ZERO PRODUCT PRESERVING LINEAR MAPS OF CCR C*-ALGEBRAS WITH HAUSDORFF SPECTRUM

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Abstract. In this paper, we try to attack a conjecture of Araujo and Jarosz that every bijective linear map $\theta$ between C*-algebras, with both $\theta$ and its inverse $\theta^{-1}$ preserving zero products, arises from an algebra isomorphism followed by a central multiplier. We show it is true for CCR C*-algebras with Hausdorff spectrum, and in general, some special C*-algebras associated to continuous fields of C*-algebras.

1. Introduction

The theory of general C*-algebras is made easy by observing the interplay between their algebraic and analytical structures. For example, the norm structure can be recovered from the *-algebraic structure in a C*-algebra. It is further shown by Gardner [10] (see also [16, Theorem 4.1.20]) that two C*-algebras are *-algebraically isomorphic if and only if they are algebraically isomorphic.

Extending results in [18, 17], they are shown in [6] for the unital case and in [19, Corollary 2.6] for the general case that two C*-algebras $A$, $B$ are algebraically isomorphic if and only if there is a continuous bijective linear map $\theta$ between them preserving zero products, that is,

$$\theta(a)\theta(b) = 0 \text{ in } B \text{ whenever } ab = 0 \text{ in } A.$$  

In this case,

$$\theta = \theta^{**}(1) \Psi,$$

where $\theta^{**}$ is the bidual map of $\theta$, and $\theta^{**}(1)$ is an invertible central multiplier of $B$, while $\Psi$ is an algebra isomorphism form $A$ onto $B$. Consequently, the topological, linear and zero product structures determine a C*-algebra.

In [2], Araujo and Jarosz show that every bijective linear map $\theta$ between unital standard operator algebras on Banach spaces, with both $\theta$ and its inverse $\theta^{-1}$ preserving zero products, carries the standard form (1.1). In particular, such maps are automatically bounded. Their results apply to those maps between standard C*-algebras, i.e., those containing compact operators. They state a conjecture in

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[2] to ask whether every such map between two arbitrary C*-algebras carries the standard form (1.1). In other words, they want to know whether the linear and the zero product structures suffice to determine a C*-algebra.

This might be a hard problem, as we do not have suitable functional calculus to use if we do not know in advance the map is bounded. As a matter of facts, the structure of unbounded zero product preserving linear functionals of C*-algebras is quite complicated (see [4]). Furthermore, we know that Banach algebra homomorphisms can be unbounded (see, e.g., [7]). One possible way to attack this problem is to decompose a general C*-algebra into a family of simple C*-algebras, e.g., the ones consist of compact operators. Together with [13], this suggests us to study continuous fields of C*-algebras whose fibers are elementary C*-algebras, which give rise to exactly all CCR C*-algebras with Hausdorff spectrum.

In Section 2, we shall develop a structure theory of zero product preserving linear maps \( \theta \) between two continuous fields of C*-algebras \((X, \{A_x\}, A)\) and \((Y, \{B_y\}, B)\). These maps carry a standard form

\[
\theta(f)(y) = H_y(f(\varphi(y))), \quad \forall f \in A, \forall y \in Y,
\]

where \( \varphi \) is a map from \( Y \) into \( X \), and each fiber linear map \( H_y : A_{\varphi(y)} \to B_y \) is zero product preserving. In Section 3, we assume, in addition, \( \theta \) is bijective and its inverse \( \theta^{-1} \) also preserves zero products. Then, \( \varphi \) is a homeomorphism. Moreover, all fiber linear maps \( H_y \) are bounded whenever \( X \) (or \( Y \)) contains no isolated points, or all the fiber C*-algebras are standard operator algebras. In these cases, \( \theta \) is bounded and thus, by results in [6, 19], carries the standard form (1.1). Eventually, we solve the open problem in affirmative for the CCR C*-algebra case; namely, two CCR C*-algebras with Hausdorff spectrum are *-isomorphic if and only if they have the same linear and zero product structures.

It might be worthwhile to mention that the group C*-algebra of a compact group is a direct sum of matrix algebras, and thus a CCR with Hausdorff spectrum (see, e.g., [8, 15.1]). Consequently, results in this paper can be applied. Of course, the most interesting part is to characterize further the group structure through this kind of maps. We hope this will be achieved in coming future.

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2. ZERO PRODUCT PRESERVERS BETWEEN CONTINUOUS FIELDS OF BANACH ALGEBRAS

We shall follow [9, 8] for notations. Let \( T \) be a locally compact Hausdorff space, called base space. For each \( t \) in \( T \) there is a (complex) Banach space \( E_t \). A vector field \( x \) is an element in the product space \( \prod_{t \in T} E_t \), that is, \( x(t) \in E_t, \forall t \in T \).
Definition 2.1. A continuous field $E = (T, \{E_t\}, A)$ of Banach spaces over a locally compact space $T$ is a family $\{E_t\}_{t \in T}$ of Banach spaces, with a set $A$ of vector fields such that

(i) $A$ is a (complex) vector subspace of $\prod_{t \in T} E_t$.
(ii) For every $t$ in $T$, the set of all $x(t)$ with $x$ in $A$ is dense in $E_t$.
(iii) For every $x$ in $A$, the function $t \mapsto \|x(t)\|$ is continuous on $T$ and vanishes at infinity.
(iv) Let $x$ be a vector field. Suppose for every $t$ in $T$ and every $\epsilon > 0$, there is a neighborhood $U$ of $t$ and a $y$ in $A$ such that $\|x(t) - y(t)\| < \epsilon$ for all $t$ in $U$.

