THE BRONOLOGICALLY SURJECTIVE HULL
OF AN OPERATOR IDEAL ON LOCALLY CONVEX SPACES

NGAI-CHING WONG
YAU-CHUEN WONG

Abstract. We provided an answer to an open problem of A. Pietsch by giving a
direct construction of the bornologically surjective hull $A_{bsur}$ of an operator ideal $A$
on LCS’s. Discussion of some extension problems of operator ideals were given.

1. Introduction and notations

In his classic [4], A. Pietsch asked for a direct construction of the injective hull
$A_{inj}$ and the surjective hull $A_{sur}$ of an operator ideal $A$ on LCS’s (locally convex
spaces) which should be similar to the ones about operator ideals on Banach spaces.
L. Franco and Piñeiro [1] answered the problem about injective hulls. In this paper,
we shall provide a direct construction of the bornologically surjective hull $A_{bsur}$
of $A$ after introducing the notion of bornological surjectivity. We shall discuss
the solvability of the original problem about surjective hulls. By the way, the
concept of bornological surjectivity was proved to be more interesting and suitable
for applications in [6, 7, 8, 9].

Throughout this paper, $A$ always denotes an operator ideal on either the class
$L$ of LCS’s or $B$ of Banach spaces in the sense of A. Pietsch [4]. $K = \mathbb{R}$ or $\mathbb{C}$
is the underlying scalar field. $X, Y, X_0, Y_0, \ldots$ denotes LCS’s and $E, F, E_0, F_0, \ldots$
denotes Banach spaces. Let $N$ be a normed space, $\cup_N$ always denotes the norm
closed unit ball of $N$. $\mathcal{L}(X, Y)$ denotes the family of all continuous (linear) operator
between $X$ and $Y$. An injection means a relatively open and one-to-one continuous

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operator and a (topological) surjection means an open continuous operator. $Q$ in $\mathcal{L}(X,Y)$ is said to be a bornological surjection if for every bounded set $B$ in $Y$ there is a bounded set $A$ in $X$ such that $QA = B$. In other words, a topological surjection induces the topology of the range space and a bornological surjection induces the bornology of the range space, cf. [2] for more information. It is easy to see that any bornological surjection from an LCS $X$ onto an infrabarrelled LCS $Y$ is a (topological) surjection, and any surjection from a normed space onto a normed space is a bornological surjection. It is also true that any surjection from a Fréchet space onto a Fréchet–Montel space is a bornological surjection (cf. Wong [8, p. 45]).

Let $N$ be an infinite–dimensional normed space and $N_\sigma$ be the LCS $(N, \sigma(N, N'))$. The canonical map $I : N \rightarrow N_\sigma$ is a bornological surjection but not a surjection. See also [5, ex. 4.9 and 4.20], in which a surjection from a Fréchet space onto a Fréchet space is not a bornological surjection. However, in the case of normed spaces there is no difference between these two concepts.

Let $\mathcal{C}$ be either $\mathcal{L}$ or $\mathcal{B}$. An operator ideal $\mathfrak{A}$ on $\mathcal{C}$ is said to be bornologically surjective if whenever $T$ is a continuous operator from $X$ into $Y$ and $Q$ is a bornological surjection from $X_0$ onto $X$ such that $TQ \in \mathfrak{A}(X_0, Y)$, we have $T \in \mathfrak{A}(X, Y)$, where $X, X_0, Y \in \mathcal{C}$. The bornologically surjective hull $\mathfrak{A}^{\text{bsur}}$ of $\mathfrak{A}$ is the intersection of all bornological surjective operator ideals containing $\mathfrak{A}$. Clearly, $\mathfrak{A}^{\text{bsur}}$ is the smallest bornologically surjective operator ideal containing $\mathfrak{A}$. If $\mathcal{C} = \mathcal{B}$, we have $\mathfrak{A}^{\text{bsur}} = \mathfrak{A}^{\text{sur}}$. But, if $\mathcal{C} = \mathcal{L}$ then they are, in general, different objects. The ideal $\mathfrak{L}$ of all continuous operators between LCS’s and the ideal $\mathfrak{F}$ of all continuous operators between LCS’s of finite rank are both simultaneously surjective and bornologically surjective. However, we have
Example. Let $K_p$ be the ideal of precompact operators between LCS’s. $K_p$ is surjective but not bornologically surjective. In fact, let $E$ be any reflexive Banach space and consider the canonical maps $E \xrightarrow{i} E_\sigma \xrightarrow{id_{E_\sigma}} E_\sigma$. It is clear that $i$ is a bornological surjection (but not a surjection) and $id_{E_\sigma}$ is precompact. However, $id_{E_\sigma}$ is not precompact unless $\dim E < \infty$. In particular, we have an example that $A_{bsur} \neq A_{sur}$ (even when $A$ is surjective).

