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Linear disjointness preservers of operator algebras and related structures

Jung-Hui Liu, Chun-Yen Chou, Ching-Jou Liao,
and Ngai-Ching Wong

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Abstract. We survey some recent studies of linear zero product or orthogonality preservers between C^*/W^*-algebras, their dual or predual spaces, and holomorphic disjointness preservers of C^*-algebras. Such maps are expected to provide algebra or linear Jordan ($*$-) homomorphisms between the underlying operator algebras. We also study orthogonality preservers between Hilbert C^*-modules and Fourier algebras. A few open problems are stated.

1. Introduction

In this note, we study disjointness preservers of (complex) C^*/W^*-algebras. Abelian C^*-algebras (resp. W^*-algebras) are $*$-isomorphic to the algebra $C_0(X)$ of continuous functions on a locally compact space (resp. compact hyper-stonean space) X vanishing at infinity. In general, a C^*-algebra (resp. W^*-algebra) A is algebra $*$-isomorphic to a norm (resp. σ-weakly) closed $*$-subalgebra of $\mathcal{B}(H)$ of bounded linear operators on a complex Hilbert space H. We note that every algebra $*$-isomorphism of a C^*-algebra is an isometry as well as an order isomorphism. Indeed, any one of the metric structure, the algebraic structure, and the order structure, determines A.

Let $\theta\colon C_0(X) \to C_0(Y)$ be a bijective isometry, that is, $\|\theta(f)-\theta(g)\| = \|f-g\|$ for all f, g in $C_0(X)$. Then, after translation, θ is real linear by the Mazur–Ulam Theorem. The Banach–Stone theorem in turn ensures that there is a homeomorphism $\sigma\colon Y \to X$, a partition $Y = Y_1 \cup Y_2$ of clopen subsets of Y, and a continuous

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unimodular scalar-valued function h on Y such that

$$\theta(f)(y) = \begin{cases} h(y) \cdot f(\sigma(y)) & \text{on } Y_1, \\ h(y) \cdot \overline{f(\sigma(y))} & \text{on } Y_2, \end{cases} \qquad (f \in C_0(X)). \tag{1.1}$$

Here $\overline{a+bi} = a - bi$ for a, b in $\mathbb{R}$ is the conjugate of complex numbers. We note that the map $f \mapsto f \circ \sigma$ is an algebra $*$-isomorphism from $C_0(X)$ onto $C_0(Y)$. Therefore, $h^{-1}\theta$ is a (direct) sum of a linear algebra $*$-isomorphism and a conjugate linear algebra $*$-isomorphism.

If $\theta\colon C_0(X) \to C_0(Y)$ is a ring isomorphism, then θ sends maximal ideals to maximal ideals. Since every maximal ideal of $C_0(X)$ assumes the form $I_x = \{f \in C_0(X) : f(x) = 0\}$ for some point x in X, we see that θ induces a bijective map $\sigma\colon Y \to X$ such that $\theta(I_{\sigma(y)}) = I_y$ for all y in Y. It then forces σ to be a homeomorphism, and thus $C_0(X), C_0(Y)$ are algebra $*$-isomorphic. If θ is also real linear, then θ assumes the standard form (1.1) with $h = 1$ being the constant 1 function. Thus θ is a sum of a linear algebra $*$-isomorphism and a conjugate linear algebra $*$-isomorphism. This is the so-called Gelfand–Kolmogorov theorem [36]. We note however that if θ is not real linear, it is not even continuous. In fact, a ring automorphism $\sigma\colon \mathbb{C} \to \mathbb{C}$ is continuous if and only if $\sigma(\mathbb{R}) = \mathbb{R}$ if and only if σ is linear or conjugate linear. See, for example, [85].

Moreover, let X, Y be compact and the bijective map $\theta\colon C(X) \to C(Y)$ be an order isomorphism, that is,

$$f \leq g \text{ in } C(X) \quad \text{if and only if} \quad \theta(f) \leq \theta(g) \text{ in } C(Y).$$

Kaplansky [49] shows that X, Y are homeomorphic, and thus $C(X), C(Y)$ are algebra $*$-isomorphic. If θ is also real linear, then θ assumes the standard form (1.1) but with h being a positive function bounded above and away from zero instead. See, for example, [28, 59, 60] for more explorations in this direction.

For the nonabelian case, Kadison [45] extends the Banach–Stone theorem to bijective (complex) linear isometries between general C^*-algebras. He shows that every bijective linear isometry $\theta\colon A \to B$ between two unital C^*-algebras arises from a linear Jordan $*$-isomorphism $J\colon A \to B$ and a unitary multiplier u of B such that $\theta(a) = uJ(a) \quad (a \in A)$. Hence the metric structure of a C^*-algebra determines its Jordan $*$-algebraic structure. We note that the nonunital case is also valid; see [69]. On the other hand, Kadison also shows in [46] that every bijective unital linear order isomorphism between two unital C^*-algebras is a linear Jordan $*$-isomorphism. Although Choi and Christensen [24] show that there is a bijective bi-completely positive linear map between two non-$*$-isomorphic C^*-algebras, a

Jordan isomorphism between C^*-algebras decomposes essentially as a direct sum of an algebra isomorphism and an algebra anti-isomorphism by a result of Brešar (see Theorem 2.2 below).

We have seen that different structures of a C^*-algebra A determine the Jordan $*$-algebraic structure of A. On the other hand, linear Jordan $*$-isomorphisms between C^*-algebras preserve many good properties. For example, they are isometries, and preserves both ordering and commutativity (see §2 for details). Apart from being an isometry or an order isomorphism, we are looking for minimum conditions to ensure a linear map between C^*-algebras providing an algebra or a linear Jordan ($*$-)homomorphism. Observing that for continuous functions f, g, the lattice property of disjointness $|f| \wedge |g| = 0$ is equivalent to the algebraic condition of having zero product $fg = 0$, we find that the disjointness structures of operator algebras can be the candidates.

In the context of operator algebras (on Hilbert spaces) there are at least four versions of disjointness:

- zero product ($ab = 0$),
- range orthogonality ($a^*b = 0$),
- domain orthogonality ($ab^* = 0$), and
- double orthogonality ($a^*b = ba^* = 0$).

Since the range and the domain orthogonality are symmetric, we do have only three different variants. Some authors might be also interested in the notion of

- double zero products ($ab = ba = 0$).

But this last case seems to be less studied than the others.

If the algebra is abelian, then all these concepts coincide. Let $\theta\colon C_0(X) \to C_0(Y)$ be a bijective linear map between abelian C^*-algebras preserving zero products, that is,

$$ab = 0 \implies \theta(a)\theta(b) = 0.$$

Then θ is automatically continuous and assumes the standard form (1.1). See Proposition 3.2 below.

Zero product and orthogonality linear preservers $\theta\colon A \to B$ between general C^*-algebras are studied in, for example, [6, 19, 23, 50, 74, 79, 81, 84]. In this case, all disjointness coincide on the set of self-adjoint elements. Suppose θ sends self-adjoint elements with zero products to (not necessarily self-adjoint) elements with zero products, that is,

$$ab = 0 \implies \theta(a)\theta(b) = 0 \qquad (a, b \in A_{sa}).$$

Assume that θ is bijective and *continuous*. Then A and B are isomorphic as Jordan algebras. If θ preserves zero products of arbitrary elements in A, then A and B are isomorphic as algebras. In both cases, $\theta^{**}(1)$ is an invertible central multiplier of B, and

$$J = \theta^{**}(1)^{-1}\theta \tag{1.2}$$

is a linear Jordan isomorphism or an algebra isomorphism. Here, $\theta^{**}\colon A^{**} \to B^{**}$ is the double dual map of θ between the enveloping W^*-algebras. The proofs make heavy uses of functional calculus, thanks to the continuity of θ ([84, Theorems 2.4 and 2.5]; see also Theorem 3.5).

Without assuming continuity, we can only utilize pure algebraic techniques. A few partial results exist in literature, for example, for properly infinite unital C^*-algebras [62] and CCR C^*-algebras with Hausdorff spectrum [58, 78]. Systematic approaches can be found in [19, 23]. In [21] and [57], it is shown that every bijective linear disjointness preserver between W^*-algebras arises from an algebra or a Jordan algebra ($*$-)isomorphism as in (1.2). Recently, we have extended some of these results for linear disjointness preservers of AW^*-algebras in [63]. The structure of such maps between general C^*-algebras is, however, still open to us.

Here is the content of this note. In §2, we recall some established results about the properties of Jordan ($*$-)homomorphisms. In §3, we present the progress of the linear disjointness preserver problems of C^*-algebras and W^*-algebras. In §4 and §5, we turn our attention to the linear orthogonality preservers of the predual (resp. dual) spaces of W^*-algebras (resp. C^*-algebras), and those between Hilbert C^*-modules, respectively. In §6, we complete this survey with the results on holomorphic disjointness preservers of C^*-algebras.

There are several related topics about disjointness preservers of operator algebras not discussed in this note. For example, we have not covered non-commutative L^p-spaces associated to W^*-algebras, which carry natural orthogonality structures. In the interesting paper [67] of Oikhberg and Peralta, a deep analysis on orthogonality preservers of non-commutative L^p-spaces is carried out. Readers are encouraged to explore more into this subject through this note and those papers in its bibliography. See also other survey articles, for example, [61, 64], about other interesting problems of preservers.

2. Jordan ($*$-)homomorphisms

As stated in §1, bijective isometries as well as order isomorphisms between operator algebras provide us Jordan $*$-isomorphisms. In this section, we give a brief review of how well Jordan ($*$-)homomorphisms behave, and how they differ from algebra ($*$-)homomorphisms.