Then $x \in A$.

Elements in $A$ are called continuous vector fields.

It is not difficult to see that $A$ becomes a Banach space under the norm $\|x\| = \sup_{t \in T} \|x(t)\|$. If $g$ is in $C_b(T)$, i.e., $g$ is a bounded continuous complex-valued function on $T$, and $x$ is in $A$ then $t \mapsto g(t)x(t)$ defines a continuous vector field $gx$ on $T$. The set $\{x(t) : x \in A\}$ coincides with $E_t$ for every $t$ in $T$. Moreover, for any distinct points $s, t$ in $T$ and any $\alpha$ in $E_s$ and $\beta$ in $E_t$, there is a continuous vector field $x$ such that $x(s) = \alpha$ and $x(t) = \beta$ (see, e.g., [9, 14]).

Definition 2.2. A continuous field of Banach algebras (resp. $C^*$-algebras) $(X, \{A_x\}, A)$ is a continuous field of Banach spaces with Banach algebra (resp. $C^*$-algebra) fibres $A_x$ such that $A$ becomes a Banach algebra (resp. $C^*$-algebra) under the point-wise algebraic (resp. $*$-algebraic) operations and norm $\|f\| = \sup \|f(x)\|$.

Example 2.3. Recall that a $C^*$-algebra $A$ is called a CCR if every irreducible representation of $A$ consists of compact operators. The spectrum $\hat{A}$ of $A$ is the family of unitary equivalence classes of non zero irreducible representations under the hull-kernel topology. This topology is always locally compact, and the spectrum of a CCR $C^*$-algebra is $T_1$. Let $A$ be a CCR $C^*$-algebras with Hausdorff spectrum $X = \hat{A}$. According to [8, Theorem 10.5.4], we can represent $A$ as a continuous field of $C^*$-algebras $(X, \{A_x\}, A)$, where $A_x$ consists of compact linear operators on a Hilbert space $H_x$ for each $x$ in $X$.

Let $(X, \{A_x\}, A)$ and $(Y, \{B_y\}, B)$ be two continuous fields of $C^*$-algebras, and let $\theta : A \rightarrow B$ be a zero product preserving linear map. Denote by $X_\infty = X \cup \{\infty\}$ and $Y_\infty = Y \cup \{\infty\}$ the one-point compactifications of $X$ and $Y$, respectively. Note that the point $\infty$ at infinity will be isolated in $X_\infty$ if $X$ is already compact. Set for each $x$ in $X$ the sets

$I_x = \{f \in A : f \text{ vanishes in a neighborhood in } X_\infty \text{ of } x\},$

$M_x = \{f \in A : f(x) = 0\}.$
In particular,

\[ I_\infty = \{ f \in A : f \text{ has a compact support} \}, \]
\[ M_\infty = A. \]

Similar conventions are also made for each \( y \) in \( Y \). Furthermore, denote by \( \delta_y \) the evaluation map at \( y \) in \( Y \), i.e.,

\[ \delta_y(g) = g(y) \in B_y, \quad \forall g \in B. \]

We call a Banach algebra \( A \) primitive if it has an (isometric) faithful irreducible representation \( \pi : A \rightarrow B(E) \) into the Banach algebra of all bounded linear operators on a Banach space \( E \). We call a linear map between Banach algebras has a primitive range if the Banach algebra generated by its range is primitive.

**Theorem 2.4.** Let \((X, \{A_x\}, A), (Y, \{B_y\}, B)\) be continuous fields of Banach algebras over locally compact Hausdorff spaces \( X, Y \), respectively. Let \( \theta : A \rightarrow B \) be a zero product preserving linear map such that \( \delta_y \circ \theta : A \rightarrow B_y \) has primitive range for every \( y \) in \( Y \).

If we set

\[ Y_0 = \{ y \in Y_\infty : \delta_y \circ \theta = 0 \}, \]

then there is a unique continuous map \( \varphi : Y \setminus Y_0 \rightarrow X_\infty \) satisfying the condition that

\[ \theta(I_{\varphi(y)}) \subseteq M_y. \]

Set

\[ Y_1 = \{ y \in Y \setminus Y_0 : \theta(M_{\varphi(y)}) \subseteq M_y \}, \]
\[ Y_2 = \{ y \in Y \setminus Y_0 : \theta(M_{\varphi(y)}) \nsubseteq M_y \}. \]

Then \( \infty \in Y_0 \) and \( Y_0 \) is compact,

\[ \theta(f)|_{Y_0} = 0, \quad \forall f \in A, \]

and \( Y_2 \) is open in \( Y_\infty \). Moreover, there is a linear map \( H_y : A_{\varphi(y)} \rightarrow B_y \) for each \( y \) in \( Y_1 \) such that

\[ \theta(f)(y) = H_y(f(\varphi(y))), \quad \forall f \in A, \forall y \in Y_1. \]

