On the Convergence Analysis of Modified Hybrid Steepest-Descent Methods with Variable Parameters for Variational Inequalities[1]

L. C. Zeng[2], N. C. Wong[3] and J. C. Yao[4]

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[2]Department of Mathematics, Shanghai Normal University, Shanghai 200234, China. This research was partially supported by the Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, China and the Dawn Program Foundation in Shanghai.

[3]Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan 804. This research was partially supported by grant from the National Science Council of Taiwan.

[4]Corresponding author. Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan 804. This research was partially supported by grant from the National Science Council of Taiwan.

Abstract. Assume that F is a nonlinear operator on a real Hilbert space H which is η-strongly monotone and κ-Lipschitzian on a nonempty closed convex subset C of H. Assume also that C is the intersection of the fixed point sets of a finite number of nonexpansive mappings on H. We construct an iterative algorithm with variable parameters which generates a sequence $\{x_n\}$ from an arbitrary initial point $x_0 \in H$. The sequence $\{x_n\}$ is shown to converge in norm to the unique solution u^* of the variational inequality

$$\langle F(u^*), v - u^* \rangle \geq 0, \quad \forall\, v \in C.$$

Applications to constrained generalized pseudoinverse are included.

Key Words: Iterative algorithms, modified hybrid steepest-descent methods with variable parameters, convergence, nonexpansive mappings, Hilbert space, constrained generalized pseudoinverses.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H and let $F : H \to H$ be a nonlinear operator. Then we consider the following variational inequality problem: find a point $u^* \in C$ such that

$$\mathrm{VI}(F, C) \qquad \langle F(u^*), v - u^* \rangle \geq 0 \ \ \forall v \in C.$$

Variational inequalities were initially studied by Stampacchia (cf. Ref, 1). These problems have been widely studied since they cover as diverse disciplines as partial differential equations, optimal control, optimization, mathematical programming, mechanics and finance. The reader is referred to Refs.1-5 and references therein.

It is well known that if F is a strongly monotone and Lipschitzian mapping on C then the VI(F, C) has a unique solution, see e.g., Ref. 6. We remark that not only the existence and uniqueness of solutions are important in the study of the VI(F, C), but also how to find a solution of the VI(F, C) is important. A great deal of effort has gone into this problem. See Refs. 2 and 7.

It is also well known that the VI(F, C) is equivalent to the following fixed-point equation:

$$u^* = P_C(u^* - \mu F(u^*)), \tag{1}$$

where P_C is the (nearest point) projection from H onto C, i.e.,

$$P_C x = \mathrm{argmin}_{y \in C} \|x - y\|, \quad \text{for } x \in H,$$

and $\mu > 0$ is an arbitrarily fixed constant. If F is a strongly monotone and Lipschitzian mapping on C and $\mu > 0$ is small enough, then the mapping determined by the right-hand side of (1) is a contraction. Hence the Banach Contraction Principle guarantees that the Picard iterates converge strongly to the unique solution of the VI(F, C). Such a method is called the projection method. It has been widely extended and developed to compute approximate solutions of various classes of variational inequalities and complementarity problems; see Zeng (Refs. 8-10). Unfortunately, the projection method involves the projection P_C which may not be easy to compute due to the complexity of convex set C.

On the other hand, in order to reduce the complexity probably caused by the projection P_C, Yamada (Ref. 11, see also Ref. 12) recently introduced a hybrid steepest-descent method for solving the VI(F, C). His idea is stated now. Let C be the fixed point set of a nonexpansive mapping $T : H \to H$; that is, $C = \{x \in H : Tx = x\}$. Recall that T is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \text{for } x, y \in H.$$

Let $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$ denote the fixed-point set of T. Let F be η-strongly monotone and κ-Lipschitzian on C. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$ and a sequence $\{\lambda_n\}$ of real numbers in (0,1) satisfying the conditions below:

(L1) $\lim\lambda_n = 0$,

(L2) $\sum \lambda_n = \infty$,

(L3) $\lim(\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$.

Starting with an arbitrary initial guess $u_0 \in H$, one can generate a sequence (u_n) by the following algorithm:

$$u_{n+1} := Tu_n - \lambda_{n+1}\mu F(Tu_n), \quad n \geq 0. \tag{2}$$

Then Yamada (Ref. 11) proved that $\{u_n\}$ converges strongly to the unique solution of the VI(F, C). An example of a sequence $\{\lambda_n\}$, which satisfies conditions (L1)-(L3), is given by $\lambda_n = 1/n^\sigma$, where $0 < \sigma < 1$. Note that condition (L3) was first used by Lions (Ref. 13). In the case when C is expressed as the intersection of fixed-point sets of N nonexpansive mappings $T_i : H \to H$ with $N \geq 1$ an integer, Yamada (Ref. 11) proposed another algorithm:

$$u_{n+1} := T_{[n+1]}u_n - \lambda_{n+1}\mu F(T_{[n+1]}u_n), \quad n \geq 0, \tag{3}$$

where $T_{[k]} := T_{k \mathrm{mod} N}$ for integer $k \geq 1$ with the mod function taking values in the set $\{1, 2, \cdots, N\}$ (i.e., if $k = jN + q$ for some integers $j \geq 0$ and $0 \leq q < N$, then $T_{[k]} = T_N$ if $q = 0$ and $T_{[k]} = T_q$ if $0 < q < N$), where $\mu \in (0, 2\eta/\kappa^2)$ and where the sequence $\{\lambda_n\}$ of parameters satisfies conditions (L1),(L2), and (L4),

(L4) $\sum |\lambda_n - \lambda_{n+N}|$ is convergent.

Under these conditions, Yamada (Ref. 11) proved the strong convergence of $\{u_n\}$ to the unique solution of the VI(F, C). Note that condition (L4) was first used by Bauschke (Ref. 14). In the special case of $N = 1$, (L4) was introduced by Wittmann (Ref. 15).