Recall that a *Jordan homomorphism* $J\colon A \to B$ between two rings (always of characteristics not equal to 2 in this note) is an additive map satisfying $J(a^2)=J(a)^2$ $(a\in A)$, or equivalently, $J(ab+ba)=J(a)J(b)+J(b)J(a)$ $(a,b\in A)$. When A, B are $*$-rings, we call J a *Jordan $*$-homomorphism* if, in addition, $J(a^*) = J(a)^*$ $(a \in A)$.

In this section, we list some known facts about Jordan ($*$-)homomorphisms.

Lemma 2.1. *Let $J\colon A \to B$ be a Jordan homomorphism between two rings, and let $R(JA)$ be the ring generated by JA.*

(a) *J preserves powers, that is, $J(a^n) = J(a)^n$.*

(b) *J preserves Jordan triple products, that is,*

$$J(abc + cba) = J(a)J(b)J(c) + J(c)J(b)J(a).$$

(c) *J preserves squares of Lie brackets, that is,*

$$J((ab - ba)^2) = (J(a)J(b) - J(b)J(a))^2.$$

(d) *J preserves commutativity, that is,*

$$ab = ba \implies J(a)J(b) = J(b)J(a) \quad (a, b \in A),$$

provided that $R(JA)$ contains no nonzero nilpotent elements in its center, especially when B is a C^*-algebra.

(e) *When A has an identity 1, the element $J(1)$ is the identity of $R(JA)$. In this case, if a is invertible in A then $J(a)$ is invertible in $R(JA)$.*

(f) *J is 'almost' multiplicative or anti-multiplicative, in the sense that*

$$\begin{aligned}&(J(ab) - J(a)J(b))(J(ab) - J(b)J(a))\\ &\qquad = (J(ab) - J(b)J(a))(J(ab) - J(a)J(b)) = 0 \quad \text{for all } a, b \text{ in } A.\end{aligned}$$

Proof. See, e.g., [38, 42]. ∎

Theorem 2.2. (Brešar [13, Theorem 2.3]) *Let A and B be algebras over any field of characteristic not equal to 2 with B semiprime and θ a Jordan homomorphism from A onto B. Then there exist ideals U and V of A and ideals U' and V' of B such that*

(i) *$U \cap V = \ker\theta$ and $U + V$ is an essential ideal of A,*

(ii) *$U' \cap V' = 0$ and $U' \oplus V'$ is an essential ideal of B,*

(iii) *$\theta(U) = U'$ and $\theta(V) = V'$,*

(iv) *$\theta(ux) = \theta(u)\theta(x)$ for all u in U and all x in A,*

(v) *$\theta(vx) = \theta(x)\theta(v)$ for all v in V and all x in A.*

Moreover, if B is a normed algebra, then the ideals U' and V' are closed.

When θ is bijective, in general, $A \neq U \oplus V$ and $B \neq U' \oplus V'$ in Theorem 2.2 (see the example in [10, p. 458]). But as pointed out by Kadison in [45], $A = U \oplus V$ and $B = U' \oplus V'$ when A, B are W^*-algebras.

The following is well known. We state it for the record.

Lemma 2.3. *Let $J\colon A \to B$ be a map between rings or $*$-rings (of characteristics not equal to 2). If J is a ring (resp. $*$-ring) homomorphism then J preserves zero products (resp. range and domain orthogonality). If J is a Jordan homomorphism (resp. Jordan $*$-homomorphism) between C^*-algebras then J preserves double zero products (resp. double orthogonality).*

Proof. Suppose $J\colon A \to B$ is a Jordan $*$-homomorphism between two C^*-algebras. We check that J preserves double orthogonality. Let $a^*b = ba^* = 0$ in A. Then

$$J(a)^*J(b) + J(b)J(a)^* = J(a^*b + ba^*) = 0.$$

Since J preserves commutativity, we have

$$J(a)^*J(b) = J(a^*)J(b) = J(b)J(a^*) = J(b)J(a)^* = 0.$$

The other cases are also plain. ∎

Proposition 2.4. (Wong [84, Lemma 2.1]) *Let $J\colon A \to B$ be a Jordan homomorphism from a C^*-algebra into a ring. Then J preserves zero products of self-adjoint elements, that is, $ab = 0 \implies J(a)J(b) = 0$ $(a, b \in A_{sa})$.*

Theorem 2.5. (Brešar, see [23, Theorem 4.14]) *Let θ be a linear Jordan isomorphism from a (complex) algebra A onto a C^*-algebra B. If θ preserves zero products of arbitrary elements, then θ is an algebra isomorphism.*

Next comes the question of the automatic continuity of a Jordan homomorphism.

Theorem 2.6.

(a) (Sinclair [75, p. 526]) *Every linear Jordan homomorphism from a Banach algebra onto a semisimple Banach algebra is continuous.*

(b) (Civin and Yood [25, Theorem 5.4]) *Every linear Jordan $*$-homomorphism from a C^*-algebra onto a dense subset of a C^*-algebra is continuous.*

We note that a ring homomorphism might *not* be continuous. Any discontinuous ring automorphism $\sigma\colon \mathbb{C} \to \mathbb{C}$ is a counterexample (see, for example, [85]). On the other hand, $*$-ring homomorphisms are continuous, as a consequence of the following result.

Proposition 2.7. (Tomforde [77, Theorem 3.6 and Proposition 3.9]) *Let $\Phi\colon A \to B$ be a unital $*$-ring homomorphism between unital C^*-algebras. Then $\|\Phi(a)\| \leq \|a\|$, for all a in A. Moreover, there exist projections p, q in B commuting with elements of $\Phi(A)$ such that $p+q = 1$, $\Phi_1 = p\Phi$ is a linear $*$-ring homomorphism, and $\Phi_2 = q\Phi$ is a conjugate linear $*$-ring homomorphism.*

To end this section we state a classical result.

Proposition 2.8. (Gardner [35, p. 395]; see also Sakai [73, Corollary 4.1.22]) *If $\pi\colon A \to B$ is an algebra isomorphism between C^*-algebras, then A, B are algebra $*$-isomorphic. Indeed, there is an invertible element u in A^{**} and an algebra $*$-isomorphism $\psi\colon A \to B$ such that $\pi(a) = \psi(uau^{-1}) \quad (a \in A)$.*

3. Linear disjointness preservers of operator algebras

We start with the abelian case. Recall that abelian C^*-algebras are $C_0(X)$, and all 5 disjointness mentioned in §1 coincide. The following summarizes results of Abramovich [1], Arendt [7], Jarosz [43], Font and Hernandez [32], and Jeang and Wong [44].

Theorem 3.1. *Let $\theta\colon C_0(X) \longrightarrow C_0(Y)$ be a linear map preserving zero products:*

$$fg = 0 \text{ on } X \quad \Longrightarrow \quad \theta(f)\theta(g) = 0 \text{ on } Y.$$

For each y in Y, we consider the functional $\delta_y \circ T\colon f \mapsto Tf(y)$ of $C_0(X)$. Let

$$\begin{cases} Y_c = \{y \in Y : \delta_y \circ T \text{ is continuous and nonzero}\}, \\ Y_d = \{y \in Y : \delta_y \circ T \text{ is discontinuous}\}, \\ Y_0 = \{y \in Y : \delta_y \circ T = 0\}. \end{cases}$$

There exist a continuous map $\varphi\colon Y_c \cup Y_d \longrightarrow X \cup \{\infty\}$ and a bounded nonvanishing continuous scalar function h on Y_c such that $Tf_{|Y_c} = h \cdot f \circ \varphi$, and $Tf_{|Y_0} = 0$. The exceptional set Y_d is open with $\varphi(Y_d)$ being finite. Moreover, θ is automatically bounded if θ is bijective.

If the map θ is bijective, we can say a little more.

Proposition 3.2. (Araujo, Beckenstein and Narici [5, Propositions 2 and 3]) *If $\theta\colon C_0(X) \to C_0(Y)$ is an additive bijective map preserving zero products in both directions, then the realcompactifications of X and Y are homeomorphic. Moreover, θ*

is real linear if and only if θ is continuous; in this case, there exist a homeomorphism $\sigma\colon Y\to X$, a partition $Y=Y_1\cup Y_2$ of clopen subsets of Y, and a bounded continuous scalar function h on Y away from zero such that

$$\theta(f)(y)=\begin{cases} h(y)\cdot f(\sigma(y)) & \text{on } Y_1,\\ h(y)\cdot \overline{f(\sigma(y))} & \text{on } Y_2\end{cases}\qquad (f\in C_0(X)).$$

If θ is linear then $Y=Y_1$, and $\theta(f)=h\cdot f\circ\sigma$ for all f in $C_0(X)$.

If θ is not bijective, then θ might not be continuous. In [43] there are given examples of discontinuous zero product preserving linear maps between algebras of continuous functions. For example, any free ultrafilter on $\mathbb{N}$ provides an unbounded zero product preserving linear functional of $C_0(X)$ if X contains infinitely many points. One can find in [14] a systematic study on unbounded zero product preserving linear functionals.