(2.1)

The exceptional set \( \varphi(Y_2) \) consists of finitely many non-isolated points in \( X_\infty \). Furthermore, \( \theta \) is bounded if and only if \( Y_2 = \emptyset \) and all \( H_y \) are bounded. In this case,

\[ \| \theta \| = \sup_y \| H_y \|. \]

Finally, the fiber maps \( H_y \) are zero product preserving if \((X, \{A_x\}, A)\) is a continuous field of \( C^* \)-algebras.
Composing \( \delta_y \circ \theta \) with a faithful irreducible representation of the Banach algebra generated by \( \{ \theta(f)(y) \in B_y : f \in \mathcal{A} \} \), we can assume that \( B_y \) is an irreducible subalgebra of the algebra \( B(E_y) \) of all bounded linear operators on some Banach space \( E_y \) and \( \delta_y \circ \theta \) is again zero-product preserving with range generating \( B_y \).

It is clear that \( Y_0 \) is compact, contains the point at infinity, and
\[
\theta(f)|_{Y_0} = 0, \quad \forall f \in \mathcal{A}.
\]
On the other hand, for each \( y \in Y \setminus Y_0 \), the range \( \theta(\mathcal{A}) \) is not trivial at \( y \). For every open subset \( U \) of \( X \), denote by \( \mathcal{A}_U \) the subalgebra of all \( f \) in \( \mathcal{A} \) vanishing outside a compact subset of \( U \). For each \( y \) in \( Y \setminus Y_0 \), denote by
\[
S_y = \left\{ x \in X_\infty : \text{for every open neighborhood } U \text{ of } x, \right. \\
\left. \text{there is an } f \text{ in } \mathcal{A}_U \text{ such that } \theta(f)(y) \neq 0 \right\}.
\]

We divide the proof of Theorem 2.4 into several lemmas.

**Lemma 2.5.** The set \( S_y \) is nonempty for each \( y \) in \( Y \setminus Y_0 \).

*Proof.* Suppose on the contrary that for each \( x \) in \( X_\infty \) there is an open neighborhood \( U_x \) of \( x \) in \( X_\infty \) such that \( \theta(f)(y) = 0 \) for all \( f \) in \( \mathcal{A}_{U_x} \). Let \( V_x \) be an open neighborhood of \( x \) with compact closure \( V \subseteq U \). By compactness,
\[
X_\infty = V_{x_0} \cup V_{x_1} \cup \cdots \cup V_{x_n}
\]
for some points \( x_0 = \infty, x_1, \ldots, x_n \) in \( X_\infty \). Let
\[
1 = h_0 + h_1 + \cdots + h_n
\]
be a continuous partition of unity such that \( h_i \) vanishes outside \( V_{x_i} \) for \( i = 0, 1, \ldots, n \). For any \( g \) in \( \mathcal{A} \), observe that
\[
(h_i g) \in \mathcal{A}_{U_{x_i}} \quad \text{implies} \quad \theta(h_i g)(y) = 0,
\]
and then \( \theta(g)(y) = 0, \forall g \in \mathcal{A} \). This gives a contradiction \( y \in Y_0 \). \( \square \)

**Lemma 2.6.** \( S_y \) consists of exactly one point for all \( y \) in \( Y \setminus Y_0 \).

*Proof.* We shall verify that \( x_1, x_2 \in S_y \) implies \( x_1 = x_2 \). Suppose \( x_2 \neq x_1 \). Let \( U_1 \) and \( U_2 \) be disjoint open neighborhoods of \( x_1 \) and \( x_2 \), respectively. Since
\[
f_1 f_2 = f_2 f_1 = 0 \quad \text{for all } f_i \text{ in } \mathcal{A}_{U_i}, \quad i = 1, 2,
\]
we have
\[
\theta(f_1)\theta(f_2) = \theta(f_2)\theta(f_1) = 0 \quad \text{in } \mathcal{B}.
\]
Let \( E_1 \) be the intersection of the kernels of all \( \theta(f_i)(y) \) with \( f_1 \) in \( \mathcal{A}_{U_1} \). Because both \( \theta|_{\mathcal{A}_{U_1}} \) and \( \theta|_{\mathcal{A}_{U_2}} \) are not trivial at \( y \), we see that \( E_1 \) is a proper nontrivial subspace of \( E_y \), that is, \( \{0\} \neq E_1 \neq E_y \).
Let \( V \) be a nonempty open set in \( Y \) such that the compact closure \( \overline{V} \subseteq U_1 \). For any \( h \) in \( A_V \), let \( g \) be in \( C(X_\infty) \) such that \( g = 1 \) on the support of \( h \) and \( g \) vanishes outside \( V \). Then for each \( f \) in \( A \), since \( fg \) vanishes outside \( V \), we have \( \theta(fg)(y)|_{E_1} = 0 \).

