On the Existence of Nonlinear Inequalities[*] [†]

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Abstract– The object of this paper is to establish an existence result for nonlinear inequalities for not necessarily pseudomonotone maps. As a consequence of our result, we give some existence results for variational and variational-like inequalities.

Keywords– Nonlinear inequalities, variational inequality problems, variational-like inequality problems, fixed points.

1. INTRODUCTION AND PRELIMINARIES

Let X be a real locally convex space with its dual X^* and K be a non-empty convex subset of X. Let $\varphi : K \times K \longrightarrow \mathbb{R}$ be a bifunction and $f \in X^*$. Gwinner [9] considered the problem of finding u_0 in K such that

$$\varphi(u_0, v) \geq \langle f, v - u_0 \rangle, \quad \text{for all} \quad v \text{ in } K, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between X^* and X. (1) is called the *nonlinear inequality.*

Such types of nonlinear inequalities model some equilibrium problems drawn from operations research as well as some unilateral boundary value problems stemming from mathematical physics. The existence theory and abstract stability analysis of (1) have been investigated by Gwinner [9] in the setting of reflexive Banach spaces.

When $\varphi(u, v) = \langle T(u), v - u \rangle$, where $A : K \longrightarrow X^*$ is an ?operator?, (1) reduces to the *classical variational inequality problem* introduced by Lions and Stampacchia [12], that is, to find a u_0 in K such that

$$\langle T(u_0), v - u_0 \rangle \geq \langle f, v - u_0 \rangle, \quad \text{for all} \quad v \text{ in } K. \tag{2}$$

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Even more specifically, when $\varphi(u,v) = \langle T(u), v-u\rangle$ and $f \equiv 0$, (1) becomes the *original variational inequality problem* (see, e.g. [4], [10]), that is, to find a u_0 in K such that

$$\langle T(u_0), v-u_0\rangle \geq 0, \quad \text{for all} \quad v \text{ in } K. \tag{3}$$

Many authors have studied this type of problems in the context of reflexive Banach spaces (see, for example [4], [14], [11]). Watson [18] established the existence of solutions to problem (3) in the setting of not necessarily reflexive Banach spaces and pseudomonotone and hemicontinuous maps. His assumptions are weaker than those needed in [6].

When $\varphi(u,v) = \langle T(u), \eta(v,u)\rangle$ and $f \equiv 0$, where $\eta : K \times K \longrightarrow X$ is a ?map?, (1) is equivalent to find a u_0 in K such that

$$\langle T(u_0), \eta(v,u_0)\rangle \geq 0, \quad \text{for all} \quad v \text{ in } K. \tag{4}$$

Inequality (4) is known as a *variational-like inequality* which has lot of applications in operations research, optimization and mathematical programming. For further details, we refer [13], [7], [19], [16].

In case $f \equiv 0$ and $\varphi(u,u) = 0$, for all u in K, (1) reduces to the *equilibrium problem* considered in [1]-[3], that is, to find a u_0 in K such that

$$\varphi(u_0, v) \geq 0, \quad \text{for all} \quad v \text{ in } K. \tag{5}$$

Apparently, our nonlinear inequality (1) contains all above variational inequalities and equilibrium problems as special cases.

The main object of this paper is to establish an existence result for the nonlinear inequality (1) for general maps without any pseudomonotone assumption. We shall employ Fan-Browder [5, 8] type fixed point theorems due to Tarafdar [17]. As a consequence of our result, we shall derive some existence results for the variational inequality (3) and the variational-like inequality (4) without any kind of monotonicity assumption.

The following fixed point theorem will be used in this paper. We denote by 2^Y the family of all non-empty subsets of a set Y.

Theorem 1. [17] Let K be a non-empty convex subset of a Hausdorff topological vector space X. Let $Q : K \longrightarrow 2^K$ be a multifunction such that

(a) for each x in K, $Q(x)$ is a non-empty convex subset of K;

(b) for each y in K, $Q^{-1}(y)$ contains relatively open subset O_y of K (O_y may be empty for some y in K) such that $\bigcup_{y\in K} O_y = K$;

(c) K contains a non-empty subset D_0 which is contained in a compact convex subset D_1 of K such that the set $D = \bigcap_{x \in D_0} O_x^c$ is either empty or compact, where O_x^c denotes the complements of O_x in K.

Then there exists a point x_0 in K such that $x_0 \in Q(x_0)$.

2. MAIN RESULTS

We now prove the main result of this paper.

Theorem 2. Let K be a non-empty convex subset of a Hausdorff topological vector space X (over the real field). Let f be a non-zero continuous linear functional on X. Let $\varphi : K \times K \longrightarrow \mathbb{R}$ be a bifunction vanishing on the diagonal, i.e. $\varphi(u, u) = 0$, for all u in K, and satisfying the following conditions.

1. φ is convex in the second variable.
2. $\liminf_{u \to u^*} \varphi(u, v) \leq \varphi(u^*, v)$ for all v in K whenever $u \to u^*$ in K.
3. There is a compact convex subset D_1 of K such that for each u in $K \setminus D_1$ there is a v in D_1 with
$$\varphi(u, v) < \langle f, v - u \rangle.$$

Then the nonlinear inequality (1) has a solution in K.