Recently, Xu and Kim (Ref. 16) continued the convergence study of the hybrid steepest-decent algorithms (2) and (3). Their major contribution is that the strong convergence of algorithms (2) and (3) holds with condition (L3) replaced by the condition

(L3)' $\lim \lambda_n/\lambda_{n+1} = 1$, or equivalently, $\lim(\lambda_n - \lambda_{n+1})/\lambda_{n+1} = 0$,

and with condition (L4) replaced by the condition

(L4)' $\lim \lambda_n/\lambda_{n+N} = 1$, or equivalently, $\lim(\lambda_n - \lambda_{n+N})/\lambda_{n+N} = 0$.

It is clear that condition (L3)' is strictly weaker than condition (L3), coupled with conditions (L1) and (L2). Moreover, (L3)' includes the important and natural choice $\{1/n\}$ for $\{\lambda_n\}$ while (L3) does not. It is easy to see that if $\lim_{n\to\infty} \lambda_n/\lambda_{n+N}$ exists then condition (L4) implies condition (L4)'. However, in general, conditions (L4) and (L4)' are not comparable; neither of them implies the other (see Ref. 17 for details). Furthermore, under conditions (L1), (L2), (L3)' and (L4)', they gave the applications of algorithms (2) and (3) to the constrained generalized pseudoinverses.

Motivated and inspired by Yamada's algorithms (2) and (3), we introduce and study the following modified hybrid steepest-descent algorithms (I) and (II) with variable parameters for computing approximate solutions of the VI(F, C):

Algorithm (I): Let $\{\lambda_n\} \subset (0, 1)$, and $\{\mu_n\} \subset (0, 2\eta/\kappa^2)$. Starting with an arbitrary initial guess $u_0 \in H$, one can generate a sequence $\{u_n\}$ by the following iterative scheme:

$$u_{n+1} := Tu_n - \lambda_{n+1}\mu_{n+1}F(Tu_n), \quad n \geq 0.$$

Algorithm (II): Let $\{\lambda_n\} \subset (0, 1)$, and $\{\mu_n\} \subset (0, 2\eta/\kappa^2)$. Starting with an arbitrary initial guess $u_0 \in H$, one can generate a sequence $\{u_n\}$ by the following iterative scheme:

$$u_{n+1} := T_{[n+1]}u_n - \lambda_{n+1}\mu_{n+1}F(T_{[n+1]}u_n), \quad n \geq 0.$$

Compared with algorithms (2) and (3) respectively, Algorithms (I) and (II) introduce the sequence $\{\mu_n\}$ of positive parameters so as to take into account possible inexact computation.

In this paper, we give two assumptions (A), (B):

(A) $\limsup_{n\to\infty}\langle Tu_n - u_{n+1}, Tu_n - u_n\rangle \leq 0$ for Algorithm (I).

(B) $\limsup_{n\to\infty}\langle T_{[n+N]}\cdots T_{[n+1]}u_n - u_{n+N}, T_{[n+N]}\cdots T_{[n+1]}u_n - u_n\rangle \leq 0$ for Algorithm (II).

Firstly, under (A), (L2) and conditions (d1), (d2):

(d1) $|\mu_n - \frac{\eta}{\kappa^2}| \leq \frac{\sqrt{\eta^2 - c\kappa^2}}{\kappa^2}$ for some $c \in (0, \eta^2/\kappa^2)$,

(d2) $\lim_{n\to\infty}(\mu_{n+1} - \frac{\lambda_n}{\lambda_{n+1}} \cdot \mu_n) = 0$,

we prove that the sequence $\{u_n\}$ generated by Algorithm (I) converges in norm to the unique solution of the VI(F, C). Secondly, under (B), (L2), (d1) and condition (d3):

(d3) $\lim_{n\to\infty}(\mu_{n+N} - \frac{\lambda_n}{\lambda_{n+N}} \cdot \mu_n) = 0$,

we prove that the sequence $\{u_n\}$ generated by Algorithm (II) converges strongly to the unique solution of the VI(F, C). Furthermore, we apply these two results to consider the constrained generalized pseudoinverses. Note that for $\mu \in (0, 2\eta/\kappa^2)$ whenever $\mu_n = \mu$ $\forall n \geq 1$, then the above condition (d1) holds. Indeed, since

$$\lim_{t\to 0^+}\frac{\eta - \sqrt{\eta^2 - t\kappa^2}}{\kappa^2} = 0 < \mu \quad \text{and} \quad \lim_{t\to 0^+}\frac{\eta + \sqrt{\eta^2 - t\kappa^2}}{\kappa^2} = \frac{2\eta}{\kappa^2} > \mu,$$

so there exist some $\delta_1, \delta_2 \in (0, \eta^2/\kappa^2)$ such that

$$\begin{aligned} \tfrac{\eta-\sqrt{\eta^2-t\kappa^2}}{\kappa^2} &< \mu, \quad \forall t \in (0, \delta_1), \\ \tfrac{\eta+\sqrt{\eta^2-t\kappa^2}}{\kappa^2} &> \mu, \quad \forall t \in (0, \delta_2). \end{aligned}$$

Hence, it is obvious that we can pick a number $c \in (0, \eta^2/\kappa^2)$ such that

$$\frac{\eta - \sqrt{\eta^2 - c\kappa^2}}{\kappa^2} < \mu < \frac{\eta + \sqrt{\eta^2 - c\kappa^2}}{\kappa^2},$$

that is,

$$|\mu - \frac{\eta}{\kappa^2}| < \frac{\sqrt{\eta^2 - c\kappa^2}}{\kappa^2}.$$

Moreover, obviously (L3)' implies that for $\mu \in (0, 2\eta/\kappa^2)$ whenever $\mu_n = \mu$, $\forall n \geq 1$, the above condition (d2) holds; (L4)' implies that for $\mu \in (0, 2\eta/\kappa^2)$ whenever $\mu_n = \mu$, $\forall n \geq 1$, the above condition (d3) holds. On the other hand, under (L1), (L2) and (L3)', Xu and Kim (Ref. 16) proved that $\{u_n\}$ is bounded and $\lim_{n\to\infty} \|u_n - Tu_n\| = 0$; see Steps 1 and 4 in the proof of their Theorem 3.1 (Ref. 16). Hence we derive

$$\limsup_{n\to\infty} \langle Tu_n - u_{n+1}, Tu_n - u_n \rangle = 0.$$

This shows that (L1), (L2) and (L3)' imply (A). Also, under (L1), (L2) and (L4)', Xu and Kim (Ref. 16) proved that $\{u_n\}$ is bounded and $\lim_{n\to\infty} \|u_n - T_{[n+N]} \cdots T_{[n+1]} u_n\| = 0$; see Steps 1 and 4 in the proof of their Theorem 3.2 (Ref. 16). Thus, we obtain

$$\limsup_{n\to\infty} \langle T_{[n+N]} \cdots T_{[n+1]} u_n - u_{n+N}, T_{[n+N]} \cdots T_{[n+1]} u_n - u_n \rangle = 0.$$

This shows that (L1), (L2) and (L4)' imply (B). In conclusion, our results improve, extend and unify the corresponding ones in Xu and Kim (Ref. 16).