On the other hand, we have \( h(f(1 - g)) = 0 \). This implies \( \theta(h)(y)\theta(f)(y)|_{E_1} = \theta(h)(y)\theta(fg)(y)|_{E_1} = 0, \forall f \in A \). Since \( V \) is an arbitrary nonempty open set with compact closure contained in \( U_1 \), we have \( \theta(h)(y)\theta(f)(y)|_{E_1} = 0 \) for all \( f \in A \) and for all \( h \in A_{U_1} \). Therefore, \( \theta(A)(y)(E_1) \subseteq E_1 \). Since \( \theta(A)(y) \) generates the irreducible algebra \( B_y \), we see that \( E_1 \) could not be proper. This is a contradiction. \( \square \)

Define a map \( \varphi \) from \( Y \setminus Y_0 \) into \( X_\infty \) by \( S_y = \{ \varphi(y) \} \).

**Lemma 2.7.** The point \( \varphi(y) \) is the unique point in \( X_\infty \) satisfying the condition that

\[
\theta(I_{\varphi(y)}) \subseteq M_y, \quad \forall y \in Y \setminus Y_0.
\]

**Proof.** Let \( f \in I_{\varphi(y)} \) vanish in an open neighborhood \( U \) of \( \varphi(y) \). For all \( x \notin U \), by the definition of \( S_y \) there is an open neighborhood \( V_x \) of \( x \) such that \( \theta(A_{V_x})(y) = \{0\} \).

By compactness, we can write \( X_\infty = U \cup V_{x_1} \cup \cdots \cup V_{x_n} \) for some \( x_1, \ldots, x_n \) in \( X_\infty \setminus U \). Let \( 1 = h + h_1 + \cdots + h_n \) be a corresponding continuous partition of unity. Note that \( \theta(h_i g)(y) = 0 \) for all \( g \) in \( A \) and \( i = 1, \ldots, n \). Hence, \( \theta(y)(y) = \theta(h g)(y) \) for all \( g \) in \( A \). As \( f(h g) = 0 \), we see that \( \theta(f)(y)\theta(g)(y) = \theta(f)(y)\theta(h g)(y) = 0 \). Since \( \delta_y \circ \theta \) has a primitive range, \( \theta(f)(y) = 0 \), or \( \theta(f) \in M_y \). Finally, the uniqueness assertion follows from the definition of \( S_y \). \( \square \)

It is clear that the map \( \varphi \) is uniquely characterized by (2.2). Now the definitions of the sets \( Y_1 \) and \( Y_2 \) make sense.

**Lemma 2.8.** \( \varphi : Y \setminus Y_0 \to X_\infty \) is continuous.

**Proof.** Suppose \( y_n \to y \) in \( Y \setminus Y_0 \), but \( x_n = \varphi(y_n) \to x \neq \varphi(y) \). By Lemma 2.7, \( \theta(I_x) \not\subseteq M_y \). Let \( U_x, U_{\varphi(y)} \) be disjoint compact neighborhoods of \( x, \varphi(y) \), respectively. Let \( g \in C(X_\infty) \) such that \( g = 1 \) on \( U_x \) and \( g = 0 \) on \( U_{\varphi(y)} \). Since \( x_n \to x \), \( (1 - g)f \in I_{x_n} \) eventually. Thus, \( \theta((1 - g)f) \in M_{y_n} \) eventually. By the continuity of the norm function, \( \theta((1 - g)f)(y) = 0 \). On the other hand, \( gf \in I_{\varphi(y)} \) implies \( \theta(gf) \in M_y \). Hence, \( \theta(f)(y) = 0, \forall f \in A \). This gives \( y \in Y_0 \), a contradiction. \( \square \)

**Lemma 2.9.** Let \( \{y_n\} \) be an infinite sequence in \( Y \setminus Y_0 \) such that \( \varphi(y_n) \) are distinct points in \( X_\infty \). Then

\[
\limsup \|\delta_{y_n} \circ \theta\| < +\infty.
\]

**Proof.** Suppose not, by passing to a subsequence if necessary, we can assume that \( \|\delta_{y_n} \circ \theta\| > n^4 \), and there is an element \( f_n \) in \( A \) such that \( \|f_n\| \leq 1 \) and \( \|\theta(f_n)(y_n)\| > n^3 \), for \( n = 1, 2, \ldots \). Let \( V_n, U_n \) be compact neighborhoods of \( x_n \) in \( X_\infty \) such that \( V_n \)
is contained in the interior of $U_n$, and $U_n \cap U_m = \emptyset$, for distinct $n, m = 1, 2, \ldots$. Let $g_n \in C(X_\infty)$ such that $g_n = 1$ on $V_n$ and $g_n = 0$ outside $U_n$ for $n = 1, 2, \ldots$. Observe

$$
\theta(f_n)(y_n) = \theta(g_n f)(y_n) + \theta((1 - g_n)f)(y_n)
$$

$$
= \theta(g_n f)(y_n), \quad \text{as } (1 - g_n)f \in I_{x_n}.
$$

So we can assume $f_n$ is supported in $U_n$, for $n = 1, 2, \ldots$. Let

$$
f = \sum_{n=1}^{\infty} \frac{1}{n^2} f_n \in \mathcal{A}.
$$

Since $n^2f - f_n \in I_{x_n}$, we have $n^2\theta(f)(y_n) = \theta(f_n)(y_n)$ by (2.2), and thus $\|\theta(f)(y_n)\| > n$, for $n = 1, 2, \ldots$. As $\theta(f)$ in $\mathcal{B}$ has a bounded norm, we arrive at a contradiction. 