For the nonabelian case, one can date back to the works of Dye [26] on the orthomorphisms between the projection lattices of W^*-algebras, Uhlhorn [80] on the generalization of the Wigner Theorem on orthogonality preservers of Hilbert spaces, and Abramovich [1] on the representation theory of linear disjointness preservers between vector lattices. However, the first result about linear disjointness preservers between general C^*-algebras we find in the literature is due to Wolff [81]; see also Schweizer [74]. Here for a subset D of a C^*-algebra B, the commutant

$$D'=\{b\in B: bd=db \text{ for all } d\in D\}$$

consists of elements in B commuting with all members in D. Moreover, we identify the multiplier algebra $M(B)$ as a C^*-subalgebra of the enveloping W^*-algebra B^{**}, which is also thought of as the double dual of B; namely,

$$M(B)=\{x\in B^{**}: xB\subseteq B \text{ and } Bx\subseteq B\}.$$

Theorem 3.3. (Wolff [81, Theorem 2.3 and Corollary 2.4]) *Let $\theta\colon A\to B$ be a bounded linear map from a unital C^*-algebra A into a C^*-algebra B. Suppose θ preserves involutions, and preserves zero products in A_{sa}, that is,*

$$\theta(x^*)=\theta(x)^*\quad (x\in A),\quad \textit{and}$$
$$ab=0\implies \theta(a)\theta(b)=0\quad (a,b\in A_{sa}).$$

Let $C=\overline{\theta(1)\{\theta(1)\}'}$. Then $\theta(A)\subseteq C$ and there is a linear Jordan $*$-homomorphism $J\colon A\to M(C)$ such that $\theta(a)=\theta(1)J(a)\quad (a\in A)$. If, in addition, B is unital and $\theta(1)$ is invertible, then we can consider $J(\cdot)=\theta(1)^{-1}\theta(\cdot)=\theta(\cdot)\theta(1)^{-1}$ as a linear Jordan $*$-homomorphism from A into B.

Chebotar, Ke, Lee and Wong [23] and Wong [84] extend Theorem 3.3 to the case that θ needs not preserve involutions and A needs not be unital. We recall first the following technical result which allows us to consider just the unital case, which especially helps to improve results in [23].

Lemma 3.4. (Wong [84, Lemma 2.2]) *Let A, B be C^*-algebras, and $M(A)$ be the multiplier algebra of A. Let $\theta\colon A_{sa} \to B$ be a bounded linear map sending zero products in A_{sa} to zero products in B. Then its double dual map θ^{**} sends zero products in $M(A_{sa})$ to zero products in B^{**}.*

Using Lemma 3.4, we can restate [23, Lemma 4.5, Theorems 4.6 and 4.7], together with [84, Theorems 2.4 and 2.5], as follows.

Theorem 3.5. *Let $\theta\colon A \to B$ be a bounded linear map between C^*-algebras. Suppose θ sends zero products in A_{sa} to zero products in B. Then $\theta^{**}(1)$ commutes with all elements in $\theta(A)$. Moreover,*

$$\theta^{**}(1)\theta(a^2) = \theta(a)^2 \quad (a \in A).$$

(a) *If $\theta^{**}(1)$ is invertible, then $\theta(\cdot)\theta^{**}(1)^{-1}$ is a linear Jordan homomorphism from A into B^{**}.*
(b) *If $\theta^{**}(1)$ is normal with support projection $p = \lim_n(\theta^{**}(1)^*\theta^{**}(1))^{1/n}$, then there is a sequence of bounded linear Jordan homomorphism J_n from A into B^{**} such that $J_n(a)\theta^{**}(1)$ converges to $\theta(a)p$ strongly for all a in $\mathcal{A}$.*
(c) *If $\theta(A) = B$ then $\theta^{**}(1)$ is invertible, and $\theta(\cdot)\theta^{**}(1)^{-1}$ is a surjective linear Jordan homomorphism from A onto B.*
(d) *If $\theta(A) = B$ and θ preserves all zero products in A, then $\theta(\cdot)\theta^{**}(1)^{-1}$ is a surjective algebra homomorphism from A onto B.*

The case of double zero product preservers is also covered in Theorem 3.5 (except (d)), thanks to the commutativity preserving property of a Jordan homomorphism.

We note that the proof for Theorem 3.5(d) makes use of Theorem 2.2. On the other hand, [23, Example 4.8] tells us that Theorem 3.5(b) is optimal as we cannot always write $\theta = \theta^{**}(1)J$ for any linear Jordan homomorphism J from A into B^{**} if the normal element $\theta^{**}(1)$ is not invertible. The difficulty arises from the fact that a linear Jordan homomorphism between C^*-algebras can have arbitrarily large norm. In particular, the linear Jordan homomorphism J_n in Theorem 3.5(b) might not converge in any operator algebra topology.

If we consider non-self-adjoint orthogonality preservers, the case is a bit further different. Parts (a) and (b) of the following Theorem 3.6 are based on [74, §4]. We state it with a proof here for completeness.

Theorem 3.6. *Let $\theta : A \to B$ be a bounded linear map between C^*-algebras preserving range orthogonality. Then we have $\theta^{**}(1)^*\theta(a^*b) = \theta(a)^*\theta(b)$ $(a, b \in A)$.*

(a) *If $\theta^{**}(1)$ is invertible, then $\theta(\cdot)\theta^{**}(1)^{-1}$ is an algebra $*$-homomorphism from A into B^{**}.*

(b) *In general, there is an algebra $*$-homomorphism π from A into B^{**} such that $\theta(a) = \pi(a)\theta^{**}(1)$ $(a \in A)$.*

(c) *Assume θ is bijective. If its inverse θ^{-1} also preserves range orthogonality or $\theta^{**}(1)$ is normal, then $\theta^{**}(1)$ is invertible and $\theta(\cdot)\theta^{**}(1)^{-1}$ is an algebra $*$-isomorphism from A onto B.*

Proof. Let a be a self-adjoint element in A with spectrum $X \subseteq [-\|a\|, \|a\|]$. Identify the abelian C^*-subalgebra of A^{**} generated by 1 and a with $C(X)$. As in Lemma 3.4, we can assume that θ^{**} is range orthogonality preserving on $C(X)$. Let

$$-\|a\| - 1 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \alpha_n = \|a\| + 1,$$

and $X = \bigcup_k X_k$ be the partition of X with $X_k = X \cap [\alpha_{k-1}, \alpha_k)$. In particular, $1 = \sum_k 1_{X_k}$, where 1_{X_k} is the indicator function of X_k for $k = 1, \ldots, n$. Note that some X_k might be empty, and thus $1_{X_k} = 0$ in these cases.

For distinct j, k, we can find two sequences of continuous non-negative functions $\{f_n\}_n$ and $\{g_n\}_n$ from $C(X) \cap A$ such that $f_m g_{m+p} = 0$ for all $m, p = 1, 2, \ldots$, $f_m \to 1_{X_j}$, and $g_m \to 1_{X_k}$ pointwise on X, and hence in the weak* topology of A^{**}. By the weak* continuity of θ^{**}, we see that

$$\theta^{**}(1_{X_k})^*\theta^{**}(f_m) = \lim_{p\to\infty} \theta^{**}(g_{m+p})^*\theta^{**}(f_m) = 0, \quad \forall\, m = 1, 2, \ldots.$$

Thus $\theta^{**}(1_{X_k})^*\theta^{**}(1_{X_j}) = \lim_{m\to\infty} \theta^{**}(1_{X_k})^*\theta^{**}(f_m) = 0$, whenever j, k are distinct.

Let $p = 1_{X_j}$ and $q = 1_{X_k}$ for distinct j, k. Observe that $p(qb) = q(pb) = 0$ for any b in A. Arguing with $\{f_n\}_n$ and $\{g_n\}_n$ as above, we have

$$\theta^{**}(p)^*\theta^{**}(qb) = \theta^{**}(q)^*\theta^{**}(pb) = 0.$$

It follows $\theta^{**}(p+q)^*\theta^{**}(qb) = \theta^{**}(q)^*\theta^{**}(qb) = \theta^{**}(q)^*\theta^{**}((p+q)b)$. Summing up with $p = 1_{X_j}$ for $j \neq k$, we have

$$\theta^{**}(1)^*\theta^{**}(qb) = \theta^{**}(q)^*\theta(b). \tag{3.1}$$

Note that a can be approximated in norm by step functions in the form of $\sum_j \lambda_j 1_{X_j}$. Thus, $\theta(a)$ can be approximated in norm by $\sum_j \lambda_j \theta^{**}(1_{X_j})$. It follows from (3.1) that $\theta^{**}(1)^*\theta(ab) = \theta(a)^*\theta(b)$, for all a, b in A with $a = a^*$. In general, we have

$$\theta^{**}(1)^*\theta(a^*b) = \theta(a)^*\theta(b) \quad (a, b \in A). \tag{3.2}$$

(a) If $\theta^{**}(1)$ is invertible, define $\pi\colon A \to B^{**}$ by $\pi(a) = \theta(a)\theta^{**}(1)^{-1}$. It follows from (3.2) that

$$\begin{aligned}\pi(a^*b) &= \theta(a^*b)\theta^{**}(1)^{-1} = (\theta^{**}(1)^*)^{-1}\left(\theta^{**}(1)^*\theta(a^*b)\right)\theta^{**}(1)^{-1} \\ &= (\theta^{**}(1)^*)^{-1}\left(\theta(a)^*\theta(b)\right)\theta^{**}(1)^{-1} = \left(\theta(a)\theta^{**}(1)^{-1}\right)^*\left(\theta(b)\theta^{**}(1)^{-1}\right) \\ &= \pi(a)^*\pi(b) \quad (a, b \in A).\end{aligned}$$

Therefore, $\pi = \theta(\cdot)\theta^{**}(1)^{-1}$ is an algebra $*$-homomorphism from A into B^{**}.