2. Preliminaries

The following lemmas will be used for the proofs of our main results in Section 3.

Lemma 2.1 (Ref. 16). Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the conditions: $s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n,\ \forall\ n \geq 0$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that

(i) $\{\alpha_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$, or equivalently,

$\prod_{n=0}^{\infty}(1-\alpha_n) := \lim_{n\to\infty} \prod_{k=0}^{n}(1-\alpha_k) = 0$;

(ii) $\text{limsup}_{n\to\infty}\beta_n \leq 0$, or

(ii)' $\sum_n \alpha_n\beta_n$ is convergent.

Then, $\lim_{n\to\infty}s_n = 0$.

Lemma 2.2 (Ref. 18). Demiclosedness Principle. Assume that T is a nonexpansive self-mapping of a closed convex subset C of a Hilbert space H. If T has a fixed point, then $I-T$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I-T)x_n\}$ strongly converges to some y, it follows that $(I-T)x = y$. Here I is the identity operator of H.

The following lemma is an immediate consequence of an inner product.

Lemma 2.3. In a real Hilbert space H, there holds the inequality:

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in H.$$

3. Modified Hybrid Steepest-Descent Algorithms with Variable Parameters

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $F : H \to H$ be an operator such that, for some constants $\kappa, \eta > 0$, F is κ-Lipschitzian and η-strongly monotone on C; that is, F satisfies the following conditions, respectively:

$$\|Fx - Fy\| \le \kappa\|x - y\|, \quad x, y \in C, \tag{4}$$

$$\langle Fx - Fy, x - y\rangle \ge \eta\|x - y\|^2, \quad x, y \in C. \tag{5}$$

Under these conditions, it is well known that the variational inequality problem

$$\text{VI}(F, C) \qquad\qquad \langle Fu^*, v - u^*\rangle \ge 0 \quad v \in C$$

has a unique solution $u^* \in C$.

Denote by P_C the projection of H onto C. Namely, for each $x \in H$, $P_C x$ is the unique element in C satisfying

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}.$$

Recall that the projection P_C is characterized by the inequality:

$$\langle x - P_C x, y - P_C x\rangle \le 0, \quad y \in C.$$

In this section, assume that $T : H \to H$ is a nonexpansive mapping with $\mathrm{Fix}(T) = C$. Note that obviously $\mathrm{Fix}(P_C) = C$. Then we propose a modified steepest-descent algorithm with variable parameters which produces a sequence converging in norm to the unique solution u^* of $VI(F, C)$. We introduce now some notation. Let λ be a number in (0,1) and let $0 < \mu < 2\eta/\kappa^2$. Associated with the nonexpansive mapping $T : H \to H$, define the mapping $T^{(\lambda,\mu)} : H \to H$ by

$$T^{(\lambda,\mu)}x := Tx - \lambda\mu F(Tx), \quad x \in H.$$

Lemma 3.1 (Ref. 11; see also Ref. 16). $T^{(\lambda,\mu)}$ is a contraction provided $0 < \mu < 2\eta/\kappa^2$. Indeed,

$$\|T^{(\lambda,\mu)}x - T^{(\lambda,\mu)}y\| \le (1 - \lambda\tau)\|x - y\|, \quad x, y \in H, \tag{6}$$

where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\kappa^2)} \in (0, 1)$.

Algorithm 3.1. Modified Hybrid Steepest-Descent Algorithm (I).

Let $\{\lambda_n\}$ be a sequence in (0,1) and let $\{\mu_n\}$ be a sequence in $(0, 2\eta/\kappa^2)$. Starting with an arbitrary initial guess $u_0 \in H$, one can generate a sequence $\{u_n\}$ by the following iterative scheme

$$u_{n+1} := T^{(\lambda_{n+1},\mu_{n+1})}u_n = Tu_n - \lambda_{n+1}\mu_{n+1}F(Tu_n), \quad n \geq 0. \tag{7}$$

We have the following convergence result.

Theorem 3.1. Let the sequence $\{u_n\}$ be generated by algorithm (7). Assume that

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$ where $\{\lambda_n\} \subset (0,1)$;

(ii) $|\mu_n - \frac{\eta}{\kappa^2}| \leq \frac{\sqrt{\eta^2 - c\kappa^2}}{\kappa^2}$ for some $c \in (0, \eta^2/\kappa^2)$;

(iii) $\lim_{n\to\infty}(\mu_{n+1} - \frac{\lambda_n}{\lambda_{n+1}} \cdot \mu_n) = 0$.

If

$$\limsup_{n\to\infty}\langle Tu_n - u_{n+1}, Tu_n - u_n\rangle \leq 0,$$

then $\{u_n\}$ converges strongly to the unique solution u^* of the VI(F,C).

Remark 3.1. In Theorem 3.1, we take $T = I,\ c = \frac{\eta^2}{2\kappa^2},\ \lambda_n = \frac{1}{2}$ and $\mu_n = \frac{\eta}{2\kappa^2}$ for all $n \geq 1$. Then it is clear that all the conditions in Theorem 3.1 are satisfied. But since $\lambda_n \to \frac{1}{2} \neq 0$ and $\mu_n \to \frac{\eta}{2\kappa^2} \neq 0$, it is known that (L1) is not satisfied. In this case Xu and Kim's Theorem 3.1 (Ref. 16) can not guarantee that the sequence $\{u_n\}$ generated by

$$u_{n+1} = u_n - \frac{\eta}{4\kappa^2}F(u_n),$$

converges strongly to the unique solution u^* of the VI(F,C).