Example. Let $K_{loc}p$ be the ideal of all locally precompact operators between LCS’s. In other words, $K_{loc}p$ consists of all such continuous operators between LCS’s sending bounded sets to precompact sets. $K_{loc}p$ is clearly bornologically surjective. Using [5, ex. 4.9], we can represent $E = \ell_1$ as a quotient space of the locally convex space $X = \bigoplus_{x \geq 0} E(B(x))$. Here $x \in E$, $x = (x_n) \geq 0$ means $x_n \geq 0$ for all $n$, and $B(x) = \{ y = (y_n) \in E : |y_n| \leq x_n, n = 1, 2, \ldots \}$. Since $B(x)$ is precompact in $E$ for every $x \geq 0$, the operator $id_E \circ Q$ belongs to $K_{loc}p(X,E)$, where $Q$ is the quotient map from $X$ onto $E$. However, $id_E$ is not locally precompact since $E$ is of infinite dimension. This shows that $K_{loc}p$ is not surjective. In particular, we have an example that $A_{bsur} \neq A_{sur}$ (even when $A$ is bornologically surjective).

A subset $B$ of a LCS $X$ is said to be a disk if $B$ is absolutely convex, i.e., $\lambda B + \beta B \subset B$ whenever $|\lambda| + |\beta| \leq 1$. A disk $B$ is said to be a $\sigma$-disk, or absolutely $\sigma$-convex if $\sum_{n} \lambda_n b_n$ converges in $X$ and the sum belongs to $B$ whenever $(\lambda_n) \in \ell_1$ and $b_n \in B$, $n = 1, 2, \ldots$. A bounded disk $B$ is said to be infracomplete (or a Banach disk) if the normed space $X(B) = \bigcup_{\lambda > 0} \lambda B$ equipped with the gauge $\gamma_B$ of $B$ as its norm is complete, where $\gamma_B(x) = \inf\{ |\lambda| : x \in \lambda B \}$, for each $x$ in $X(B)$. Any continuous image of a $\sigma$-disk or an infracomplete bounded disk is still
a $\sigma$–disk or an infracomplete bounded disk, respectively. It is well–known that
a bounded disk is infracomplete if $B$ is sequentially complete under some locally
convex topology which is compatible with the dual pair $(X, X')$. In particular, if $X$
is quasi–complete then every closed and bounded disk is infracomplete. We call a
$LCS X$ to be infracomplete if the von Neumann bornology $\mathcal{M}_{\text{von}}(X)$, i.e. the original
bornology induced by the topology of $X$, has a basis consisting of infracomplete
subsets of $X$, or equivalently, $\sigma$–disked subsets of $X$. Hence a quasi–complete
LCS is infracomplete. The converse is not true, in general, as $(\ell_1, \sigma(\ell_1, \ell_\infty))$
is sequentially complete (because $\ell_1$ is the predual of the $W^*$–algebra $\ell_\infty$) but not
quasi–complete (because $(\ell_1, \| \cdot \|_{\ell_1})$ is not reflexive).

2. Bornologically surjective hulls of operator ideals on LCS’s

Let $X$ be a LCS and $\mathcal{D}(X)$ be the family of all bounded disks in $X$. To each
$B$ in $\mathcal{D}(X)$ we associate a normed subspace $L_1(B)$ of $\ell_1(B)$ defined by $L_1(B) =$
$\{(\lambda_b)_{b\in B} : \sum_b \lambda_b \cdot b \text{ converges in } X\}$. In case $X$ is infracomplete, $L_1(B) = \ell_1(B)$.
Define $X^1$ to be the locally convex direct sum $X^1 = \oplus\{L_1(B) : B \in \mathcal{D}(X)\}$
equipped with the direct sum topology. Define $Q_X^1 : X^1 \to X$ by $Q_X^1(\bigoplus_B \lambda_B) =$
$\sum\sum_B \lambda_{B,b} \cdot b$ where $B \in \mathcal{D}(X)$ and $\lambda_B = (\lambda_{B,b})_{b\in B} \in L_1(B)$.

Lemma 2.1. $Q_X^1$ is a bornological surjection of $X^1$ onto $X$.

Proof. It is apparent that $Q_X^1$ is linear and surjective. Since the mapping $L_1(B) \to X$
sending $\lambda_B = (\lambda_{B,b})_{b\in B}$ to $\sum_b \lambda_{B,b} \cdot b$ is continuous for each $B$ in $\mathcal{D}(X)$, $Q_X^1$ is
continuous. Moreover, if $B$ is a bounded disk in $X$ then $\cup_{L_1(B)}$ is a bounded disk
in $X^1$ and $Q_X^1(\cup_{L_1(B)}) \supset B$. That is, $Q_X^1$ is a bornological surjection.  $\square$
**Lemma 2.2.** Let $X$ and $Y$ be LCS’s and $T \in \mathcal{L}(X, Y)$. Then we have a $T_1$ in $\mathcal{L}(X^1, Y^1)$ such that $TQ_X^1 = Q_Y^1 T_1$.