\[\square\]

**Lemma 2.10.** $\varphi(Y_2)$ is a finite set of non-isolated points in $X_\infty$.

**Proof.** Let $x = \varphi(y)$ with $y$ in $Y_2$. Then by (2.2) we have

$$
\theta(I_x) \subseteq M_y \quad \text{but} \quad \theta(M_x) \notin M_y.
$$

This implies the linear operator $\delta_y \circ \theta$ is unbounded, since $I_x$ is dense in $M_x$ by Uryshon’s Lemma. By Lemma 2.9, we can have only finitely many of such $x$'s. So $\varphi(Y_2)$ is a finite set. Moreover, if $x$ is an isolated point in $X_\infty$ then $I_x = M_x$, and thus $x \notin \varphi(Y_2)$. \[\square\]

**Proof of Theorem 2.4.** Let $y \in Y_1$, we have $\theta(M_{\varphi(y)}) \subseteq M_y$. Hence, there is a linear operator $H_y : E_{\varphi(y)} \to F_y$ such that

$$
(2.3) \quad \theta(f)(y) = H_y(f(\varphi(y))), \quad \forall f \in \mathcal{A}.
$$

Next we want to see that $Y_2$ is open, or equivalently, $Y_0 \cup Y_1$ is closed in $Y_\infty$. Let $y_\lambda \to y$ with $y_\lambda$ in $Y_0 \cup Y_1$. We want to show that $y \in Y_0 \cup Y_1$. Since $Y_0$ is compact, we may assume $y_\lambda \in Y_1$ for all $\lambda$. Suppose $y \notin Y_0$. By Lemma 2.8, we see that $\varphi(y_\lambda) \to \varphi(y)$. In the case there is a subnet of $\{\varphi(y_\lambda)\}$ consisting of only finitely many points, we can assume $\varphi(y_\lambda) = \varphi(y)$ for all $\lambda$. Then for all $f$ in $\mathcal{A}$, $f(\varphi(y)) = 0$ implies $f(\varphi(y_\lambda)) = 0$, and thus $\theta(f)(y_\lambda) = 0$ for all $\lambda$ by (2.3). By continuity, $\theta(f)(y) = 0$. Consequently, $\theta(M_{\varphi(y)}) \subseteq M_y$, and thus $y \in Y_1$. In the other case, every subnet of $\{\varphi(y_\lambda)\}$ contains infinitely many points. Lemma 2.9 asserts that $M = \lim \sup \|H_{y_\lambda}\| < +\infty$. This gives

$$
\|\theta(f)(y)\| = \lim \|\theta(f)(y_\lambda)\| = \lim \|H_{y_\lambda}(f(\varphi(y_\lambda)))\| \leq M \|f(\varphi(y))\|.
$$

Thus, if $f(\varphi(y)) = 0$ we have $\theta(f)(y) = 0$. Consequently, $y \in Y_1$. 


Now observe that the boundedness of $\theta$ implies $Y_2 = \emptyset$. Moreover,

$$\|\theta\| = \sup\{\|\theta(f)\| : f \in \mathcal{A} \text{ with } \|f\| = 1\}$$

(2.4)

$$= \sup\{\|H_y(f(\varphi(y)))\| : f \in \mathcal{A} \text{ with } \|f\| = 1, y \in Y_1\}$$

$$\leq \sup\{\|H_y\| : y \in Y_1\}.$$ 

The reverse inequality is plain. Conversely, we suppose $Y_2 = \emptyset$ and all $H_y$ are bounded. We claim that $\sup \|H_y\| < +\infty$. For else, there is a sequence $\{y_n\}$ in $Y_1$ such that $\lim_{n \to \infty} \|H_{y_n}\| = +\infty$. By Lemma 2.9, we can assume all $\varphi(y_n) = x$ in $X$. Let $e \in A_x$ and $f \in \mathcal{A}$ such that $f(x) = e$. Then

$$\|H_{y_n}(e)\| = \|\theta(f)(y_n)\| \leq \|\theta(f)\|, \quad n = 1, 2, \ldots.$$ 

It follows from the uniform boundedness principle that $\sup \|H_{y_n}\| < +\infty$, a contradiction. It then follows from (2.4) that $\theta$ is bounded.

Finally, suppose $(X, \{A_x\}, \mathcal{A})$ is a continuous field of C*-algebras, and in particular, $\mathcal{A}$ is a C*-algebra. Let $\alpha, \beta = 0$ in $A_x$ for some $x$ in $\varphi(Y_1)$. Consider the closed two-sided ideal $I = \{c \in \mathcal{A} : c(x) = 0\}$ of $\mathcal{A}$. Let $a, b$ in $\mathcal{A}$ be such that $a(x) = \alpha, b(x) = \beta$. Then $ab \in I$. By a result of Akemann and Pedersen [1] (see also [6, Lemma 4.14]), we shall have $a', b'$ in $\mathcal{A}$ such that $a'(x) = \alpha, b'(x) = \beta$ and $a'b' = 0$. Now $\theta(a')\theta(b') = 0$ implies $H_y(\alpha)H_y(\beta) = 0$. So each $H_y$ preserves zero products. \hfill \Box

3. ZERO PRODUCT PRESERVERS BETWEEN CCR C*-ALGEBRAS

Recall that an algebra $A$ of continuous linear operators on some locally convex space $E$ is called standard if $A$ contains all finite rank operators. Note that we do not assume $A$ contains the identity map on $E$ or $A$ is closed under any topology. The following result belongs to Araujo and Jarosz [2, Theorem 1]. They verify the case of unital standard operator algebras on Banach spaces. The arguments below slightly simplify theirs.