Proof of Theorem 3.1. We divide the proof into several steps.

Step 1. $\{u_n\}$ is bounded. Indeed, we have (note that

$$T^{(\lambda_{n+1},\mu_{n+1})}u^* = u^* - \lambda_{n+1}\mu_{n+1}F(u^*)\ )$$

$$\begin{aligned}\|u_{n+1} - u^*\| &= \|T^{(\lambda_{n+1},\mu_{n+1})}u_n - u^*\| \\ &\le \|T^{(\lambda_{n+1},\mu_{n+1})}u_n - T^{(\lambda_{n+1},\mu_{n+1})}u^*\| + \|T^{(\lambda_{n+1},\mu_{n+1})}u^* - u^*\| \\ &\le (1 - \lambda_{n+1}\tau_{n+1})\|u_n - u^*\| + \lambda_{n+1}\mu_{n+1}\|F(u^*)\|, \end{aligned} \tag{8}$$

where $\tau_{n+1} := 1 - \sqrt{1 - \mu_{n+1}(2\eta - \mu_{n+1}\kappa^2)}$. By virtue of condition (ii) we claim that $\tau_{n+1} \ge \tau$ where $\tau = 1 - \sqrt{1-c}$. Indeed, it follows from condition (ii) that

$$\frac{\eta - \sqrt{\eta^2 - c\kappa^2}}{\kappa^2} \le \mu_{n+1} \le \frac{\eta + \sqrt{\eta^2 - c\kappa^2}}{\kappa^2} < \frac{2\eta}{\kappa^2} := \mu$$

and hence

$$(\mu_{n+1} - \frac{\eta}{\kappa^2} + \frac{\sqrt{\eta^2 - c\kappa^2}}{\kappa^2}) \cdot (\mu_{n+1} - \frac{\eta}{\kappa^2} - \frac{\sqrt{\eta^2 - c\kappa^2}}{\kappa^2}) \le 0.$$

This implies that

$$\kappa^2\mu_{n+1}^2 - 2\eta\mu_{n+1} + c \le 0.$$

Observe that

$$\mu_{n+1}(2\eta - \mu_{n+1}\kappa^2) \ge c = 1 - [1 - (1 - \sqrt{1-c})]^2 = 1 - (1-\tau)^2,$$

where $\tau := 1 - \sqrt{1-c}$. Hence, we derive

$$\tau_{n+1} = 1 - \sqrt{1 - \mu_{n+1}(2\eta - \mu_{n+1}\kappa^2)} \ge \tau.$$

Therefore, it follows from (8) that

$$\|u_{n+1} - u^*\| \leq (1 - \lambda_{n+1}\tau)\|u_n - u^*\| + \lambda_{n+1}\mu\|F(u^*)\|.$$

By induction, it is easy to see that

$$\|u_n - u^*\| \leq \max\{\|u_0 - u^*\|, \quad (\mu/\tau)\|F(u^*)\|\} \quad n \geq 0.$$

Step 2. $\|u_{n+1} - u_n\| \to 0, n \to \infty$. Indeed, we have

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \|T^{(\lambda_{n+1},\mu_{n+1})}u_n - T^{(\lambda_n,\mu_n)}u_{n-1}\| \\
&\leq \|T^{(\lambda_{n+1},\mu_{n+1})}u_n - T^{(\lambda_{n+1},\mu_{n+1})}u_{n-1}\| + \|T^{(\lambda_{n+1},\mu_{n+1})}u_{n-1} - T^{(\lambda_n,\mu_n)}u_{n-1}\| \\
&\leq (1 - \lambda_{n+1}\tau_{n+1})\|u_n - u_{n-1}\| + |\lambda_{n+1}\mu_{n+1} - \lambda_n\mu_n| \cdot \|F(Tu_{n-1})\| \\
&\leq (1 - \lambda_{n+1}\tau)\|u_n - u_{n-1}\| + |\lambda_{n+1}\mu_{n+1} - \lambda_n\mu_n| \cdot \|F(Tu_{n-1})\|.
\end{aligned}$$

Note that by Step 1 $\{u_n\}$ is bounded. Then, $\{F(Tu_n)\}$ is bounded since

$$\|F(Tu_n) - F(u^*)\| \leq \kappa\|u_n - u^*\|.$$

Putting $M = \sup\{\|F(Tu_n)\| : n \geq 0\}$, we obtain

$$\|u_{n+1} - u_n\| \leq (1 - \lambda_{n+1}\tau)\|u_n - u_{n-1}\| + (\lambda_{n+1}\tau)\beta_{n+1}$$

where

$$\beta_{n+1} = M|\lambda_{n+1}\mu_{n+1} - \lambda_n\mu_n|/(\tau\lambda_{n+1}) \to 0 \quad \text{(using condition (iii))}.$$

By Lemma 2.1, we deduce that $\|u_{n+1} - u_n\| \to 0$.

Step 3. $\|Tu_n - u_n\| \to 0, n \to \infty$. Indeed, observe that

$$\begin{aligned}\|u_n - u_{n+1}\|^2 &= \|u_n - Tu_n + \lambda_{n+1}\mu_{n+1}F(Tu_n)\|^2 \\ &= \|u_n - Tu_n\|^2 + 2\lambda_{n+1}\mu_{n+1}\langle F(Tu_n), u_n - Tu_n\rangle + \lambda_{n+1}^2\mu_{n+1}^2\|F(Tu_n)\|^2 \\ &\geq \|u_n - Tu_n\|^2 + 2\lambda_{n+1}\mu_{n+1}\langle F(Tu_n), u_n - Tu_n\rangle \\ &= \|u_n - Tu_n\|^2 + 2\langle Tu_n - u_{n+1}, u_n - Tu_n\rangle,\end{aligned}$$

and hence

$$\|Tu_n - u_n\|^2 \leq \|u_{n+1} - u_n\|^2 + 2\langle Tu_n - u_{n+1}, Tu_n - u_n\rangle.$$

Combining Step 2 with the assumption that $\limsup_{n\to\infty}\langle Tu_n - u_{n+1}, Tu_n - u_n\rangle \leq 0$, we conclude that

$$\begin{aligned}\limsup_{n\to\infty}\|Tu_n - u_n\|^2 \quad & leq \limsup_{n\to\infty}[\|u_{n+1} - u_n\|^2 + 2\langle Tu_n - u_{n+1}, Tu_n - u_n\rangle] \\ &\leq \limsup_{n\to\infty}\|u_{n+1} - u_n\|^2 + 2leq\limsup_{n\to\infty}\langle Tu_n - u_{n+1}, Tu_n - u_n\rangle \\ &\leq 0.\end{aligned}$$

Thus $\|Tu_n - u_n\| \to 0$.