**Proof.** For each $B$ in $\mathcal{D}(X)$, $TB \in \mathcal{D}(Y)$. Define $T_B : L_1(B) \to L_1(TB)$ by $T_B(\lambda) = \beta$ where $\lambda = (\lambda_b)_{b \in B}$ and $\beta = (\beta_c)_{c \in TB}$ with $\beta_c = \sum_{T_b = c} \lambda_b$. Note $|\beta_c| \leq \sum_{T_b = c} |\lambda_b| \leq \|\lambda\|_{L_1(B)} < \infty$ and $\|\beta\|_{L_1(TB)} = \sum |\beta_c| = \sum |\lambda_b| = \|\lambda\|_{L_1(B)}$. So $T_B$ is a well-defined continuous operator. We define $T_1$ in $\mathcal{L}(X^1, Y^1)$ by the commutative diagrams

$$
\begin{array}{ccc}
X^1 & \xrightarrow{T_1} & Y^1 \\
\uparrow & & \uparrow \\
L_1(B) & \xrightarrow{T_B} & L_1(TB)
\end{array}
$$

where the vertical arrows represent the corresponding canonical embeddings and $B$ runs through all members in $\mathcal{D}(X)$. Finally if $\lambda = \oplus \lambda_B \in X^1$ with $\lambda_B = (\lambda_{B,b})_{b \in B} \in L_1(B)$,

$$
TQ_X^1(\lambda) = T\left(\sum_B \sum_{b \in B} \lambda_{B,b} \cdot b\right) \\
= \sum_B \sum_{b \in B} \lambda_{B,b} \cdot Tb \\
= \sum_B \sum_{c \in TB} \beta_{TB,c} \cdot c
$$

where $\beta_{TB,c} = \sum_{T_b = c} \lambda_{B,b}$, and

$$
Q_Y^1 T_1(\lambda) = Q_Y^1 \left(\oplus_B (\beta_{TB,c})_{c \in TB}\right) \\
= \sum_B \sum_{c \in TB} \beta_{TB,c} \cdot c.
$$

Hence $TQ_X^1 = Q_Y^1 T_1$. □

**Lemma 2.3.** Let $X$ and $Y$ be LCS’s and $T$ be a bornological surjection from $X$ onto $Y$. Then there is a $T_{-1}$ in $\mathcal{L}(Y^1, X^1)$ such that $T_1 T_{-1} = id_{Y^1}$.
Proof. Let $C \in \mathcal{D}(Y)$. Since $T$ is a bornological surjection there exists a $B_C$ in $\mathcal{D}(X)$ such that $TB_C = C$. Let $\delta_C$ be a (set-theoretical) bijection from $C$ onto a subset $\delta_C(C)$ of $B_C$ such that $T\delta_C(c) = c$ for every $c$ in $C$. Define $T_C$ from $L_1(C)$ into $L_1(B_C)$ by $T_C(\beta) = \lambda$ where $\beta = (\beta_c)_{c \in C}$ and $\lambda = (\lambda_b)_{b \in B_C}$ with $\lambda_b = \beta_c$ if $b = \delta_C(c)$ for some $c$ in $C$ and $\lambda_b = 0$, otherwise. Clearly $T_C$ is linear. The equalities
\[ \|\lambda\| = \sum_{b \in B_C} |\lambda_b| = \sum_{c \in C} |\beta_c| = \|\beta\| \]
say that $T_C$ is continuous. We define a continuous operator $T_{-1}$ from $Y^1$ into $X^1$ such that the diagrams
\[
\begin{array}{ccc}
X^1 & \xrightarrow{T_{-1}} & Y^1 \\
\uparrow & & \uparrow \\
L_1(B_C) & \xrightarrow{T_C} & L_1(C)
\end{array}
\]
are all commutative for each $C$ in $\mathcal{D}(Y)$. It is not difficult to see that $T_1T_{-1} = \text{id}Y^1$.

Theorem 2.4. Let $\mathfrak{A}$ be an operator ideal on LCS’s. The bornologically surjective hull of $\mathfrak{A}$ is given by
\[ \mathfrak{A}^{\text{bsur}}(X, Y) = \{ T \in \mathcal{L}(X, Y) : TQ_X^1 \in \mathfrak{A}(X, Y^1) \} \]
for every pair $X$ and $Y$ of LCS’s.