(b) We represent B through the GNS construction as a C^*-subalgebra of $\mathcal{B}(H)$ of bounded linear operators on a Hilbert space H. Let H_0 be the closed subspace of H generated by $\theta(A)H$. Define $\pi\colon A \to \mathcal{B}(H)$ by

$$\pi(a)\Big(\sum_i \theta(x_i)\xi_i\Big) = \sum_i \theta(ax_i)\xi_i$$

for any finite sum in $\theta(A)H$, and extend $\pi(a)$ to the whole H by setting $\pi(a) = 0$ on the orthogonal complement of H_0 in H. It can be shown that $\pi(a)$ is a bounded linear operator on H with $\|\pi(a)\| \le \|a\|$. Moreover, $\pi(a)$ belongs to the weak operator closure of B in $\mathcal{B}(H)$. Hence we can say that $\pi(a) \in B^{**}$.

Let a, b, x be in A and ξ in H; we have

$$\pi(ab)\theta(x)\xi = \theta(abx)\xi = \pi(a)\theta(bx)\xi = \pi(a)\pi(b)\theta(x)\xi.$$

This amounts to saying that π is multiplicative. Let $a_\lambda \uparrow 1$ be an approximate identity in A. We see that $\theta(a) = \lim_\lambda \theta(aa_\lambda) = \lim_\lambda \pi(a)\theta(a_\lambda) = \pi(a)\theta^{**}(1)$.

Finally, observe that for any x, y, z in A we can choose a positive scalar λ such that $\lambda - x$ is invertible in A^{**}. Applying (3.2) to the orthogonality preserving bounded linear map $u \mapsto \theta(u(\lambda - x))$ for $a = y$ and $b = z(\lambda - x)^{-1}$, we get

$$\theta(\lambda - x)^*\theta(y^*z) = \theta(y(\lambda - x))^*\theta(z).$$

Applying (3.2) to θ with $a = x$ and $b = y$, we get $\theta(x)^*\theta(y^*z) = \theta(yx)^*\theta(z)$ $(x, y, z \in A)$. Now observe

$$\begin{aligned}\langle\theta(x)\xi, \pi(a^*)\theta(y)\eta\rangle &= \langle\xi, \theta(x)^*\theta(a^*y)\eta\rangle = \langle\xi, \theta(ax)^*\theta(y)\eta\rangle \\ &= \langle\theta(ax)\xi, \theta(y)\eta\rangle = \langle\pi(a)\theta(x)\xi, \theta(y)\eta\rangle \\ &= \langle\theta(x)\xi, \pi(a)^*\theta(y)\eta\rangle \quad (a, x, y \in A,\ \xi, \eta \in H).\end{aligned}$$

This says $\pi(a^*) = \pi(a)^*$ $\quad (a \in A)$. Hence $\pi\colon A \to B^{**}$ is the $*$-homomorphism we are looking for.

(c) Suppose first that $\psi = \theta^{-1}$ also preserves range orthogonality. Write $1_A, 1_B$ for the identities of A^{**}, B^{**}, respectively. Let $h = \theta^{**}(1_A)$ and $k = \psi^{**}(1_B)$. By (b) we can write $\theta(\cdot) = \pi_\theta(\cdot)h$ and $\psi(\cdot) = \pi_\psi(\cdot)k$ for a $*$-homomorphism π_θ from A into B^{**} and a $*$-homomorphisms π_ψ from B into A^{**}.

For any self-adjoint b in B, by the surjectivity of θ, we have an a in A such that $b = \theta(a) = \pi_\theta(a)h$. Consequently,

$$\pi_\theta^{**}(1_A)b = \pi_\theta^{**}(1_A)\pi_\theta(a)h = \pi_\theta(a)h = b = b^* = b\pi_\theta^{**}(1_A)^* = b\pi_\theta^{**}(1_A).$$

Therefore, the projection $\pi_\theta^{**}(1_A) = 1_B$. Similarly, $\pi_\psi^{**}(1_B) = 1_A$. Observe

$$b = \theta(\psi(b)) = \pi_\theta(\psi(b))h = \pi_\theta^{**}(\pi_\psi(b))\pi_\theta^{**}(k)h \quad (b \in B).$$

Thus by putting $b = 1_B$ we have $1_B = \pi_\theta^{**}(k)h$, and hence $\pi_\theta^{**}(\pi_\psi^{**}(b)) = b$ $(b \in B^{**})$. By symmetry, we have $1_A = \pi_\psi^{**}(h)k$, and $\pi_\psi^{**}(\pi_\theta^{**}(a)) = a \quad (a \in A^{**})$. This gives again

$$\begin{aligned} 1_A &= \pi_\psi^{**}(\pi_\theta^{**}(k))\pi_\psi^{**}(h) = k\pi_\psi^{**}(h), \\ 1_B &= \pi_\theta^{**}(\pi_\psi^{**}(h))\pi_\theta^{**}(k) = h\pi_\psi^{**}(k). \end{aligned}$$

So both h, k are invertible.

By (3.2), the surjectivity of θ and the Cohen factorization theorem we have

$$h^*B = h^*\theta(A) = B^2 = B = B^* = \theta(A)^* = h^*\pi_\theta(B)$$

and thus $B = \pi_\theta(B)$ since h is invertible. On the other hand, if $\pi_\theta(a) = \theta(a)h^{-1} = 0$ then $\theta(a) = 0$, which in turn forces $a = 0$. Therefore, π_θ is a $*$-isomorphism from A onto B.

Finally, we suppose $h = \theta^{**}(1)$ is normal. We claim that θ^{-1} preserves range orthogonality and thus the assertion follows from the above arguments. Let $q = \lim_{n\to\infty}(h^*h)^{1/n} = \lim_{n\to\infty}(hh^*)^{1/n}$ be the support projection of h. By (3.2), we see that $q\theta(a)^*\theta(a) = qh^*\theta(a^*a) = h^*\theta(a^*a) = \theta(a)^*\theta(a)$ $(a \in A)$. Due to $\theta(A) = B$, we conclude that $q = 1_B$. Assume that $\theta(a)^*\theta(b) = 0$ for some a, b in A. By (3.2) we have $h^*\theta(a^*b) = 0$. Hence $\theta(a^*b) = q\theta(a^*b) = \lim_{n\to\infty}(hh^*)^{1/n}\theta(a^*b) = 0$. Since θ is injective, we have $a^*b = 0$. Therefore, θ^{-1} is range orthogonality preserving as asserted. ■

Remark 3.7. In Theorems 3.3 and 3.5, $\theta^{**}(1)$ commutes with all members in $\theta(A)$. But in Theorem 3.6, $\theta^{**}(1)$ does not necessarily commute with $\theta(A)$. For a counterexample, let h be an arbitrary invertible bounded linear operator on a Hilbert space H, and consider the bijective range orthogonality preserving bounded linear map $\theta\colon \mathcal{K}(H) \to \mathcal{K}(H)$ of compact operators, defined by $\theta(a) = ah$ ($a \in \mathcal{K}(H)$). In this case, $\theta^{**}(1) = h$ does not commute with $\mathcal{K}(H)$, unless it is a scalar multiple of the identity.

We note that in both Theorems 3.3 and 3.5, we have indeed two conditions stated as

$$ab = 0 \quad \Longrightarrow \quad \theta(a)\theta(b) = 0 \;\; \textit{and} \;\; \theta(b)\theta(a) = 0 \quad (a, b \in A_{sa}).$$

Using both conditions, we can show that $\theta^{**}(1)$ commutes with all members in $\theta(A)$. But for the range orthogonality preserver θ in Theorem 3.6, we have only one condition, namely, $a^*b = 0 \implies \theta(a)^*\theta(b) = 0$ $(a, b \in A)$.

In [83], it is shown that every bounded linear double orthogonality preserver θ between C^*-algebras preserves the triple products $\{a, b, c\} := ab^*c + cb^*a$ whenever $\theta^{**}(1)$ is a partial isometry. Without any assumption on $\theta^{**}(1)$, a complete description of such maps is given by M. Burgos, F. J. Fernández-Polo, J. J. Garcés, J. Martínez Moreno and A. M. Peralta in [19, 20] in terms of JB*-algebras and JB*-triples. We quote below a C^*-algebra version.

Theorem 3.8. (Lau and Wong [52, Theorem 2.2]) *Let $\theta\colon A \to B$ be a bounded linear map between two C^*-algebras A and B with dense range. Then θ is double orthogonality preserving on positive elements if and only if $\theta = \theta^{**}(1)J$, where $\theta^{**}(1)$ is a multiplier of B with $\theta^{**}(1)^*\theta^{**}(1) = \theta^{**}(1)\theta^{**}(1)^*$ being central in $M(B)$, and $J\colon A \to M(B)$ is a linear Jordan $*$-homomorphism, such that $\theta^{**}(1)J(A) \subseteq B$. Moreover, $\theta^{**}(1)$ is invertible if and only if θ is surjective. In this case, $J(A) = B$.*

The following example borrowed from [52, Example 2.3(b)] shows that $\theta^{**}(1)$ might not be central. Consider the bijective linear map $\theta\colon M_2 \to M_2$ between 2×2 complex matrices defined by

$$\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

Then θ is double orthogonality preserving with $\theta(1)^*\theta(1) = \theta(1)\theta(1)^* = 1$. But $\theta(1)$ is not central.

The unbounded case is much more difficult, as we do not have access to any functional calculus tools. Our ultimate goal is to verify

Conjecture 3.9. Every bijective disjointness linear preserver between C^*-algebras is automatically continuous, and thus arises from a linear Jordan ($*$-)isomorphism.

Let us start with the beautiful result of Araujo and Jarosz [6]. Recall that a standard operator algebra A on a locally convex space E is the one containing all continuous finite rank operators on E. If θ is continuous, then θ can be just assumed to be surjective and the assertion follows from Chebotar, Ke, Lee and Wong [23].