Step 4. $\text{limsup}_{n\to\infty}\langle -F(u^*), u_n - u^*\rangle \leq 0$. To prove this, we pick a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ so that

$$\limsup_{n\to\infty}\langle -F(u^*), u_n - u^*\rangle = \lim_{i\to\infty}\langle -F(u^*), u_{n_i} - u^*\rangle.$$

Without loss of generality, we may further assume that $u_{n_i} \to \tilde{u}$ weakly for some $\tilde{u} \in H$. But by Lemma 2.2 and Step 3, we derive $\tilde{u} \in \text{Fix}(T) = C$. Since u^* is the unique solution of the VI(F, C), we obtain

$$\limsup_{n\to\infty}\langle -F(u^*), u_n - u\rangle = \langle -F(u^*), \tilde{u} - u^*\rangle \leq 0.$$

Step 5. $u_n \to u^*$ in norm. Indeed, by applying Lemma 2.3, we get

$$\begin{aligned} \|u_{n+1} - u^*\|^2 &= \|(T^{(\lambda_{n+1},\mu_{n+1})}u_n - T^{(\lambda_{n+1},\mu_{n+1})}u^*) + (T^{(\lambda_{n+1},\mu_{n+1})}u^* - u^*)\|^2 \\ &\leq \|T^{(\lambda_{n+1},\mu_{n+1})}u_n - T^{(\lambda_{n+1},\mu_{n+1})}u^*\|^2 + 2\langle T^{(\lambda_{n+1},\mu_{n+1})}u^* - u^*, u_{n+1} - u^*\rangle \\ &\leq (1 - \lambda_{n+1}\tau_{n+1})\|u_n - u^*\|^2 + 2\lambda_{n+1}\mu_{n+1}\langle -F(u^*), u_{n+1} - u^*\rangle \\ &\leq (1 - \lambda_{n+1}\tau)\|u_n - u^*\|^2 + (\lambda_{n+1}\tau) \cdot \frac{2\ mu_{n+1}}{\tau}\langle -F(u^*), u_{n+1} - u^*\rangle. \end{aligned}$$

Since $\{\mu_n\}$ is a positive and bounded sequence, an application of Lemma 2.1 combined with Step 4 yields that $\|u_n - u^*\| \to 0$. □

Next we consider a more general case when

$$C = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i),$$

with $N \geq 1$ an integer and $T_i : H \to H$ being nonexpansive for each $1 \leq i \leq N$.

We introduce now another modified hybrid steepest-descent algorithm with variable parameters for solving the VI(F, C).

Algorithm 3.2. Modified Hybrid Steepest-descent Algorithm (II).

Let $\{\lambda_n\}$ be a sequence in (0,1) and let $\{\mu_n\}$ be a sequence in $(0, 2\eta/\kappa^2)$. Starting with an arbitrary initial guess $u_0 \in H$, one can generate a sequence $\{u_n\}$ by the following iterative scheme

$$u_{n+1} = T_{[n+1]}^{(\lambda_{n+1},\mu_{n+1})}u_n = T_{[n+1]}u_n - \lambda_{n+1}\mu_{n+1}F(T_{[n+1]}u_n), \quad n \geq 0. \tag{9}$$

We prove now the main result of this paper.

Theorem 3.2. Let the sequence $\{u_n\}$ be generated by algorithm (9). Suppose that

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$ where $\{\lambda_n\} \subset (0,1)$;

(ii) $|\mu_n - \frac{\eta}{\kappa^2}| \leq \frac{\sqrt{\eta^2 - c\kappa^2}}{\kappa^2}$ for some $c \in (0, \eta^2/\kappa^2)$;

(iii) $\lim_{n\to\infty}(\mu_{n+N} - \frac{\lambda_n}{\lambda_{n+N}} \cdot \mu_n) = 0$.

Assume in addition that

$$C = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 T_2 \cdots T_N) = \mathrm{Fix}(T_N T_1 \cdots T_{N-1}) = \cdots = \mathrm{Fix}(T_2 T_3 \cdots T_N T_1). \quad (10)$$

If

$$\limsup_{n\to\infty} \langle T_{[n+N]} \cdots T_{[n+1]} u_n - u_{n+N}, T_{[n+N]} \cdots T_{[n+1]} u_n - u_n \rangle \leq 0,$$

then $\{u_n\}$ converges in norm to the unique solution u^* of the VI(F,C).

Remark 3.2. In Theorem 3.2 we take $T_1 = \cdots = T_N = I,\ c = \frac{\eta^2}{2\kappa^2},\ \lambda_n = \frac{1}{2}$ and $\mu_n = \frac{\eta}{2\kappa^2}$ for all $n \geq 1$. Then it is clear that all the conditions in Theorem 3.2 are satisfied. But since $\lambda_n \to \frac{1}{2} \neq 0$ and $\mu_n \to \frac{\eta}{2\kappa^2} \neq 0$, it is known that (L1) is not satisfied. In this case, Xu and Kim's Theorem 3.2 (Ref. 16) can not guarantee that the sequence $\{u_n\}$ generated by

$$u_{n+1} = u_n - \frac{\eta}{4\kappa^2} F(u_n)$$

converges strongly to the unique solution u^* of the VI(F,C).

Proof of Theorem 3.2. We shall again divide the proof into several steps.