Proof. We first check that $\mathfrak{A}^{\text{bsur}}$ is an operator ideal on LCS’s. Let $X$ and $Y$ be LCS’s. It is obvious that $\mathfrak{A}^{\text{bsur}}(X, Y)$ contains all continuous operators of finite rank from $X$ into $Y$ since $\mathfrak{A}(X, Y^1)$ does. If $S$ and $T$ belong to $\mathfrak{A}^{\text{bsur}}(X, Y)$ then so do $S + T$. Hence $\mathfrak{A}^{\text{bsur}}(X, Y)$ is a nonempty linear subspace of $\mathcal{L}(X, Y)$. Let $S \in \mathcal{L}(X_0, X)$, $T \in \mathfrak{A}^{\text{bsur}}(X, Y)$ and $R \in \mathcal{L}(Y, Y_0)$ for some LCS’s $X_0, X, Y_0$ and
The commutative diagram

\[
\begin{array}{ccccccc}
X^1 & \xrightarrow{Q_X^1} & X & \xrightarrow{T} & Y \\
\uparrow s_1 & & \uparrow s & & \downarrow R \\
X_0^1 & \xrightarrow{Q_{X_0}^1} & X_0 & \xrightarrow{RTS} & Y_0
\end{array}
\]

shows that \( RTS \in \mathcal{A}^{bsur}(X_0, Y_0) \).

Next we check that \( \mathcal{A}^{bsur} \) is bornologically surjective. Let \( X_0, X \) and \( Y \) be LCS's, \( T \in \mathcal{L}(X, Y) \) and \( Q \in \mathcal{L}(X_0, X) \) be a bornological surjection such that \( TQ \in \mathcal{A}^{bsur}(X_0, Y) \). Then \( TQ^1_X = TQ^1_X id_{X^1} = TQ^1_X Q_1 Q^{-1} = ((TQ)Q^1_{X_0}) Q^{-1} \in \mathcal{A}(X^1, Y) \) by the commutative diagram

\[
\begin{array}{ccccccc}
X_0 & \xrightarrow{Q} & X & \xrightarrow{T} & Y \\
\uparrow Q_{X_0}^1 & & \uparrow Q_X^1 & & \uparrow Q_X^1 \\
X_0^1 & \xrightarrow{Q_1} & X^1 & & & & \uparrow id_{X^1} \\
& & \downarrow Q^{-1} & & & & \downarrow \
\end{array}
\]

Hence \( T \in \mathcal{A}^{bsur}(X, Y) \) and thus \( \mathcal{A}^{bsur} \) is bornologically surjective.

Finally, if \( \mathcal{A}_0 \) is another bornologically surjective operator ideal containing \( \mathcal{A} \) and \( T \in \mathcal{A}^{bsur}(X, Y) \) for some LCS’s \( X \) and \( Y \) then \( TQ_X^1 \in \mathcal{A}(X^1, Y) \subset \mathcal{A}_0(X^1, Y) \). The bornological surjectivity of \( Q_X^1 \) implies \( T \in \mathcal{A}_0(X, Y) \). Therefore, \( \mathcal{A}^{bsur} \subset \mathcal{A}_0 \) and thus \( \mathcal{A}^{bsur} \) is the bornologically surjective hull of \( \mathcal{A} \). \( \square \)
The following result ensures that we can safely substitute the surjectivity for the bornological surjectivity in many cases. Let \( N \) be a normed space. Similar to the case of Banach spaces, we define \( N^{\text{sur}} \) to be the normed space \( L_1(\cup N) \) and \( Q_N : N^{\text{sur}} \rightarrow N \) to be the surjection defined by \( Q_N((\lambda_x)_{x \in \cup N}) = \sum \lambda_x \cdot x \).

**Proposition 2.5.** Let \( \mathfrak{A} \) be an operator ideal on LCS’s, \( N \) a normed space, \( Y \) a LCS and \( T \in \mathcal{L}(N, Y) \). Then \( TQ_N^1 \in \mathfrak{A}(N^1, Y) \) if and only if \( TQ_N \in \mathfrak{A}(N^{\text{sur}}, Y) \).

**Proof.** For each \( B \) in \( sD(N) \), let \( \lambda_B > 0 \) such that \( B \subset \lambda_B \cup N \). Associate to each \( f_B \) in \( L_1(B) \) a \( g_B \) in \( L_1(\cup N) \) such that \( g_B(b) = \lambda_B f_B(\lambda_B b) \) for all \( b \) in \( \cup N \) such that \( \lambda_B b \in B \) and \( g_B(b) = 0 \), otherwise. Define \( P \) in \( \mathcal{L}(N^1, N^{\text{sur}}) \) by \( P(\oplus f_B) = \sum_B g_B \) and \( J : N^{\text{sur}} \rightarrow N^1 \) be the canonical embedding. It is easy to see that \( Q_N P = Q_N^1 \) and \( Q_N = Q_N^1 J \). The desired assertion follows from this. \( \square \)

**Corollary 2.6.** Let \( \mathfrak{A} \) be an operator ideal on LCS’s and \( N \) be a normed space. Then

\[
\mathfrak{A}^{\text{bsur}}(N, Y) \subset \mathfrak{A}^{\text{sur}}(N, Y), \quad \forall \text{LCS } Y.
\]

They are equal if \( \mathfrak{A} \) is surjective.