Proposition 3.10. (Araujo and Jarosz [6, Theorem 1]; see also Leung and Wong [58, Proposition 3.1]) *Let A, B be standard operator algebras on locally convex spaces E, F, respectively. Let $\theta\colon A \to B$ be a bijective linear map such that both θ, θ^{-1} preserve zero products. Then there exist a scalar $\lambda \neq 0$ and an invertible bi-weak-weak continuous operator $S\colon E \to F$ such that $\theta(a) = \lambda S a S^{-1}$ $(a \in A)$.*

As pointed out by Fell in [29, p. 243] (see also [72, §5.1]), a CCR C^*-algebra A has Hausdorff spectrum if and only if we can represent it as a continuous field $C_0(X, \{\mathcal{K}(H_x)\}_{x\in X}, \mathcal{A})$ of C^*-algebras $\mathcal{K}(H_x)$ of compact operators over a locally compact Hausdorff base space X, with a continuous structure $\mathcal{A}$ consisting of continuous operator fields vanishing at infinity. In this way, we can identify A with $\mathcal{A}$, as consisting of continuous fields of compact operators on X. Combining the technique developed in the abelian case and the standard operator algebra case, we have the following

Proposition 3.11. *Let $\theta\colon A \to B$ be a bijective linear map between C^*-algebras.*

(1) (Leung and Wong [58, Theorem 3.3]) *Let A and B be CCR C^*-algebras with Hausdorff spectrum. Suppose*

$$ab = 0 \text{ in } A \quad \text{if and only if} \quad \theta(a)\theta(b) = 0 \text{ in } B.$$

Then θ is automatically bounded. More precisely, $\theta(\cdot) = \pi(\cdot)\theta^{**}(1)$, and $\theta^{**}(1)$ is an invertible central multiplier of B and π is an algebra isomorphism from A onto B.

(2) (Tsai [78, Theorem 3]) *Let A, B be two C^*-algebras with continuous traces. Suppose*

$$a^*b = 0 \text{ in } A \quad \text{if and only if} \quad \theta(a)^*\theta(b) = 0 \text{ in } B.$$

Then θ is automatically bounded. More precisely, $\theta(\cdot) = \pi(\cdot)\theta^{**}(1)$, and $\theta^{**}(1)$ is an invertible right multiplier of B and π is an algebra $*$-isomorphism from A onto B.

The above results apply especially to finite type I W^*-algebras. On the other hand, every W^*-algebra without a finite type I summand is generated by its projections as a rational linear space by, for example, a result of Goldstein and Paszkiewicz [37, Theorem 3(3)]. With pure algebraic arguments, we can show that bijective linear maps between W^*-algebras without finite type I summands, preserving zero products or orthogonality in both directions, are also automatically continuous. By writing a general W^*-algebra as the direct sum of its finite type I part and its non finite type I part, we obtain

Theorem 3.12. (Leung, Tsai and Wong [57, Theorem 1.3]) *Let M, N be two W^*-algebras. Let $\theta\colon M \to N$ be a bijective linear map. Then M, N are isomorphic as W^*-algebras, provided that any one of the following conditions holds.*

(1) *θ preserves zero products in both directions, that is,*

$$ab = 0 \text{ in } M \quad \text{if and only if} \quad \theta(a)\theta(b) = 0 \text{ in } N.$$

In this case, $\theta(1)$ is a central invertible element and $\theta(\cdot)\theta(1)^{-1}$ is an algebra isomorphism.

(2) *θ preserves range orthogonality in both directions, that is,*

$$a^*b = 0 \text{ in } M \quad \text{if and only if} \quad \theta(a)^*\theta(b) = 0 \text{ in } N.$$

In this case, $\theta(1)$ is an invertible element and $\theta(\cdot)\theta(1)^{-1}$ is an algebra $*$-isomorphism.

Utilizing a similar type decomposition idea, in [21] the following theorem is proved independently.

Theorem 3.13. (Burgos, Garcés and Peralta [21, Theorem 19]) *Every linear surjection between W^*-algebras preserving double orthogonality in both directions is automatically continuous. Consequently, two W^*-algebras are isomorphic as JB^*-triples if and only if they carry equivalent linear and double orthogonality structures.*

We note that a double orthogonality preserving linear surjection $\theta\colon M \to N$ between W^*-algebras is automatically injective. As θ being bounded by Theorem 3.13, $\theta(1)$ is invertible, $\theta(1)^*\theta(1) = \theta(1)\theta(1)^*$ is central, and $\theta(1)^{-1}\theta$ is a linear Jordan $*$-isomorphism from M onto N, via [21, Corollary 6] or Theorem 3.8.

As a new attempt to verifying Conjecture 3.9, we have extended Theorem 3.12 recently to the case of AW^*-algebras. Recall that an AW^*-algebra A is a unital C^*-algebra such that the right annihilator of every subset S of A is a (norm closed) left ideal arising from some projection p in A, that is,

$$S_r^{\perp} = \{a \in A : aS = 0\} = Ap.$$

It is plain that every W^*-algebra is an AW^*-algebra.

Theorem 3.14. (Liu, Chou, Liao and Wong [63]) *Let $\theta\colon A \to B$ be a bijective additive map between AW^*-algebras.*

(a) *If θ preserves zero products in both directions, then A, B are ring isomorphic. If θ is also assumed linear, then $\theta(1)$ is a central invertible element, and $\theta(\cdot)\theta(1)^{-1}$ is an algebra isomorphism from A onto B.*

(b) *If θ preserves range orthogonality in both directions, then A, B are $*$-ring isomorphic. If θ is also assumed linear, then $\theta(1)$ is an invertible element, and $\theta(\cdot)\theta(1)^{-1}$ is an algebra $*$-isomorphism from A onto B.*

A sketch of the proof. Let z be the central projection in A such that the ideal $A_1 = (1 - z)A$ is abelian, and the ideal $A_2 = zA$ contains no abelian summand. Similarly, we write $B = B_1 + B_2$.

By a result of Berberian [12, Theorem 1], A_2 is a ring generated by its projections. Applying [23, Theorem 2.6] and following the proof of [57, Theorem 1.3], we see that $\theta(A_2)$ is a two-sided annihilator ideal of B. Inherited from A_2, the image $\theta(A_2)$ contains no abelian summand. We thus have $\theta(A_2) \subseteq B_2$. Arguing with θ^{-1} in a similar way, we see that $\theta^{-1}(B_2) \subseteq A_2$, and thus $\theta(A_2) = B_2$. This also forces that $\theta(A_1) = B_1$.

Then we can see that $\theta(a) = \pi_2(a)\theta(z)$, where π_2 is a ring isomorphism in case (a), or a $*$-ring isomorphism in case (b), from $A_2 = Az$ onto B_2. On the other hand, Proposition 3.2 tells us that the abelian W^*-algebras A_1 and B_1 have homeomorphic spectrum, and thus they are algebra $*$-isomorphic. Therefore, A and B are ring or $*$-ring isomorphic in case (a) or (b).

If θ is linear, we see that the disjointness preserver θ induces an algebra or an algebra $*$-isomorphism π_1 between the abelian C^*-algebras $A_1 = A(1 - z)$ and B_1 such that $\theta(a) = \pi_1(a)\theta(1 - z)$. Meanwhile, π_2 is also linear. Consequently, $\pi = \pi_1 + \pi_2 = \theta(\cdot)\theta(1)^{-1}$ is an algebra or an algebra $*$-isomorphism from A onto B. ∎

It is expected that we would have a result for double orthogonality preservers of AW^*-algebras as those stated in Theorem 3.14. To end this section, we make a conjecture to respond to this concern.

Conjecture 3.15. Let $\theta\colon A \to B$ be a bijective additive map between AW^*-algebras preserving double orthogonality in both directions, that is,

$$a^*b = ba^* = 0 \text{ in } A \iff \theta(a)^*\theta(b) = \theta(b)\theta(a)^* = 0 \text{ in } B.$$

Then A, B are Jordan $*$-isomorphic. If θ is also assumed linear, then $\theta(1)$ is an invertible multiplier of B such that $\theta(1)^*\theta(1) = \theta(1)\theta(1)^*$ is central, and $\theta(\cdot)\theta(1)^{-1}$ is a linear Jordan $*$-isomorphism from A onto B.

4. Disjointness preservers of duals/preduals of C^*/W^*-algebras

The dual A^* of a C^*-algebra A or the predual M_* of a W^*-algebra M carries an orthogonality structure. Note that $A^* \cong M_*$ with $M = A^{**}$. Positive normal linear functionals $\varphi, \tau \in (M_*)_+$ are *orthogonal* if they have orthogonal support projections $s(\varphi) \perp s(\tau)$ in M, or equivalently, $\|\varphi \pm \tau\| = \|\varphi\| + \|\tau\|$. Normal linear functionals $\varphi, \tau \in M_*$ are *left* (resp. *right*) *orthogonal* if they have orthogonal left (resp. right) support projections

$$s_l(\varphi) = s(|\varphi|) \quad \text{and} \quad s_l(\tau) = s(|\tau|),$$
$$(\text{resp. } s_r(\varphi) = s(|\varphi^*|) \quad \text{and} \quad s_r(\tau) = s(|\tau^*|)).$$

Note that $s_l(\varphi)$ and $s_r(\varphi)$ are the smallest projections in M such that

$$\varphi(x) = \varphi(s_l(\varphi)x) = \varphi(xs_r(\varphi)) \quad (x \in M).$$

A linear map $\varphi \mapsto \varphi'$ between preduals of W^*-algebras is left (resp. right, left-to-right, right-to-left) orthogonality preserving if

$$\begin{aligned}
s_l(\varphi)s_l(\tau) = 0 &\implies s_l(\varphi')s_l(\tau') = 0, \\
s_r(\varphi)s_r(\tau) = 0 &\implies s_r(\varphi')s_r(\tau') = 0, \\
s_l(\varphi)s_l(\tau) = 0 &\implies s_r(\varphi')s_r(\tau') = 0, \\
s_r(\varphi)s_r(\tau) = 0 &\implies s_l(\varphi')s_l(\tau') = 0.
\end{aligned}$$

If the map is positive, that is, $\varphi \geq 0 \implies \varphi' \geq 0$, then all these orthogonality preserving properties coincide on self-adjoint elements with the *orthogonal decomposition preserving property* of Bunce and Wright [18], namely, $(\varphi_+)' = (\varphi')_+$.