Step 1. $\{u_n\}$ is bounded. Indeed, we have (note that $T_{[n]}^{(\lambda_n,\mu_n)}u^* = u^* - \lambda_n\mu_n F(u^*)\ \ n \geq 1$)

$$\begin{aligned}\|u_{n+1} - u^*\| &= \|T_{[n+1]}^{(\lambda_{n+1},\mu_{n+1})}u_n - u^*\| \\ &\leq \|T_{[n+1]}^{(\lambda_{n+1},\mu_{n+1})}u_n - T_{[n+1]}^{(\lambda_{n+1},\mu_{n+1})}u^*\| + \|T_{[n+1]}^{(\lambda_{n+1},\mu_{n+1})}u^* - u^*\| \\ &\leq (1 - \lambda_{n+1}\tau_{n+1})\|u_n - u^*\| + \lambda_{n+1}\mu_{n+1}\|F(u^*)\|\end{aligned}$$

where $\tau_{n+1} := 1-\sqrt{1 - \mu_{n+1}(2\eta - \mu_{n+1}\kappa^2)}$. As in Step 1 of the proof of Theorem 3.1, according to condition (ii) we can derive $\tau_{n+1} \geq \tau$, where $\tau = 1 - \sqrt{1-c}$. Therefore we conclude that

$$\|u_{n+1} - u^*\| \leq (1 - \lambda_{n+1}\tau)\|u_n - u^*\| + \lambda_{n+1}\mu\|F(u^*)\|$$

where $\mu = 2\eta/\kappa^2$. From this we get by induction

$$\|u_n - u^*\| \leq \max\{\|u_0 - u^*\|, (\mu/\tau)\|F(u^*)\|\} \quad n \geq 0.$$

Step 2. $\|u_{n+N} - u_n\| \to 0$. As a matter of fact observing that $T_{[n+N]} = T_{[n]}$, we have

$$\begin{aligned}&\|u_{n+N} - u_n\| \\ &= \|T_{[n+N]}^{(\lambda_{n+N},\mu_{n+N})}u_{n+N-1} - T_{[n]}^{(\lambda_n,\mu_n)}u_{n-1}\| \\ &\leq \|T_{[n+N]}^{(\lambda_{n+N},\mu_{n+N})}u_{n+N-1} - T_{[n+N]}^{(\lambda_{n+N},\mu_{n+N})}u_{n-1}\| + \|T_{[n+N]}^{(\lambda_{n+N}\mu_{n+N})}u_{n-1} - T_{[n]}^{(\lambda_n,\mu_n)}u_{n-1}\| \\ &\leq (1 - \lambda_{n+N}\tau_{n+N})\|u_{n+N-1} - u_{n-1}\| + |\lambda_{n+N}\mu_{n+N} - \lambda_n\mu_n| \cdot \|F(T_{[n]}u_{n-1})\| \\ &\leq (1 - \lambda_{n+N}\tau)\|u_{n+N-1} - u_{n-1}\| + |\lambda_{n+N}\mu_{n+N} - \lambda_n\mu_n| \cdot \|F(T_{[n]}u_{n-1})\|.\end{aligned}$$

Note that by Step 1 $\{u_n\}$ is bounded. Then $\{F(T_{[n]}u_{n-1})\}$ is bounded since

$$\|F(T_{[n]}u_{n-1}) - F(u^*)\| \leq \kappa\|u_n - u^*\|.$$

Putting $M = \sup\{\|F(T_{[n]}u_{n-1})\| : n \geq 1\}$, we obtain

$$\|u_{n+N} - u_n\| \le (1 - \lambda_{n+N}\tau)\|u_{n+N-1} - u_{n-1}\| + (\lambda_{n+N}\tau)\beta_{n+N},$$

where

$$\beta_{n+N} = M|\lambda_{n+N}\mu_{n+N} - \lambda_n\mu_n|/(\tau\lambda_{n+N}) \to 0 \quad \text{(using condition (iii))}.$$

Now we apply Lemma 2.1 to get that $\|u_{n+N} - u_n\| \to 0$.

Step 3. $T_{[n+N]} \cdots T_{[n+1]}u_n - u_n \to 0$ in norm. Indeed observe that

$$\begin{aligned}
&\|u_{n+N} - u_n\|^2 \\
&= \|u_n - T_{[n+N]} \cdots T_{[n+1]}u_n + T_{[n+N]} \cdots T_{[n+1]}u_n - u_{n+N}\|^2 \\
&\ge \|u_n - T_{[n+N]} \cdots T_{[n+1]}u_n\|^2 + 2\langle T_{[n+N]} \cdots T_{[n+1]}u_n - u_{n+N}, u_n - T_{[n+N]} \cdots T_{[n+1]}u_n\rangle,
\end{aligned}$$

and hence

$$\begin{aligned}
&\|u_n - T_{[n+N]} \cdots T_{[n+1]}u_n\|^2 \\
&\le \|u_{n+N} - u_n\|^2 + 2\langle T_{[n+N]} \cdots T_{[n+1]}u_n - u_{n+N}, T_{[n+N]} \cdots T_{[n+1]}u_n - u_n\rangle.
\end{aligned}$$

According to Step 2 and the assumption that

$$\limsup_{n\to\infty}\langle T_{[n+N]} \cdots T_{[n+1]}u_n - u_{n+N}, T_{[n+N]} \cdots T_{[n+1]}u_n - u_n\rangle \le 0,$$

we deduce that $\lim_{n\to\infty} \|u_n - T_{[n+N]} \cdots T_{[n+1]}u_n\| = 0$.