**Proposition 3.1** ([2]). Let $\theta : A \to B$ be a bijective linear map between standard operator algebras $A, B$ on locally convex spaces $M, N$, respectively, such that both $\theta$ and its inverse $\theta^{-1}$ preserve zero products. Then there is a nonzero scalar $\lambda$ and a weak-weak bi-continuous invertible linear map $S : M \to N$ such that

$$\theta(a) = \lambda SaS^{-1}, \quad \forall a \in A.$$ 

In case both $M, N$ are Frechet spaces, $S$ is bi-continuous in the metric topologies. In particular, $\theta$ is bounded if both $M, N$ are Banach spaces.

**Proof.** Put

$$a^\perp = \{c \in A : ca = 0\}, \quad \text{for all nonzero } a \text{ in } A.$$
We see that \( a^\perp \subseteq b^\perp \) if and only if the closure of the range space of \( a \) contains that of \( b \). Consequently, \( a^\perp \) is maximum among all \( b^\perp \) if and only if \( a \) is of rank one. By the zero product preserving property of \( \theta \) and \( \theta^{-1} \), we see that \( \theta \) preserves the order of \( a^\perp \)'s, and thus sends the maxima onto the maxima. In other words, \( \theta \) sends rank one operators onto rank one operators. It then follows from the Fundamental Theorem of Affine Geometry that there exist linear maps \( S : M \to N \) and \( T : N \to M \) such that

\[
\theta(a) = SaT, \quad \forall a \in \mathcal{F}(M),
\]

where \( \mathcal{F}(M) \) is the algebra of all continuous finite rank operators on \( M \). In particular,

\[
\theta(x \otimes y') = Sx \otimes T'y',
\]

for every rank one operator \( x \otimes y' \) with \( x \) in \( M \), \( y' \) in the topological dual space \( M' \) of \( M \). Here, \( T' \) is the (algebraic) dual map of \( T \), and \( (x \otimes y')(z) = y'(z)x \) defines a rank at most one continuous operator on \( M \). Consequently, \( T'M' \subseteq N' \) and thus \( T \) is weak-weak continuous. Dealing with the inverse \( \theta^{-1} \), we see that \( T^{-1} \) is also weak-weak continuous. Moreover, if \( y'_2(x_1) = 0 \) then \( (x_2 \otimes y'_2)(x_1 \otimes y'_1) = 0 \). Thus,

\[
\theta(x_2 \otimes y'_2)\theta(x_1 \otimes y'_1) = 0.
\]

In other words,

\[
y'_2(x_1) = 0
\]

implies \( (T'y'_2)(Sx_1)(Sx_2 \otimes T'y'_1) = 0 \), \( \forall x_1, x_2 \in M, y'_1, y'_2 \in M' \),

implies \( y'_2(T'Sx_1) = 0 \), \( \forall x_1 \in M, y'_2 \in M' \).

By linearity, \( T = \lambda S^{-1} \) for some nonzero scalar \( \lambda \), and

\[
\theta(a) = \lambda SaS^{-1}, \quad \forall a \in \mathcal{F}(M).
\]

In general, let \( a \in A \). For any \( x \neq 0 \) in \( M \), let \( x' \in M' \) such that \( x'(x) = 1 \). Set \( b = a - (ax \otimes x') \). Observe \( b(x \otimes x') = 0 \). Thus,

\[
\theta(b)\theta(x \otimes x') = \lambda(\theta(b)Sx) \otimes (S^{-1})'x' = 0.
\]

This implies

\[
\theta(a)Sx = \lambda(Sax \otimes (S^{-1})'x')(Sx) = \lambda Sax, \quad \forall x \in M.
\]

Hence,

\[
\theta(a) = \lambda SaS^{-1}, \quad \forall a \in A.
\]

In case \( M, N \) are Frechet spaces, the Closed Graph Theorem ensures that both \( S, S^{-1} \) are continuous in the metric topology. If they are Banach spaces, then \( \theta \) is automatically bounded.
Theorem 3.2. Let $\{X, \{A_x\}, A\}$, $\{Y, \{B_y\}, B\}$ be continuous fields of primitive Banach algebras over locally compact base spaces. Let $\theta : A \to B$ be a bijective linear map such that both $\theta$, $\theta^{-1}$ preserve zero products. Suppose, in addition, at least one of the following conditions hold.

(1) $X$ (or $Y$) contains no isolated points.
(2) All fibers $A_x$ and $B_y$ are standard operator algebras.