**Proof.** Let \( T \in \mathcal{L}(N, Y) \). Observe that

\[
T \in \mathfrak{A}^{\text{bsur}}(N, Y)
\]

\( \Rightarrow \) \( TQ_N^1 \in \mathfrak{A}(N^1, Y) \) by Theorem 2.4

\( \Rightarrow \) \( TQ_N \in \mathfrak{A}(N^{\text{sur}}, Y) \) by Proposition 2.5

\( \Rightarrow \) \( T \in \mathfrak{A}^{\text{sur}}(N, Y) \) since \( Q_N \) is a surjection.

The asserted equality is trivial. \( \square \)

**Remark.** The construction of \( \mathfrak{A}^{\text{bsur}} \) is deeply influenced by [1]. A direct construction of \( \mathfrak{A}^{\text{sur}} \) of the similar sort for an operator ideal \( \mathfrak{A} \) on LCS’s seems to be impossible.
For example, locally convex direct sums and quotients of Mackey spaces are still Mackey, see e.g. [3]. It forces us to remain in the category of Mackey spaces.

For comparison and later uses we describe the construction of \( A^{\text{inj}} \). Let \( X \) be a LCS and \( \mathcal{E}(X') \) be the collection of all \( \sigma(X', X) \)-closed and equicontinuous disks in \( X' \) and let \( X^\infty = \Pi\{\ell_\infty(D) : D \in \mathcal{E}(X')\} \) be the product space equipped with the product topology. Define \( J_X^\infty : X \to X^\infty \) by setting \( J_X^\infty(x) = (J_{X,D}(x))_{D \in \mathcal{E}(X')} \), where \( J_{X,D}(x) \in \ell_\infty(D) \) is a bounded scalar function on \( D \) with values \( J_{X,D}(x)(d') = \langle x, d' \rangle, \forall d' \in D \).

**Theorem 2.7** (Franco and Piñeiro [1]). The map \( J_X^\infty \in \mathcal{L}(X, X^\infty) \) is an injection for every LCS \( X \). Let \( X \) and \( Y \) be LCS’s and \( T \in \mathcal{L}(X, Y) \). There is a \( T_\infty \) in \( \mathcal{L}(X^\infty, Y^\infty) \) such that \( J_Y^\infty T = T_\infty J_X^\infty \). If, in addition, \( T \) is an injection then there is a \( T_{-\infty} \) in \( \mathcal{L}(Y^\infty, X^\infty) \) such that \( T_{-\infty} T_\infty = \text{id}_{X^\infty} \). Moreover, the injective hull \( A^{\text{inj}} \) of an operator ideal \( A \) on LCS’s is given by

\[
A^{\text{inj}}(X, Y) = \{ T \in \mathcal{L}(X, Y) : J_Y^\infty T \in A(X, Y^\infty) \}
\]

for every pair \( X \) and \( Y \) of LCS’s.

Associate to each normed space \( N \) the Banach space \( N^{\text{inj}} = \ell_\infty(\bigcup N') \) and the injection \( J_N \) in \( \mathcal{L}(N, N^{\text{inj}}) \) defined by \( J_N(x) = (\langle x, a \rangle)_{a \in \bigcup N'} \). Analogous to Proposition 2.5, we have

**Proposition 2.8.** Let \( A \) be an operator ideal on LCS’s, \( X \) be a LCS, \( N \) be a normed space and \( T \in \mathcal{L}(X, N) \). \( J_N T \in A(X, N^{\text{inj}}) \) if and only if \( J_N^\infty T \in A(X, N^\infty) \).

**Proof.** Define \( \pi \) to be the canonical projection from \( N^\infty \) onto \( N^{\text{inj}} \). Let \( D \) be a closed and bounded disk in \( N' \) and \( \lambda_D > 0 \) such that \( D \subset \lambda_D \bigcup N' \). Associate to
each $f$ in $ℓ∞(∪N′)$ an $f_D$ in $ℓ∞(D)$ such that $f_D(d) = λ_D \left( \frac{d}{λ_D} \right)$, $∀d ∈ D$. Define a $j$ in $L(N^{inj}, N^∞)$ by $j(f) = (f_D)_{D ∈ ℓ(N′)}$. It is easy to see that $jJ_N = J_N^∞$ and $J_N = πJ_N^∞$. The assertion is now clear.

Proposition 2.9. Let $A$ be an operator ideal on LCS’s. We have

$$(A^{bsur})^{inj} = (A^{inj})^{bsur}$$

Proof. Follows easily from Theorems 2.4 and 2.7.