Enriching results of Araki [3, Theorem 1.1] and Bunce and Wright [18, Corollary 2.9], by relaxing the assumption that Ψ preserves positivity, one has

Theorem 4.1. (Lau and Wong [52, Theorem 2.8]) *Let $\Psi\colon M_* \to N_*$ be a bounded bijective linear map between preduals of W^*-algebras with dual map $\Psi^*\colon N \to M$. The following are all equivalent.*

(1) *Ψ is right biorthogonality preserving, that is,*

$$s_r(\varphi)s_r(\tau) = 0 \iff s_r(\Psi(\varphi))s_r(\Psi(\tau)) = 0 \quad (\varphi, \tau \in M_*).$$

(2) *Ψ^* is domain biorthogonality preserving, that is,*

$$ab^* = 0 \iff \Psi^*(a)(\Psi^*(b))^* = 0 \quad (a, b \in M).$$

(3) *$\Psi^* = z\pi$, where $z = \Psi^*(1)$ is an invertible element in M and $\pi\colon N \to M$ is a weak* continuous algebra $*$-isomorphism.*

In case $\Psi = \theta^*$ is the dual map of a bounded bijective linear map $\theta\colon B \to A$ between C^*-algebras, the above statements are also equivalent to

(4) *θ is domain biorthogonality preserving.*

(5) *$\theta = z\pi$, where z is an invertible multiplier of A and $\pi\colon B \to A$ is an algebra $*$-isomorphism.*

We note that the classical Lamperti theorem of zero product preservers of the Banach space $L^1(X, \mathcal{A}, \mu)$ of integrable functions on a measure space (X, μ) is of different nature than the above theorem. In [51, Proof of Theorem 3.1], Lamperti studies the structure of bijective isometries of $L^p(X, \mathcal{A}, \mu)$ spaces, where $1 \le p < +\infty$. He verifies that all such maps are zero product preserving (when $p = 2$, it needs to assume also that the isometry sends positive functions to positive functions). Then he shows that every bijective zero product preserving bounded linear map $\Psi\colon L^p(X, \mathcal{A}, \mu) \to L^p(Y, \mathcal{B}, \nu)$ is given by a σ-set isomorphism $\Phi\colon \mathcal{A} \to \mathcal{B}$ between the measure σ-algebras in the sense that

$$\Psi(\mathbf{1}_E) = h\mathbf{1}_{\Phi(E)} \quad (E \in \mathcal{A}).$$

Here, h is a measurable function on Y and $\mathbf{1}_E$ is the indicator function of E.

Lessard [53, Theorem 5.4] shows that, when μ is a tight σ-finite measure on the Baire σ-algebra $\mathcal{A}$ of a topological space X, the σ-set isomorphism Φ can be realized as a point-to-point map $\varphi\colon Y \to X$ such that $\Phi(E) = \varphi^{-1}(E)$, and hence

$$\Psi(f)(y) = h(y)f(\varphi(y)) \quad (\text{for a.e. } y \in Y),$$

as in the continuous function space case. Moreover, φ is $\mathcal{B}^*$-measurable in the sense that $\varphi^{-1}(E)$ belongs to the completion $\mathcal{B}^*$ of the σ-algebra $\mathcal{B}$ for every Baire subset E of X.

When the measure μ is localizable, $L^\infty(X, \mathcal{A}, \mu) = L^1(X, \mathcal{A}, \mu)^*$, and thus $L^1(X, \mathcal{A}, \mu)$ can be considered as the predual of a W^*-algebra (see, for example, [73, Proposition 11.18.1]). What Lamperti [51] and Lessard [53] make use of is the pointwise disjointness of the function space $L^1(X, \mathcal{A}, \mu)$, but not the orthogonality structure arising from the W^*-algebra $L^\infty(X, \mathcal{A}, \mu)$. It is not very clear to us how to apply the orthogonality structure of $L^1(X, \mathcal{A}, \mu)$ to study the measure space $(X, \mathcal{A}, \mu)$ in general. However, if it is the Haar measure space carried by a locally compact group G, then we can say quite much.

Let G be a locally compact group with a fixed left Haar measure m. Let $C_b(G)$ be the space of all bounded continuous functions on G, and let $C_{00}(G)$ be the subspace of $C_b(G)$ consisting of all functions vanishing outside a compact set. Let $P(G)$ be the collection of all continuous positive definite functions on G. Recall that a continuous complex-valued function f on G is called *positive definite* if for any complex numbers $\lambda_1, \dots, \lambda_n$ and any $a_1, \dots, a_n$ in G, we have

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{\lambda_i}\lambda_j f(a_i^{-1}a_j) \geq 0.$$

Let $B(G)$ denote the linear span of $P(G)$, called the *Fourier–Stieltjes algebra* of G. Let $L^p(G)$, $1 \leq p < +\infty$, denote the Banach space of p-integrable functions with respect to m, and let $L^\infty(G)$ be the space of essentially bounded measurable functions on G. Then G is *amenable* if there exists a positive linear functional ϕ of norm one on $C_b(G)$ satisfying $\phi(af) = \phi(f)$ for all f in $C_b(G)$ and a in G, where $(af)(t) = f(at)$ for all t in G. The class of amenable groups includes all locally compact abelian groups and all compact groups.

For a in G, let $\rho(a) \in B(L^2(G))$, the space of bounded linear operators on $L^2(G)$, defined by $\rho(a)h(x) = h(a^{-1}x)$. Let $VN(G)$ denote the von Neumann algebra in $B(L^2(G))$ generated by $\{\rho(a) : a \in G\}$. The *Fourier algebra* $A(G)$ of G is the closed linear span of $P(G) \cap C_{00}(G)$ in $B(G)$. Then $A(G)$ can be identified with the unique predual of $VN(G)$ with

$$\langle f, \rho(a)\rangle = f(a) \quad (a \in G, f \in A(G)); \tag{4.1}$$

also $f \in A(G)$ if and only if there are ζ, η in $L^2(G)$ such that

$$f(x) = \langle \rho(x)\zeta, \eta\rangle_{L^2(G)} \quad (x \in G).$$

When G is abelian, $A(G)$ is the image of the Fourier transform of $L^1(\widehat{G})$, where $\widehat{G}$ is the dual group of G. The Fourier algebra $A(G)$ is a closed ideal in $B(G) \subseteq C_b(G)$ with spectrum G given by (4.1); $A(G)$ has a bounded approximate identity if and only if G is amenable. Furthermore, $A(G)$ has an identity if and only if G is compact. See [27] or [47] for a full exposition.

In [82], Walter shows that if $\Psi\colon A(G_1) \to A(G_2)$ is an isometry as well as an algebra isomorphism then G_1 and G_2 are topologically and algebraically isomorphic. This suggests that we need to consider two structures of the Fourier algebra $A(G)$ to determine the topological group structure of G. A Fourier algebra $A(G)$ carries indeed two disjointness structures. One is the zero product structure inherited from $C_b(G)$, and the other is the orthogonality structure of normal functionals as $A(G) = VN(G)_*$.

Font [30, 31] shows that two locally compact amenable groups G_1 and G_2 are homeomorphic if there exists a zero product preserving linear bijection

$\Psi\colon A(G_1) \to A(G_2)$. Monfared [65] shows that the amenability condition can be dropped if Ψ is assumed bounded. Arendt and De Cannière [8, 9] show that G_1 and G_2 are topologically and algebraically isomorphic if Ψ preserves both the pointwise order of functions on G_1, G_2 and the positive definite order of normal linear functionals of $VN(G_1)_*, VN(G_2)_*$. It is then natural to see what happens if Ψ preserves disjointness in 'two senses'.

