1}$ is either always x or always x^t. Consider the idempotents a, b in M_m with the top left 2×2 blocks given below and zero elsewhere, respectively:

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then $ab = 0$ but $ba \neq 0$. Since θ preserves zero products, so does $\frac{1}{\lambda}S\theta S^{-1}$. Because $a^t b^t = (ba)^t \neq 0$, we conclude that $\frac{1}{\lambda}S\theta(x)S^{-1} = x$ for all x in M_m. This gives us the desired assertion. □

Proof of Theorem 1.3. It is not difficult to see that the two cases stated in the conclusions are exclusive. Assume from now on $H(d)$ is a matrix in M_s of nonzero trace for some d in $B_{M_m}(0; r)$.

Lemma 2.1 ensures that each summand P_n is an orthogonally additive and orthogonally multiplicative n-homogeneous polynomial, and the constant term $P_0 = H(0) = 0$. Lemma 2.2 provides a linear map $T_n : M_m \to M_s$ for each n such that

$$P_n(x) = T_n(x^n), \quad \forall x \in M_m.$$

Inherited from $\{P_n\}$, the family $\{T_n\}$ satisfies the orthogonality preserving property stated in (2.1).

Since $H(d) = \sum_n T_n(d^n)$, the continuity of the trace functional ensures that some $T_k(d^k)$ in the sum has nonzero trace. In particular, $T_k(d^k)$ is not a nilpotent. Lemma 2.4 ensures $m = s$ and provides an invertible matrix S_k and a nonzero scalar λ_k such that

$$P_k(x) = \lambda_k S_k^{-1} x^k S_k, \quad \forall x \in B_{M_m}(0; r),$$

or

$$P_k(x) = \lambda_k S_k^{-1} (x^t)^k S_k, \quad \forall x \in B_{M_m}(0; r).$$

We claim that all other P_n either carries a similar form or constantly zero. Redefining $H(x)$ with $\frac{1}{\lambda_k} S_k H(x) S_k^{-1}$ or $\frac{1}{\lambda_k} S_k H(x^t) S_k^{-1}$, we can assume

$$P_k(x) = x^k, \quad \forall x \in B_{M_m}(0; r).$$

Suppose that with a nonzero scalar λ_n and an invertible S_n in M_m we have

$$P_n(x) = \lambda_n S_n^{-1} x^n S_n, \quad \forall x \in B_{M_m}(0; r),$$

or

$$P_n(x) = \lambda_n S_n^{-1} (x^t)^n S_n, \quad \forall x \in B_{M_m}(0; r).$$

By Lemma 2.1(b),

$$x S_n^{-1} y S_n = S_n^{-1} y S_n x = 0, \quad \text{whenever } x, y \text{ are orthogonal projections.}$$

This forces

$$S_n^{-1} y S_n = \alpha_y y$$

with some scalar α_y for every rank one projection y in M_m. In particular, every nonzero vector is an eigenvector of S_n. Thus $S_n = \alpha I$ for some nonzero scalar α. In other words, we can assume that $P_n(x) = \lambda_n x^n$ for all x in $B_{M_m}(0; r)$, or

$P_n(x) = \lambda_n(x^t)^n$ for all x in $B_{M_m}(0;r)$. However, Lemma 2.1(b) ruins out the possibility of the second case. For example, try the pair of orthogonal projections

$$a = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{-1} \\ -\sqrt{-1} & 1 \end{pmatrix} \oplus 0 \quad \text{and} \quad b = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{-1} \\ \sqrt{-1} & 1 \end{pmatrix} \oplus 0.$$

While $ab = 0$, we have $ab^t \neq 0$.

Suppose next that there is a P_n whose range consists of nilpotent elements. We will verify that $P_n = 0$. Arguing as above, we have

$$xT_n(y) = T_n(y)x = 0, \quad \text{whenever } x, y \text{ are orthogonal projections.}$$

This forces

$$T_n(y) = \alpha_y y$$

with some scalar α_y for every rank one projection y in M_m. Since $\beta^{mn}T_n(y)^m = P_n(\beta y)^m = 0$ for all small scalar $\beta > 0$, we see that $\alpha_y = 0$. Since every self-adjoint matrix is an orthogonal sum of rank one projections, the linear map $T_n = 0$, and thus $P_n = 0$, on M_m.

The claim is established. It follows that there is a scalar sequence $\{\lambda_n\}$ and an invertible S in M_m such that

$$H(x) = \sum_{n\geq 1} \lambda_n S^{-1}x^n S, \quad \forall x \in B_{M_m}(0;r).$$

Translating back to the original situation, there is also another possible case that

$$H(x) = \sum_{n\geq 1} \lambda_n S^{-1}(x^t)^n S, \quad \forall x \in B_{M_m}(0;r).$$

Finally, assume that H also preserves zero products, and thus so does every P_k. Consequently, the linearization T_k sends two rank one idempotents with zero products to a pair of elements with zero products. By Lemma 2.5, we have $T_k(x) = \lambda_k S^{-1}x^k S$ for all x in M_m. This forces H carries the first form as in (‡). This completes the proof. □

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The authors are grateful to the referee for his/her careful reading and critical comments. In particular, the referee questions about the application of Lemma 2.3 and a statement in a previous version of this note concerning the validity of our arguments in the case when the underlying field is the real numbers. Due to lack of sufficient tools, we are unable to extend the results in this note to the real case at this moment. We hope to finish this task in a subsequent project.

Added in proofs. In a very recent paper [12], J. J. Garcés, A. M. Peralta, D. Puglisi and R. M. Isabel obtain another satisfactory result about orthogonally additive holomorphic functions $H : B_A(0;r) \to B$, where A, B are general C*-algebras. Assume that the range of H contains an invertible element, and

$$H(a)^*H(b) = H(a)H(b)^* = 0 \quad \text{whenever} \quad ab = 0 \text{ for self-adjoint } a, b \in B_A(0,r).$$

They prove that there exist a sequence $\{h_n\}$ in B^{**} and Jordan *-homomorphisms $J, \tilde{J}$ from the multiplier algebra $M(A)$ of A into B^{**} such that

$$H(a) = \sum_{n\geq 1} h_n J(a^n) = \sum_{n\geq 1} \tilde{J}(a^n) h_n, \quad \forall a \in B_A(0; r).$$

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