Step 4. $\limsup_{n\to\infty}\langle -F(u^*), u_n - u^*\rangle \le 0$. To see this, we pick a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that

$$\limsup_{n\to\infty}\langle -F(u^*), u_n - u^*\rangle = \lim_{i\to\infty}\langle -F(u^*), u_{n_i} - u^*\rangle.$$

Since $\{u_n\}$ is bounded, we may also assume that $u_{n_i} \to \tilde{u}$ weakly for some $\tilde{u} \in H$. Since the pool of mapping $\{T_i : 1 \leq i \leq N\}$ is finite, we may further assume (passing to a further subsequence if necessary) that for some integer $k \in \{1, 2, ..., N\}$, $T_{[n_i]} \equiv T_k \ \forall i \geq 1$. Then it follows from Step 3 that $u_{n_i} - T_{[i+N]} \cdots T_{[i+1]} u_{n_i} \to 0$. Hence by Lemma 2.2 we deduce that

$$\tilde{u} \in \mathrm{Fix}(T_{[i+N]} \cdots T_{[i+1]}).$$

Together with assumption (10) this implies that $\tilde{u} \in C$. Now since u^* solves the VI(F, C), we obtain

$$\limsup_{n\to\infty} \langle -F(u^*), u_n - u^* \rangle = \langle -F(u^*), \tilde{u} - u^* \rangle \leq 0.$$

Step 5. $u_n \to u^*$ in norm. Indeed, applying Lemma 2.3, we get

$$\begin{aligned}
&\|u_{n+1} - u^*\|^2 \\
&= \|T_{[n+1]}^{(\lambda_{n+1},\mu_{n+1})} u_n - u^*\|^2 \\
&= \|(T_{[n+1]}^{(\lambda_{n+1},\mu_{n+1})} u_n - T_{[n+1]}^{(\lambda_{n+1},\mu_{n+1})} u^*) + (T_{[n+1]}^{(\lambda_{n+1},\mu_{n+1})} u^* - u^*)\|^2 \\
&\leq \|T_{[n+1]}^{(\lambda_{n+1},\mu_{n+1})} u_n - T_{[n+1]}^{(\lambda_{n+1},\mu_{n+1})} u^*\|^2 + 2\langle T_{[n+1]}^{(\lambda_{n+1},\mu_{n+1})} u^* - u^*, u_{n+1} - u^* \rangle \\
&\leq (1 - \lambda_{n+1}\tau_{n+1})\|u_n - u^*\|^2 + 2\lambda_{n+1}\mu_{n+1}\langle -F(u^*), u_{n+1} - u^* \rangle \\
&\leq (1 - \lambda_{n+1}\tau)\|u_n - u^*\|^2 + (\lambda_{n+1}\tau) \cdot \frac{2\mu_{n+1}}{\tau}\langle -F(u^*), u_{n+1} - u^* \rangle.
\end{aligned}$$

Since $\{\mu_n\}$ is a positive and bounded sequence by Lemma 2.1 and Step 4, we conclude that $\|u_n - u^*\| \to 0$. □

Remark 3.3. Recall that a self-mapping of a closed convex subset K of a Hilbert space H is said to be attracting nonexpansive (Refs. 14, 19) if T is nonexpansive and if for $x, p \in K$ with $x \notin$ Fix(T) and $p \in \mathrm{Fix}(T)$, $\|Tx - p\| < \|x - p\|$. Recall also that T is firmly nonexpansive

(Refs. 14, 19) if

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in K.$$

It is known that assumption (10) in Theorem 3.2 is automatically satisfied if each T_i is attracting nonexpansive. Since a projection is firmly nonexpansive, we have the following consequence of Theorem 3.2.

Corollary 3.1. Suppose that there hold conditions (i), (ii) and (iii) in Theorem 3.2. Let $u_0 \in H$ and let the sequence $\{u_n\}$ be generated by iterative algorithm

$$u_{n+1} := P_{[n+1]}u_n - \lambda_{n+1}\mu_{n+1}F(P_{[n+1]}u_n), \quad n \geq 0,$$

where $P_k = P_{C_k},\ 1 \leq k \leq N$. If

$$\limsup_{n\to\infty}\langle P_{[n+N]}\cdots P_{[n+1]}u_n - u_{n+N}, P_{[n+N]}\cdots P_{[n+1]}u_n - u_n\rangle \leq 0,$$

then $\{u_n\}$ converges strongly to the unique solution u^* of the $VI(F, C)$ with $C = \bigcap_{k=1}^N C_k$. In particular, let $\mu \in (0, 2\eta/\kappa^2)$. Then the sequence $\{u_n\}$ determined by the algorithm

$$u_{n+1} := P_{[n+1]}u_n - (\mu/(n+1))F(P_{[n+1]}u_n), \quad n \geq 0,$$

converges in norm to the unique solution u^* of the VI(F, C).

Proof. The former part of the conclusion follows immediately from Theorem 3.2. Next, we prove only the latter part of the conclusion. Put $\lambda_n = 1/(n+1)$ and $\mu_n = \mu$. As in Steps 1 and 2 of the proof of Theorem 3.2, we can see that $\{u_n\}$ is bounded and $\|u_{n+N} - u_n\| \to 0$. Now, note that

$$\|u_{n+1} - P_{[n+1]}u_n\| = \frac{\mu}{n+1}\|F(P_{[n+1]}u_n)\| \to 0.$$

Hence, $u_{n+1} - P_{[n+1]}u_n \to 0$ in norm. Since each P_k is nonexpansive, we get the finite table

$$\begin{array}{l} u_{n+N} - P_{[n+N]}u_{n+N-1} \to 0, \\ P_{[n+N]}u_{n+N-1} - P_{[n+N]}P_{[n+N-1]}u_{n+N-2} \to 0, \\ \vdots \\ P_{[n+N]} \cdots P_{[n+2]}u_{n+1} - P_{[n+N]} \cdots P_{[n+1]}u_n \to 0. \end{array}$$

Adding up this table yields $u_{n+N} - P_{[n+N]} \cdots P_{[n+1]}u_n \to 0$ in norm. Thus, $P_{[n+N]} \cdots P_{[n+1]}u_n - u_n \to 0$ in norm. Consequently the latter part of the conclusion follows from the former one.

□

4. Applications to Constrained Generalized Pseudoinverse

Let K be a nonempty closed convex subset of a real Hilbert space H. Let A be a bounded linear operator on H. Given an element $b \in H$, consider the minimization problem

$$\min_{x \in K} \|Ax - b\|^2. \tag{11}$$

Let S_b denote the solution set. Then, S_b is closed and convex. It is known that S_b is nonempty if and only if $P_{\overline{A(K)}}(b) \in A(K)$ where $\overline{A(K)}$ is the closure of $A(K)$. In this case, S_b has a unique element with minimun norm; that is, there exists a unique point $\hat{x} \in S_b$ satisfying

$$\|\hat{x}\|^2 = \min\{\|x\|^2 : x \in S_b\}. \tag{12}$$

Definition 4.1 (Ref. 20). The K-constrained pseudoinverse of A (symbol $\hat{A}_K$) is defined as

$$D(\hat{A}_K) = \{b \in H : P_{\overline{A(K)}}(b) \in A(K)\}, \quad \hat{A}_K(b) = \hat{x} \;\text{ and }\; b \in D(\hat{A}_K),$$

where $\hat{x} \in S_b$ is the unique solution to (12).