In [52], Theorem 4.1 is applied to study double disjointness preserving linear maps between Fourier algebras. Observe that if G is a locally compact abelian group with dual group $\widehat{G}$, then the orthogonality of f, g in $A(G)$ is equivalent to their inverse Fourier transforms $\widehat{f}, \widehat{g}$ in $L^1(\widehat{G})$ having zero product; namely,

$$f \perp g \text{ in } A(G) \quad \Longleftrightarrow \quad \widehat{f}\,\widehat{g} = 0 \text{ in } L^1(\widehat{G}).$$

Proposition 4.2. (Lau and Wong [52, Corollary 4.7]) *Let G_1 and G_2 be two locally compact abelian groups with dual groups $\widehat{G_1}$ and $\widehat{G_2}$, respectively. Let $\Psi\colon A(G_1) \to A(G_2)$ be a bijective linear map, and let $\widehat{\Psi}\colon L^1(\widehat{G_1}) \to L^1(\widehat{G_2})$ be the associated bijective linear map defined through Fourier transforms. Suppose that both Ψ and $\widehat{\Psi}$ preserves zero products, that is,*

$$\begin{aligned} fg = 0 \text{ in } A(G_1) \quad &\Longrightarrow \quad \Psi(f)\Psi(g) = 0 \text{ in } A(G_2), \text{ and} \\ \widehat{f}\,\widehat{g} = 0 \text{ in } L^1(\widehat{G_1}) \quad &\Longrightarrow \quad \widehat{\Psi}(\widehat{f})\,\widehat{\Psi}(\widehat{g}) = 0 \text{ in } L^1(\widehat{G_2}). \end{aligned}$$

Then G_1 and G_2 are isomorphic as topological groups. More precisely, there are a nonzero complex number α, a character β of G_2, an element w in G_2, and a topological group isomorphism $\sigma\colon G_2 \to G_1$ such that

$$\Psi(f)(s) = \alpha\beta(s) f(w\sigma(s)) \quad (f \in A(G_1), s \in G_2).$$

When G is a compact abelian group, we can perform convolutions in the Fourier algebra $A(G)$, as it is contained in $L^1(G)$. Applying Fourier and inverse Fourier transforms we see that the orthogonality of f, g in $A(G)$ is equivalent to that they have zero convolution product; namely,

$$f \perp g \quad \Longleftrightarrow \quad f * g = 0.$$

Corollary 4.3. (Lau and Wong [52, Corollary 4.8]) *Let G_1, G_2 be compact abelian groups. Suppose a bijective linear map $\Psi\colon A(G_1) \to A(G_2)$ preserves both zero pointwise products and zero convolution products, i.e.,*

$$\begin{aligned} fg = 0 \quad &\Longrightarrow \quad \Psi(f)\Psi(g) = 0, \text{ and} \\ f * g = 0 \quad &\Longrightarrow \quad \Psi(f) * \Psi(g) = 0 \quad (f, g \in A(G_1)). \end{aligned}$$

Then G_1 and G_2 are isomorphic as topological groups, and the conclusions in Corollary 4.2 hold.

For general locally compact groups, one has

Theorem 4.4. (Lau and Wong [52, Theorem 4.1]) *Let G_1, G_2 be locally compact groups and $A(G_1), A(G_2)$ the associated Fourier algebras. Let $\Psi\colon A(G_1) \to A(G_2)$ be a linear bijection preserving pointwise disjointness. Then Ψ is orthogonality decomposition preserving (resp. left, right, left-to-right, or right-to-left biorthogonality preserving) if and only if*

(a) *there is a positive (resp. nonzero complex) number α,*
(b) *there is a character β of the group G_2,*
(c) *$w = e_1$, the identity element of G_1 (resp. there is a w in G_1), and*
(d) *there is a homeomorphic map $\sigma\colon G_2 \to G_1$, which is a group isomorphism or anti-isomorphism (resp. group isomorphism for left or right biorthogonality preservers, and group anti-isomorphism for left-to-right or right-to-left biorthogonality preservers),*

such that for all s in G_2 we have $\Psi(f)(s) = \alpha\beta(s)f(w\sigma(s))$ (resp. $\Psi(f)(s) = \alpha\beta(s)f(\sigma(s)w)$ for left or left-to-right biorthogonality preservers).

5. Disjointness preservers of Hilbert C^*-modules

A (right) *Hilbert C^*-module* or a *Hilbert A-module E*, over a C^*-algebra A, is a right A-module equipped with an A-valued inner product $\langle \cdot, \cdot \rangle\colon E \times E \to A$ such that

(1) $\langle x, ya \rangle = \langle x, y \rangle a$,
(2) $\langle x, y \rangle^* = \langle y, x \rangle$,
(3) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ exactly when $x = 0$,
(4) E is a Banach space with $\|x\| = \|\langle x, x \rangle\|^{1/2}$.

We call E a *full* Hilbert A-module if the closed two-sided ideal J_E generated by the inner products of elements of E is A. A linear map $\theta\colon E \to F$ between Hilbert A-modules is an *A-module homomorphism* or *A-linear* if $\theta(xa) = \theta(x)a$ $(a \in A, x \in E)$.

If Ω is a locally compact Hausdorff space and H is a Hilbert space, then the Banach space $C_0(\Omega, H)$ of H-valued continuous functions vanishing at infinity is a Hilbert $C_0(\Omega)$-module. In general, a Hilbert $C_0(\Omega)$-module $\mathcal{H}$ can be considered as the space of continuous sections in a Hilbert bundle $(\Omega, \{H_\omega\}_{\omega\in\Omega})$, or equivalently, the continuous structure of a continuous field $C_0(\Omega, \{H_\omega\}_{\omega\in\Omega}, \mathcal{H})$ of Hilbert spaces H_ω over the base space Ω. The $C_0(\Omega)$-inner product is given by

$$\langle f, g \rangle(\omega) = \langle f(\omega), g(\omega) \rangle_{H_\omega} \quad (f, g \in \mathcal{H}, \omega \in \Omega).$$

In [39], which indeed deals with the more general case of Banach bundles, Hsu and Wong show that any surjective isometry $\theta\colon C_0(X, \{H_x\}_{x\in X}, \mathcal{H}) \to C_0(Y, \{K_y\}_{y\in Y}, \mathcal{K})$ between two continuous fields of Hilbert spaces is given by a homeomorphism $\sigma\colon Y \to X$ of the base spaces and a field of unitaries $J_y\colon H_{\sigma(y)} \to K_y$ such that $\theta(f)(y) = J_y(f(\sigma(y)))$ $(f \in \mathcal{H},\, y \in Y)$. Consequently,

$$\langle \theta(f), \theta(g)\rangle(y) = \langle f, g\rangle(\sigma(y)) \quad (f, g \in \mathcal{H},\, y \in Y).$$

This says that surjective isometries between Hilbert C^*-modules over abelian C^*-algebras are unitaries. A similar conclusion can be made for 2-isometries between general Hilbert A-modules over a non-commutative C^*-algebra A. See [40] for details. Thus, the norm (and linear) structure determines the unitary structure in this situation.

Motivated by the Uhlhorn theorem asserting that orthogonality preservers of Hilbert spaces (of dimension at least three) arise from unitaries, we ask if the orthogonality structure of a Hilbert C^*-module determines its unitary structure. If $\theta\colon E \to F$ is a linear map preserving orthogonality:

$$\langle \theta(x), \theta(y)\rangle_A = 0 \text{ whenever } \langle x, y\rangle_A = 0,$$

we ask whether there exists a central multiplier u of A such that

$$\langle \theta(e), \theta(f)\rangle_A = u\, \langle e, f\rangle_A \quad (e, f \in E).$$

A linear map $\theta\colon E \to F$ between Hilbert A-modules is called *local* if

$$\theta(e)a = 0 \quad \text{whenever} \quad ea = 0 \qquad (e \in E, a \in A).$$

Readers should find the idea of local linear maps familiar. For example, local linear maps of the space of smooth functions defined on a manifold modeled on $\mathbb{R}^n$ are exactly the linear differential operators (see, for example, [66, 70]). See also [4, 48] for the vector-valued case, and [2] for the Banach $C^1[0,1]$-module setting. Every module map is local, but local linear maps, for example, linear differential operators, might not be module maps. Nevertheless, *bounded* local maps between Hilbert C^*-modules are module maps [54, Proposition A.1].

Conjecture 5.1. Let A be a C^*-algebra. Let E and F be Hilbert A-modules with E being full. If $\theta\colon E \to F$ is a linear local map preserving orthogonality, that is,

$$\langle x, y\rangle = 0 \quad \text{implies} \quad \langle \theta(x), \theta(y)\rangle = 0 \quad (x, y \in E),$$

then there is a central positive multiplier u of A such that

$$\langle \theta(x), \theta(y)\rangle = u\langle x, y\rangle \quad (x, y \in E).$$

Here is a counterexample if we do not assume locality. Let H be an infinite-dimensional Hilbert space, and $A = \mathcal{K}(H)$ the algebra of all compact operators on H. Suppose $\bar{H}$ is a vector space conjugate-linear isomorphic to H. Then $\bar{H}$ is a Hilbert A-module if we set

$$\langle \overline{\eta_1}, \overline{\eta_2} \rangle := \eta_1 \otimes \eta_2 \quad \text{and} \quad \overline{\eta_1} T := \overline{T^* \eta_1} \quad (\overline{\eta_1}, \overline{\eta_2} \in \bar{H},\, T \in A).$$

Let $\theta\colon \bar{H} \to \bar{H}$ be any unbounded bijective linear map. Since

$$\langle x, y \rangle = 0 \quad \text{if and only if} \quad x = 0 \text{ or } y = 0,$$

both θ and θ^{-1} preserve orthogonality.

In [54], the abelian case is completely solved.

Theorem 5.2. (Leung, Ng and Wong [54, Theorem 3.5]) *Let Ω be a locally compact Hausdorff space, and let $C_0(\Omega, \{H_\omega\}_{\omega\in\Omega}, \mathcal{H})$ and $C_0(\Omega, \{K_\omega\}_{\omega\in\Omega}, \mathcal{K})$ be two full Hilbert $C_0(\Omega)$-modules. Suppose that $\theta\colon \mathcal{H} \to \mathcal{K}$ is a local linear map preserving orthogonality. The following assertions hold.*

(1) *θ is a bounded $C_0(\Omega)$-module map.*

(2) *There is a bounded non-negative continuous function φ on Ω such that*

$$\langle \theta(e), \theta(g) \rangle = \varphi \cdot \langle e, g \rangle \quad (e, g \in \mathcal{H}).$$

(3) *There exist a strictly positive element ψ_0 in $C_b(\Omega)_+$ and a bounded $C_0(\Omega)$-module map $J\colon \mathcal{H} \to \mathcal{K}$ such that the fiber map $J_\omega\colon H_\omega \to K_\omega$ is an isometry for each ω in Ω and*

$$\theta(e)(\omega) = \psi_0(\omega) J(e)(\omega) = \psi_0(\omega) J_\omega(e) \quad (e \in \mathcal{H},\, \omega \in \Omega).$$

For the general case, Ilišević and Turnšek [41] verify Conjecture 5.1 when A is standard (that is, $\mathcal{K}(H) \subseteq A \subseteq \mathcal{B}(H)$) and θ is A-linear. In [55], positive answers to Conjecture 5.1 are given in the following cases.

i. A is a standard C^*-algebra.

ii. A is a properly infinite unital C^*-algebra of real rank zero.

iii. A is a W^*-algebra.

iv. A is a C^*-algebra of real rank zero and θ is an A-module map.