Then $\theta$ is automatically bounded and $X, Y$ are homeomorphic. Indeed, $\theta$ assumes the standard form $(1.2)$ with all fiber linear maps being bounded.

If the case (2) holds, and $A$ (resp. $B$) is a continuous field of standard $C^*$-algebras $A_x$ (resp. $B_y$) on Hilbert spaces $H_x$ (resp. $K_y$), then there exist a homeomorphism $\varphi : Y \to X$, a bounded and away from zero continuous scalar function $\lambda$ on $Y$, a bounded invertible linear map $S_y$ from $H_{\varphi(y)}$ onto $K_y$ for each $y$ in $Y$ such that
\[
\theta(f)(y) = \lambda(y) S_y f(\varphi(y)) S_y^{-1}, \quad \forall f \in A, y \in Y.
\]

In other words, the standard form $(1.1)$ holds:
\[
\theta = \theta^{**}(1)\Psi,
\]

where the invertible central multiplier $\theta^{**}(1)$ of $B$ is represented by the operator field $y \mapsto \lambda(y) I_y$ with $I_y$ being the identity map on each fiber Hilbert space $K_y$, and the algebra isomorphism $\Psi$ is given by $\Psi(f)(y) = S_y f(\varphi(y)) S_y^{-1}$.

Proof. We first note that $Y_0 = \{\infty\}$. Moreover, it follows from $(2.2)$ that $\varphi(Y) = \varphi(Y_1) \cup \varphi(Y_2)$ is dense in $X$. Since $\varphi(Y_2)$ is a finite set of non-isolated points in $X$, we see that $\varphi(Y_1)$ alone is dense in $X$. On the other hand, let $y \in Y_1$ with $\varphi(y) = x$ in $X$, and $\psi(x) = z$ in $Y_\infty$. Here, the map $\psi : X \to Y_\infty$, and the decomposition $X = X_1 \cup X_2$ is induced by $\theta^{-1}$ in an analogous way. In particular, we have
\[
\theta(M_x) \subseteq M_y \quad \text{and} \quad \theta^{-1}(I_z) \subseteq M_z.
\]

Consequently, $I_z \subseteq \theta(M_x) \subseteq M_y$ gives $y = z \in \psi(X)$. In case $y \in \psi(X_1)$, we have $\theta(M_x) = M_y$. Since $\psi(X_2)$ is a finite set of non-isolated points in $Y$, we have $\theta(M_{\varphi(y)}) = M_y$ for all but at most finitely many $y$ in $Y_1$. Therefore, the linear map $H_y$ is bijective for all but at most finitely many $y$ in $Y_1$, which are non-isolated points in $Y$. Hence, if $\theta(f)$ vanishes on $Y_1$ then $f$ vanishes on the dense set $\varphi(Y_1)$ by $(2.1)$, and thus $f = 0$. Therefore, $Y_1$ is dense in $Y$ by the surjectivity of $\theta$. The openness of $Y_2$ forces itself to be empty.

Now, $Y = Y_1$ and $X = X_1$ imply that both $\theta$ and $\theta^{-1}$ can be written as weighted composition operators:
\[
\theta(f)(y) = H_y(f(\varphi(y))), \quad \forall f \in A, \forall y \in Y,
\]
\[
\theta^{-1}(g)(x) = T_x(g(\psi(x))), \quad \forall g \in B, \forall x \in X.
\]
It is easy to see that the linear map \( H_y : E_{\varphi(y)} \rightarrow F_y \) has \( T_y \) as the inverse for every \( y \) in \( Y \), and thus it is bijective. By Lemma 2.9, at most finitely many \( H_y \) are unbounded.

Let \( y \) be a non-isolated point in \( Y \). We shall show that the linear map \( H_y \) is bounded. Suppose not, then for each \( n = 1, 2, \ldots \) there is an \( f_n \) in \( A \) of norm one such that \( \|\theta(f_n)(y)\| = \|H_y(f_n(\varphi(y)))\| > n^4 \). By the continuity of the norm of \( \theta(f_n) \), there are all distinct points \( y_n \) in \( Y \) nearby \( y \) such that \( \|\theta(f_n)(y_n)\| > n^3 \). Let \( x_n = \varphi(y_n) \) in \( X \) for \( n = 1, 2, \ldots \). Since \( \varphi \) is a homeomorphism, we can assume also that all \( x_n \) are distinct with disjoint compact neighborhoods \( U_n \). By multiplying with a norm one continuous scalar function, we can assume each \( f_n \) is supported in \( U_n \). Let \( f = \sum_n \frac{1}{n^2}f_n \) in \( A \). Since \( n^2 f - f_n \in I_{x_n} \), we have \( n^2\theta(f)(y_n) = \theta(f_n)(y_n) \) and thus \( \|\theta(f)(y_n)\| > n \) for \( n = 1, 2, \ldots \). This absurdity tells us that \( H_y \) is bounded for all non-isolated \( y \) in \( Y \).

For the case (1), if \( Y \) (or equivalently, its homeomorphic image \( X \)) contains no isolated points then all fiber linear maps \( H_y \) are bounded. By Theorem 2.4, we have \( \|\theta\| = \sup \|H_y\| < +\infty \).