Now we recall the K-constrained generalized pseudoinverse of A; see Refs. 11, 16.

Let $\theta : H \to R$ be a differentiable convex function such that θ' is a κ-Lipschitzian and η-strongly monotone operator for some constants $\kappa > 0$ and $\eta > 0$. Under these assumptions, there exists a unique point $\hat{x_0} \in S_b$ for $b \in D(\hat{A_K})$ such that

$$\theta(\hat{x_0}) = \min\{\theta(x) : x \in S_b\}. \tag{13}$$

Definition 4.2 (Ref. 16). The K-constrained generalized pseudoinverse of A associated with θ (symbol $\hat{A}_{K,\theta}$ is defined as

$$D(\hat{A}_{K,\theta}) = D(\hat{A}_K), \quad \hat{A}_{K,\theta}(b) = \hat{x}_0, \quad \text{and} \;\; b \in D(\hat{A}_{K,\theta})$$

where $\hat{x}_0 \in S_b$ is the unique solution to (13). Note that if $\theta(x) = (1/2)\|x\|^2$, then the K-constrained generalized pseudoinverse $\hat{A}_{K,\theta}$ of A associated with θ reduces to the K-constrained pseudoinverse $\hat{A}_k$ of A in Definition 4.1.

We now apply the results in Section 3 to construct the K-constrained generalized pseudoinverse $\hat{A}_{K,\theta}$ of A. But first, observe that $\tilde{x} \in K$ solves the minimization problem (11) if and only if there holds the following optimality condition:

$$\langle A^*(A\tilde{x} - b), x - \tilde{x}\rangle \geq 0, \quad x \in K,$$

where A^* is the adjoint of A. This is equivalent to, for each $\lambda > 0$,

$$\langle [\lambda A^* b + (I - \lambda A^* A)\tilde{x}] - \tilde{x}, x - \tilde{x}\rangle \geq 0, \quad x \in K,$$

or

$$P_K(\lambda A^* b + (I - \lambda A^* A)\tilde{x}) = \tilde{x}. \tag{14}$$

Define a mapping $T : H \to H$ by

$$Tx = P_K(A^* b + (I - \lambda A^* A)x), \quad x \in H. \tag{15}$$

Lemma 4.1 (Ref. 16). If $\lambda \in (0, 2\|A\|^{-2})$ and if $b \in D(\hat{A}_k)$, then T is attracting nonexpansive and $\mathrm{Fix}(T) = S_b$.

Theorem 4.1. Assume that there hold conditions (i), (ii) and (iii) in Theorem 3.1. Given an initial guess $u_0 \in H$, let $\{u_n\}$ be the sequence generated by algorithm

$$u_{n+1} = Tu_n - \lambda_{n+1}\mu_{n+1}\theta'(Tu_n), \quad n \geq 0, \tag{16}$$

where T is given in (15). If

$$\limsup_{n\to\infty} \langle Tu_n - u_{n+1}, Tu_n - u_n\rangle \leq 0,$$

then $\{u_n\}$ strongly converges to $\hat{A}_{K,\theta}(b)$.

Proof. The minimization problem (13) is equivalent to the following variational inequality problem:

$$\langle \theta'(\hat{x}_0), x - \hat{x}_0 \rangle \geq 0, \quad x \in S_b. \tag{17}$$

Since $\mathrm{Fix}(T) = S_b$ and θ' is κ-Lipschitzian and η-strongly monotone, using Theorem 3.1 with $F := \theta'$, we infer that $\{u_n\}$ converges in norm to $\hat{x}_0 = \hat{A}_{K,\theta}(b)$. □

Lemma 4.2 (Refs. 14, 19). Let N be a positive integer and let $\{T_i\}_{i=1}^N$ be N attracting nonexpansive mappings on H with a common fixed point. Then, $\bigcap_{i=1}^N \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 T_2 \cdots T_N)$.

Suppose $\{S_b^1, ..., S_b^N\}$ is a family of N closed convex subsets of K such that $S_b = \bigcap_{i=1}^N S_b^i$. For each $1 \leq i \leq N$, we define $T_i : H \to H$ by

$$T_i x = P_{S_b^i}(A^* b + (I - \lambda A^* A)x), \quad x \in H,$$

where $P_{S_b^i}$ is the projection from H onto S_b^i.

Theorem 4.2. Assume that there hold conditions (i), (ii) and (iii) in Theorem 3.2. Given an initial guess $u_0 \in H$, let $\{u_n\}$ be the sequence generated by the algorithm

$$u_{n+1} = T_{[n+1]}^{(\lambda_{n+1}, \mu_{n+1})} u_n = T_{[n+1]} u_n - \lambda_{n+1} \mu_{n+1} \theta'(T_{[n+1]} u_n), \quad n \geq 0. \tag{18}$$

If

$$\limsup_{n \to \infty} \langle T_{[n+N]} \cdots T_{[n+1]} u_n - u_{n+N}, T_{[n+N]} \cdots T_{[n+1]} u_n - u_n \rangle \leq 0,$$

then $\{u_n\}$ converges in norm to $\hat{A}_{K,\theta}(b)$.

Proof. In the proof of (Ref. 16, Theorem 4.2), Xu and Kim have proved that

$$S_b = \text{Fix}(T) = \cap_{i=1}^{N} \text{Fix}(T_i). \tag{19}$$

By Lemmas 4.1 and 4.2, we see that assumption (10) in Theorem 3.2 holds. By virtue of (19), Theorem 3.2 ensures that the sequence $\{u_n\}$ generated by (18) converges strongly to the unique solution $\hat{x}_0 = \hat{A}_{K,\theta}(b)$ of (17). □

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