Assuming that θ is bounded and A-linear, Frank, Mishchenko and Pavlov [33] verify Conjecture 5.1. Independently, with a very different approach from theirs, the unbounded A-linear case is solved in [56].

Theorem 5.3. (Leung, Ng and Wong [56, Theorem 3.2]) *Let E, F be Hilbert C^*-modules over a C^*-algebra A. Suppose that $\theta\colon E \to F$ is a linear map (not assumed* *to be bounded). Then $\theta\colon E \to F$ is an orthogonality preserving A-module map if and only if there exists a central positive multiplier u of J_E such that $\langle\theta(x),\theta(y)\rangle = u\langle x,y\rangle$ $(x,y\in E)$. In this case, u is unique and θ is automatically bounded.*

6. Holomorphic disjointness preservers of C^*-algebras

Let E, F be (complex) Banach spaces, and n a positive integer. A map $P\colon E\to F$ is called a *bounded n-homogeneous polynomial* if there exists a bounded symmetric n-linear operator $L\colon E\times\cdots\times E\to F$ such that $P(x)=L(x,\dots,x)$ $(x\in E)$. Let $B_E(a;r)=\{x\in E : \|x-a\|<r\}$. A map $H\colon U\to F$ is called *holomorphic* on an open subset U of E if for each point a in U, there exist an open ball $B_E(a;r)$ in U and bounded n-homogeneous polynomials $P_n\colon E\to F$ such that

$$H(x)=\sum_{n=0}^{\infty}P_n(x-a),\quad \text{uniformly for } x\in B_E(a;r).$$

Here, P_0 is the constant function with value $H(a)$. After translation, we can assume $a=0$, and a holomorphic function $H\colon B_E(0;r)\to F$ has its Taylor series at zero:

$$H(x)=\sum_{n=0}^{\infty}P_n(x),\quad \text{uniformly for } x\in B_E(0;r).$$

The nth derivative P_n of a holomorphic function $H\colon B_E(0;r)\to F$ is given by the vector-valued integral

$$P_n(x)=\frac{1}{2\pi i}\int_{|\lambda|=1}\frac{H(\lambda x)}{\lambda^{n+1}}\,d\lambda \qquad (n=0,1,2,\dots).$$

When E, F are function spaces or Banach operator algebras, a nonlinear map $\Phi\colon E\to F$ is said to be

- *orthogonally additive* if
$$fg=gf=0 \implies \Phi(f+g)=\Phi(f)+\Phi(g)\quad (f,g\in E);$$
- *double orthogonality preserving* if
$$f^*g=gf^*=0 \implies \Phi(f)^*\Phi(g)=\Phi(g)\Phi(f)^*=0\quad (f,g\in E);$$
- *zero product preserving* if
$$fg=0 \implies \Phi(f)\Phi(g)=0\quad (f,g\in E).$$

A holomorphic map H is called

- *conformal* (at 0) if its derivative P_1 (at 0) is a bounded invertible linear operator.

In [76], Sundaresan characterizes the linearization of orthogonally additive n-homogeneous polynomials on L^p-spaces. Several authors have extended his results to, for example, $C(K)$-spaces [71], Banach lattices [11, 15], and C^*-algebras in [68]. The following are the main tools in studying polynomail/holomorphic maps of C^*-algebras. The abelian case is obtained in [71, Theorem 2.1]; see also [11, 22].

Theorem 6.1. ([20, 68]) *Let A be a C^*-algebra, F a complex Banach space, and $P\colon A \to F$ a bounded n-homogeneous polynomial. The following are equivalent.*

(1) *P is orthogonally additive on A.*

(2) *P is orthogonally additive on A_{sa}, that is,*

$$ab = 0 \implies P(a+b) = P(a) + P(b) \quad (a, b \in A_{sa}).$$

(3) *There exists a bounded linear operator $T\colon A \to F$ such that*

$$P(a) = T(a^n) \quad (a \in A).$$

One can see that if P is an orthogonally additive n-homogeneous polynomial, then P is double orthogonality preserving (resp. zero product preserving) on A_{sa} exactly when P is so on A_+. One can also see that the bounded linear operator T in Theorem 6.1 preserves disjointness whenever the polynomial P does. Under some assumptions, we have $P(x) = T(x^n) = hJ(x^n) = hJ(x)^n$ for a linear Jordan $(*$-)homomorphism J, as results in Section 3 state. If a holomorphic function $H = \sum_n P_n$ between C^*-algebras is orthogonally additive and disjointness preserving, then so is each polynomial P_n (see [34, Proposition 6] or [17, Lemma 2.1]); especially the constant term $P_0 = 0$. Thus, one would expect $H(x) = \sum_{n\geq 1} h_n J_n(x)^n$ for a sequence $\{J_n\}$ of linear Jordan $(*$-)homomorphisms. The following theorems say that all such J_n coincide, provided some mild extra conditions are assumed. However, it is not always the case, as counterexamples in [16, Example 3.9] show.

Theorem 6.2. (Garcés, Peralta, Puglisi and Ramírez [34, Theorems 16 and 18]) *Let A, B be C^*-algebras, and let $r > 0$. Let $H\colon B_A(0; r) \to B$ be a holomorphic map. Let H be orthogonally additive and double orthogonality preserving on self-adjoint elements. Suppose that either*

(1) *B is commutative, or*

(2) *B is unital and $H(B_A(0; r))$ contains an invertible element in B.*

Then there exist a sequence $\{h_n\}$ in B^{**} and linear Jordan $*$-homomorphisms J, J' from the multiplier algebra $M(A)$ of A into B^{**} such that

$$H(a) = \sum_{n\geq 1} h_n J(a)^n = \sum_{n\geq 1} J'(a)^n h_n$$

uniformly for all a in $B_A(0; r)$.

Theorem 6.3. (Bu, Hsu and Wong [16, Theorems 3.3 and 3.11]) *Let $r > 0$.*

(1) *Let $H\colon B_{C_0(X)}(0;r) \to C_0(Y)$ be an orthogonally additive and zero product preserving holomorphic function. Then there exist a sequence $\{h_n\}$ of bounded scalar functions on Y in which each h_n is continuous on its cozero set, which is open, and a map $\varphi\colon Y \to X$ such that*

$$H(f)(y) = \sum_{n\geq 1} h_n(y)(f(\varphi(y)))^n \quad (y \in Y),$$

uniformly for all f in $B_{C_0(X)}(0;r)$. Here, φ is continuous wherever any h_n is nonvanishing.

(2) *Let A, B be C^*-algebras, and $H\colon B_A(0;r) \to B$ be an orthogonally additive conformal holomorphic map. Suppose H is zero product preserving on positive (resp. all) elements in $B_A(0;r)$. Then there exist a sequence $\{h_n\}$ of central multipliers of B and a linear Jordan (resp. algebra) isomorphism $J\colon A \to B$ such that*

$$H(a) = \sum_{n\geq 1} h_n J(a)^n \quad (a \in B_A(0;r)).$$

As an example (see [16, Corollary 3.12]), let A and B be standard C^*-algebras on the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $H\colon B_A(0;r) \to B$ be an orthogonally additive conformal holomorphic map. Suppose H is zero product preserving on positive elements. Since a linear Jordan isomorphism between standard operator algebras is either an algebra isomorphism or an algebra anti-isomorphism, there exist a sequence $\{\lambda_n\}$ of scalars and an invertible operator $S\colon \mathcal{H}_2 \to \mathcal{H}_1$ such that either

$$H(x) = \sum_{n\geq 1} \lambda_n S^{-1} x^n S \quad \text{or} \quad H(x) = \sum_{n\geq 1} \lambda_n S^{-1} (x^t)^n S \quad (x \in B_A(0;r)).$$

Here, x^t is the transpose of the operator x with respect to a fixed basis of the underlying Hilbert space. If H is zero product preserving on all elements in $B_A(0;r)$, then exactly the first case holds. Indeed, H is defined through an analytic function $h(z) = \sum_{n\geq 1} \lambda_n z^n$ around zero, modulo a similarity and a transposition; namely, either $H(x) = S^{-1}h(x)S$ or $H(x) = S^{-1}h(x)^t S$.

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J.-H. Liu, School of Information Engineering, Sanming University, No. 25, Jing Dong Road, San Yuan District, Sanming City, Fujian 365004, China; *e-mail*: 18760259501@163.com

Ch.-Y. Chou, Department of Applied Mathematics, A330 Science and Engineering Building I, National Dong Hwa University, Hualien 97401, Taiwan; *e-mail*: choucy@mail.ndhu.edu.tw

Ch.-J. Liao, Department of Mathematics, Hong Kong Baptist University, Hong Kong; *e-mail*: cjliao@hkbu.edu.hk

N.-Ch. Wong, Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan; *e-mail*: wong@math.nsysu.edu.tw