Proposition 2.10. Let $A$ be an operator ideal on LCS’s. We have

$$(A^{inj})^{sur} ⊂ (A^{sur})^{inj}$$

Proof. By Theorem 2.7, it is easy to see that the injective hull of a surjective operator ideal is still surjective. The asserted inclusion is a direct consequence of this.

3. Injectivity and surjectivity under extensions

Let $A$ be an operator ideal on LCS’s. We denote by $A_B$ the restriction of $A$ to Banach spaces.

Lemma 3.1. Let $A$ be an operator ideal on LCS’s. We have

(a) $(A^{inj})_B = (A_B)^{inj}$, and
(b) $(A^{bsur})_B = (A_B)^{sur} ⊂ (A^{sur})_B$,

where the injective hull and the surjective hull of $A_B$ are, of course, taken within the category of Banach spaces.

Proof. Follows easily from Propositions 2.5 and 2.8 and Corollary 2.7.
Corollary 3.2. Let $\mathfrak{A}$ be an operator ideal on Banach spaces.

(a) If $\mathfrak{A}$ is injective then $\mathfrak{A}^{\text{sup}}$ is injective, too.

(b) If $\mathfrak{A}$ is surjective then $\mathfrak{A}^{\text{sup}}$ is bornologically surjective, too.

Proposition 3.3. Let $\mathfrak{A}$ be an operator ideal on Banach spaces. We have

(a) $(\mathfrak{A}^{\text{inj}})^{\text{lup}} \subset (\mathfrak{A}^{\text{lup}})^{\text{inj}}$ and $(\mathfrak{A}^{\text{inj}})^{\text{lup}}(X, Y) = (\mathfrak{A}^{\text{lup}})^{\text{inj}}(X, Y)$ for every LCS $X$ and every sequentially complete LCS $Y$.

(b) $(\mathfrak{A}^{\text{sur}})^{\text{rup}} \subset (\mathfrak{A}^{\text{rup}})^{\text{bsur}}$, $(\mathfrak{A}^{\text{sur}})^{\text{rup}}(X, Y) = (\mathfrak{A}^{\text{rup}})^{\text{bsur}}(X, Y)$ for every bornological LCS $X$ and every LCS $Y$.

Proof. Let $X$ and $Y$ be LCS’s and $T \in \mathcal{L}(X, Y)$. For (a), assume that $T \in (\mathfrak{A}^{\text{inj}})^{\text{lup}}$ and verify $T \in (\mathfrak{A}^{\text{lup}})^{\text{inj}}$, or equivalently, $J_Y^\infty T \in \mathfrak{A}^{\text{lup}}(X, Y^\infty)$. Let $S \in \mathcal{L}(E, X)$ where $E$ is a Banach space. Since $T \in (\mathfrak{A}^{\text{inj}})^{\text{lup}}$, we have a Banach space $F$, an $S_0$ in $\mathfrak{A}^{\text{inj}}(E, F)$ and an $R$ in $\mathcal{L}(F, Y)$ such that $TS = RS_0$. Consider the following commutative diagram:

\[
\begin{array}{ccccccc}
E & \overset{S}{\longrightarrow} & X & \overset{T}{\longrightarrow} & Y & \overset{J_Y^\infty}{\longrightarrow} & Y^\infty \\
\downarrow{S_0} & & \downarrow{R} & & \downarrow{R_\infty} & & \\
F & \overset{J_F^\infty}{\longrightarrow} & F^\infty & \overset{j}{\longrightarrow} & F^{\text{inj}} \\
\end{array}
\]

where the map $j$ is defined in Proposition 2.8 and $R_\infty$ is the one in Theorem 2.7 (cf. [1]). Now we have $(J_Y^\infty T)S = (R_\infty j)(J_F S_0)$ and $J_F S_0 \in \mathfrak{A}(E, F^{\text{inj}})$ by Proposition
2.8. Thereby, we can infer that $J_Y^\infty T \in \mathfrak{A}^\text{lup}(X, Y^\infty)$.

Conversely, assume $T \in (\mathfrak{A}^\text{lup})^\text{inj}(X, Y)$ and $Y$ is sequentially complete. Let $E$ be a Banach space and $S \in \mathcal{L}(E, X)$. We have a factorization of $J_Y^\infty TS = RS_0$ for some $R$ in $\mathcal{L}(F, Y^\infty)$ and $S_0$ in $\mathfrak{A}(E, F)$ where $F$ is a Banach space. The goal is to establish a similar factorization of $TS$. Consider the following commutative diagram:

\[
\begin{array}{c}
E \xrightarrow{S} X \xrightarrow{T} Y \\
| \downarrow S_0 \hspace{1cm} \downarrow R_0 \\
S_0E \xrightarrow{J} \mathfrak{A}(E, F) \xrightarrow{J_Y^\infty} Y^\infty \\
| \downarrow J \hspace{1cm} \downarrow R_2 \\
F \xrightarrow{R} Y^\infty
\end{array}
\]