Proof. We define $A(v) = \{u \in K : \varphi(u, v) \geq \langle f, v - u \rangle\}$ for each v in K. Then the solution set of (1) is $S = \bigcap_{v \in K} A(v)$. We note that for each v in K, $A(v)$ is closed. Indeed, let $\{u_\lambda\}_{\lambda \in \Lambda}$ be a net in $A(v)$ such that $u_\lambda \longrightarrow u$ in K. Then for all f in X^* we have $\langle f, u_\lambda - v \rangle \longrightarrow \langle f, u - v \rangle$. Since $u_\lambda \in A(v)$ and $\liminf_{u_\lambda \to u} \varphi(u_\lambda, v) \leq \varphi(u, v)$, for all v in K, we have

$$\varphi(u, v) \geq \liminf_{u_\lambda \to u} \varphi(u_\lambda, v) \geq \liminf_{u_\lambda \to u} \langle f, u_\lambda - v \rangle = \langle f, u - v \rangle.$$

Hence $u \in A(v)$. So for all v in K, $A(v)$ is closed.

Now we shall prove that the solution set S is non-empty. Assume contrary that $S = \emptyset$. Then for each u in K, the set

$$B(u) = \{v \in K : u \notin A(v)\} = \{v \in K : \varphi(u, v) < \langle f, v - u \rangle\} \neq \emptyset.$$

Since φ is convex in the second variable, we have for each u in K, $B(u)$ is convex. Thus $B : K \longrightarrow 2^K$ defines a multifunction such that for each u in K, $B(u)$ is non-empty and convex. Now for each v in K, the set

$$\begin{aligned} B^{-1}(v) &= \{u \in K : v \in B(u)\} \\ &= \{u \in K : \varphi(u,v) < \langle f, v-u \rangle\} \\ &= \{u \in K : \varphi(u,v) \geq \langle f, v-u \rangle\}^c \\ &= [A(v)]^c \\ &= O_v \end{aligned}$$

is open in K. We claim that $\bigcup_{v \in K} O_v = \bigcup_{v \in K} B^{-1}(v) = K$. To see this, let $u \in K$. As $B(u) \neq \emptyset$, we can choose a v from $B(u)$. Hence $u \in B^{-1}(v) = O_v$.

From the last condition of the theorem, for each u in $K \setminus D_1$ there is a v in D_1 with $\varphi(u,v) < \langle f, v-u \rangle$, that is, $u \notin A(v)$. This implies that $D = \bigcap_{v \in D_1} O_v^c = \bigcap_{v \in D_1} A(v) \subset D_1$. Since for each v in K, $A(v)$ is closed, D is a closed subset of the compact set D_1 and hence D is compact. Thus the multifunction $B : K \longrightarrow 2^K$ satisfy all the conditions of Theorem 1, so there exists a point u_0 in K such that $u_0 \in B(u_0)$, that is, $0 = \varphi(u_0, u_0) < \langle f, u_0 - u_0 \rangle = 0$ which is a contradiction. Hence the solution set S is non-empty. Therefore, the nonlinear inequality (1) has a solution in K. $\square$

In case K is compact, the last condition in Theorem 2 is automatically satisfied since we can set $D_1 = K$. Meanwhile, if a Hausdorff locally convex space X is barreled then every weak* closed and bounded subset K of X^* is weak* compact (see, for example, [15, p. 141]).

Corollary 1. Let X be a Hausdorff locally convex space with dual X^*. Let K be a weak* compact convex subset of X^* and $f \in X$. Let $\varphi : K \times K \longrightarrow \mathbb{R}$ be a bifunction vanishing on the diagonal, convex in the second variable, and satisfying the condition that $\liminf_{u \to u^*} \varphi(u,v) \leq \varphi(u^*, v)$, for all $v \in K$ whenever $u \to u^*$ in K. Then the nonlinear inequality (1) has a solution in K.

Corollary 2. Let K be a weak* compact convex subset of the dual space X^* of a Hausdorff locally convex space X. Let $\eta : K \times K \longrightarrow X^*$ be a bifunction vanishing on the diagonal. Let T be a function from K into X such that $h(v) := \langle \eta(v,u), T(u) \rangle$ is convex in v, for each fixed u in K, and $\liminf_{u \to u^*} \langle \eta(v,u), T(u) \rangle \leq \langle \eta(v, u^*), T(u^*) \rangle$ for each v in K whenever $u \to u^*$ in the weak* topology of K. Then the variational like inequality (4) has a solution in K.

Corollary 3. Let K be a closed convex subset of the dual space X^* of a Hausdorff locally convex barreled space X. Let T be a function from K into X such that $\liminf_{u \to u^*} \langle v - u, T(u) \rangle \leq \langle v - u^*, T(u^*) \rangle$, for each v in K, whenever $u \to u^*$ in the weak* topology of K. Further assume that there is a compact subset D of K such that for each u in $K \setminus D$

there is a v in D such that $\langle v-u, T(u)\rangle < 0$. Then the variational inequality problem (3) has a solution in K.

Proof. We note that the convex hull of a totally bounded subset of any locally convex space is totally bounded as well. On the other hand, the dual space X^* of the Hausdorff locally convex barreled space X is quasi-complete in its weak* topology, that is, closed and bounded subsets of X^* are complete (in fact, weak* compact). As a result, the convex hull D_1 of the weak* compact subset D of X^* is still weak* compact. Consequently, Theorem 2 applies. □

Corollary 4. Let X be a Hausdorff locally convex space with dual X^*. Let T be a function from X^* into X such that

1. $\liminf_{u\to u^*}\langle v-u, T(u)\rangle \leq \langle v-u^*, T(u^*)\rangle$, for each v in X^*, whenever $u \to u^*$ in the weak* topology of X^*;
2. there exists a weak* compact convex subset D_1 of X^* such that for each u not in D_1 there is a v in D_1 with $\langle v-u, T(u)\rangle < 0$.

Then T has a zero u_0 in X^*, i.e. $T(u_0) = 0$. In case X is barreled, the convexity assumption on D_1 can be dropped.

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