Here $J$ is the natural embedding of the norm closure of the range space $S_0E$ of $S_0$ into $F$ and $S_2$ in $\mathcal{L}(E, S_0E)$ and $R_2$ in $\mathcal{L}(S_0E, Y^\infty)$ are the maps induced by $S_0$ and $R$, respectively. Since $J_Y^\infty TS = RS_0$, $Y$ is sequentially complete and $J_Y^\infty$ is an injection, we can define an $R_0$ in $\mathcal{L}(S_0E, Y)$ such that $J_Y^\infty R_0 = R_2$. Now $TS = R_0S_2$ and $S_2 \in \mathfrak{A}^\text{inj}(E, S_0E)$ since $JS_2 = S_0 \in \mathfrak{A}(E, F)$ and $J$ is an injection. It implies that $T \in (\mathfrak{A}^\text{inj})^\text{lup}(X, Y)$.

The proof of (b) is similar to the above except we shall use the map $P$ defined in the proof of Proposition 2.6 instead of $j$. For the second part, we refer the readers to the following commutative diagram and ask them to fill in the detail.
Proposition 3.4. Let $\mathfrak{A}$ be an operator ideal on Banach spaces. We have

(a) $(\mathfrak{A}^{\text{rup}})^{\text{inj}} \subset (\mathfrak{A}^{\text{inj}})^{\text{rup}}$, and

(b) $(\mathfrak{A}^{\text{lup}})^{\text{bsur}} \subset (\mathfrak{A}^{\text{sur}})^{\text{lup}}$.

Proof. (a) follows easily from [4, p. 398] but we would like to provide another proof. Let $T \in (\mathfrak{A}^{\text{rup}})^{\text{inj}}(X,Y)$ and $S \in \mathcal{L}(Y,F)$ for some LCS's $X$ and $Y$ and Banach space $F$. We use the following commutative diagram to obtain a factorization of $ST = S_2R_2$ with $S_2$ in $\mathfrak{A}^{\text{inj}}$ and hence $T \in (\mathfrak{A}^{\text{inj}})^{\text{rup}}$. Note that $J_Y^{\infty}T \in \mathfrak{A}^{\text{rup}}(X,Y^{\infty})$ ensures a factorization of $\pi S_\infty J_Y^{\infty}T = S_0R_0$ for some $R_0$ in $\mathcal{L}(X,E)$, $S_0$ in $\mathfrak{A}(E,F^{\text{inj}})$ and Banach space $E$. 

\[
\begin{array}{ccccccc}
X & \xrightarrow{T} & Y & \xrightarrow{S} & F \\
& & \downarrow{R_0} & & \downarrow{S_2} \\
E/\text{Ker}S_0 & \xrightarrow{Q_\mathcal{X}} & S_0 \\
& \downarrow{R_2} & & \downarrow{Q} \\
X^1 & \xrightarrow{R} & E
\end{array}
\]
Here $J$ is the canonical embedding from the norm closure $R_0 X$ of the range space of $R_0$ in $E$, $S_\infty$ and $J_F^\infty$ are defined in Theorem 2.7 (cf. [1]), $\pi$ is defined in Proposition 2.8, $R_2$ and $S_2$ are induced by $R_0$ and $S_0$, respectively. Now $J_F S_2 = S_0 J \in \mathfrak{A}(\overline{RE}, F^{\text{inj}})$, and thus $S_2 \in \mathfrak{A}^{\text{inj}}(\overline{RE}, F)$ by Proposition 2.8, as asserted.

(b) is essentially identical except that we shall use the following commutative diagram instead.

The detail is left to the readers. $\square$

**Proposition 3.5.** Let $\mathfrak{A}$ be an operator ideal on Banach spaces and $E$ and $F$ be Banach spaces. Then

(a) $(\mathfrak{A}^{\text{rup}})^{\text{inj}}(X, F) = (\mathfrak{A}^{\text{inj}})^{\text{rup}}(X, F)$, and
\[(b) \ (\mathfrak{A}^{\text{sur}})^{\text{bsur}}(E,Y) = (\mathfrak{A}^{\text{lup}})^{\text{lup}}(E,Y) \text{ hold for all LCS's } X \text{ and } Y.\]

Proof. We prove (b) only and (a) is similar. In view of Proposition 3.4, it suffices to verify that every \( T \) in \((\mathfrak{A}^{\text{sur}})^{\text{lup}}(E,Y)\) belongs to \((\mathfrak{A}^{\text{lup}})^{\text{bsur}}(E,Y)\) whenever \( E \) is a Banach space and \( Y \) is a LCS. By Proposition 2.5, it is equivalent to that \( TQ_E \in \mathfrak{A}^{\text{lup}}(E^{\text{sur}},Y) \). Since \( E \) is a Banach space, we have a factorization of \( Tid_E = RK \) for some \( K \) in \( \mathfrak{A}^{\text{sur}}(E,F) \) and \( R \) in \( \mathcal{L}(F,Y) \) where \( F \) is a Banach space. Now \( TQ_E = Tid_EQ_E = RKQ_E \) and \( KQ_E \in \mathfrak{A}(E^{\text{sur}},F) \) ensure the assertion. \( \square \)

**Proposition 3.6.** Let \( \mathfrak{A} \) be an operator ideal on Banach spaces. We have

\[(a) \ (\mathfrak{A}^{\text{inf}})^{\text{inj}} \subset (\mathfrak{A}^{\text{inf}})^{\text{inj}}; (\mathfrak{A}^{\text{inf}})^{\text{inj}}(X,Y) = (\mathfrak{A}^{\text{inf}})^{\text{inj}}(X,Y) \text{ for every LCS } X \text{ and every infracomplete LCS } Y.\]

\[(b) \ (\mathfrak{A}^{\text{sur}})^{\text{inf}} \subset (\mathfrak{A}^{\text{inf}})^{\text{bsur}}; (\mathfrak{A}^{\text{sur}})^{\text{inf}}(X,Y) = (\mathfrak{A}^{\text{inf}})^{\text{bsur}}(X,Y) \text{ for every bornological LCS } X \text{ and every LCS } Y.\]

Proof. The inclusions in (a) and (b) follow easily from Lemma 3.1. For (a), let \( T \in (\mathfrak{A}^{\text{inf}})^{\text{inj}}(X,Y) \). We need to show that there is a continuous seminorm \( q \) on \( X \) and a bounded \( \sigma \)-disk \( B \) in \( Y \) such that the induced map \( T_Bq \) by \( T \) belongs to \( \mathfrak{A}^{\text{inj}}(\tilde{X}_q,Y(B)) \). By assumption there is a continuous seminorm \( q \) on \( X \) and a bounded \( \sigma \)-disk \( C \) in \( Y^{\infty} \) such that \( J_{\infty}TV_q \subset C \) and the induced map \( R \) from \( \tilde{X}_q \) into \( Y^{\infty}(C) \) by \( J_{\infty}T \) belongs to \( \mathfrak{A}(\tilde{X}_q,Y^{\infty}(C)) \) where \( V_q = \{ x \in X : q(x) \leq 1 \} \). Since \( J_{\infty} \) is an injection, and \( Y \) is assumed to be infracomplete, the bounded disk \( B = (J_{\infty})^{-1}C \) is \( \sigma \)-disked in \( Y \). Moreover, it is clear that \( TV_q \subset B \). Let \( T_Bq \) in \( \mathcal{L}(\tilde{X}_q,Y(B)) \) and \( S \) in \( \mathcal{L}(Y(B),Y^{\infty}(C)) \) be the maps induced by \( T \) and \( J_{\infty} \), respectively. Since \( ST_{Bq} = R \) belongs to \( \mathfrak{A}(\tilde{X}_q,Y^{\infty}(C)) \) and \( S \) is an injection, \( T_{Bq} \in \mathfrak{A}^{\text{inj}}(\tilde{X}_q,Y(B)) \), as asserted. For (b), let \( T \in (\mathfrak{A}^{\text{inf}})^{\text{bsur}}(X,Y) \). We want to
verify that $T \in (\mathfrak{A}^{\mathrm{sur}})^{\mathrm{inf}}$. By assumption, $TQ_X^1$ has a factorization $TQ_X^1 = ST_0R$ for some $R$ in $\mathcal{L}(X^1, E)$, $S$ in $\mathcal{L}(F, Y)$ and $T_0$ in $\mathfrak{A}(E, F)$, where $E$ and $F$ are Banach spaces. Let $E_0 = E/\mathrm{Ker}ST_0$ and $Q$ be the quotient map from $E$ onto $E_0$. Define a linear operator $R_0$ from $X$ into $E_0$ by the relation $R_0x = QRy$ where $Q_X^1y = x$. Since $Q_X^1$ is bornologically surjective, $R_0$ is locally bounded. $R_0$ is thus continuous as $X$ is assumed to be bornological. Let $T_2$ in $\mathcal{L}(E_0, F)$ be the map induced by $T_0$. $T_2Q = T_0 \in \mathfrak{A}$ implies $T_2 \in \mathfrak{A}^{\mathrm{sur}}$. Now, $T = ST_2R_0 \in (\mathfrak{A}^{\mathrm{sur}})^{\mathrm{inf}}$, as asserted.

We leave the cases of left and right inferior extensions of operator ideals on Banach spaces to interested readers.

**References**