Suppose now the case (2) holds. By Proposition 3.1, each fiber linear map assumes the form \( H_y(a) = \lambda(y)S_yaS_y^{-1} \), and \( \theta \) is uniformly bounded. To see that \( \lambda \) is continuous on \( Y \), we make use of a result of Lee [14, Lemma 2] which asserts that the multiplier algebras \( M(A) \) and \( M(B) \) can be represented as families of bounded operator fields in \( (X, \{M(A_x)\}) \) and \( (Y, \{M(B_x)\}) \), respectively. By restricting the double dual map of \( \theta \) to \( M(A) \), we see that the invertible central multiplier \( \theta^{**}(1)(y) = \lambda(y)I_y \). It follows from the Dauns-Hofmann theorem (see, e.g., [15, Theorem A.34]) that \( \lambda \) is a continuous function on \( X \). Since \( \theta^{**}(1) \) is invertible, we see that \( \lambda \) is bounded and away from zero. It is also plain that the algebra isomorphism \( \Psi = \theta^{**}(1)^{-1}\theta \) is given by sending a continuous operator field \( \{f(y)\} \) to \( \{S_yf(\varphi(y))S_y^{-1}\} \).

As a special case of Theorem 3.2(2), here comes

**Theorem 3.3.** Let \( A \) and \( B \) be CCR C*-algebras with Hausdorff spectrum \( X = \hat{A} \) and \( Y = \hat{B} \), respectively. Let \( \theta : A \rightarrow B \) be a bijective linear map such that

\[
(3.1) \quad ab = 0 \text{ in } A \quad \text{if and only if} \quad \theta(a)\theta(b) = 0 \text{ in } B.
\]

Then \( \theta \) is automatically bounded. Indeed, \( \theta = m\Psi \) where \( m = \theta^{**}(1) \) is an invertible central multiplier of \( B \) and \( \Psi \) is an algebra isomorphism from \( A \) onto \( B \).

**Corollary 3.4.** Two CCR C*-algebras with Hausdorff spectrum are isomorphic as C*-algebras if and only if they have the same linear and zero product structures.
Proof. It follows from Theorem 3.3 that if there is a bijective linear map \( \theta : \mathcal{A} \to \mathcal{B} \) between two CCR C*-algebras with Hausdorff spectrum, then \( \mathcal{A} \) and \( \mathcal{B} \) are algebraically isomorphic (via the map \( \Psi = \theta^{**}(1)^{-1}\theta \)). As shown in [10] (see also [16, Theorem 4.1.20]), \( \mathcal{A} \) and \( \mathcal{B} \) are also *-isomorphic. On the other hand, the norm of an element \( a \) of a C*-algebra equals the square root of the spectral radius of \( a^*a \), which is a *-algebraic property. So \( \mathcal{A} \) and \( \mathcal{B} \) are isometrically *-isomorphic. \( \square \)

Remark 3.5. (a) The two way zero product preserving assumption (3.1) in Theorem 3.3 cannot be dropped easily. For example, abelian C*-algebras \( C_0(X) \) are CCR. In [4], there are many examples of unbounded zero product preserving linear functionals of \( C_0(X) \), provided \( X \) is an infinite set. In [12], an unbounded zero product preserving linear map from \( c \) onto \( \ell_\infty \) is given, where both \( c \), the C*-algebra of convergent scalar sequences, and \( \ell_\infty \) are CCR with Hausdorff spectrum. (b) In [9], Fell defines the notion of a full algebra of operator fields \( \mathcal{A} \) as those satisfying conditions (i), (ii), (iii) in Definition 2.1 and \( \mathcal{A} \) becomes a C*-algebra in Definition 2.2. Fell calls those satisfied in addition condition (iv) in Definition 2.1 a maximal full algebra of operator fields. He has pointed out that \( \mathcal{A} \) is maximal if and only if for all \( \alpha_x \) in a fiber algebra \( A_x \) and \( \beta_y \) in another fiber \( A_y \) there is a continuous field \( a \) in \( \mathcal{A} \) such that \( a(x) = \alpha_x \) and \( a(y) = \beta_y \). This is also equivalent to saying that for all \( a \) in \( \mathcal{A} \), and for all bounded complex scalar continuous function \( g \) on \( X \), we have \( ga \in \mathcal{A} \). In our discussion, we follow the usage of notations of Dixmier [8] and simply assume that all continuous fields are maximal. (c) We note that every C*-algebra with Hausdorff spectrum can be represented as a continuous field of primitive C*-algebras over the spectrum [15, §5.1]. Hence, Theorems 2.4 and 3.2 apply to every zero product preserving linear map between two C*-algebras with Hausdorff spectrum. (d) It is pointed out by Fell in [9, p. 243] that a CCR C*-algebra has Hausdorff spectrum if and only if it can be represented as a (maximal) continuous field of primitive C*-algebras over some locally compact Hausdorff base space. (e) One might observe that Theorem 3.3 can be extended to GCR C*-algebras. However, for a GCR C*-algebra \( \mathcal{A} \) with Hausdorff spectrum, \( \mathcal{A} \) is automatically a CCR, and thus nothing new can be achieved in this plausible generality. Indeed, a separable C*-algebra is a GCR (resp. CCR) if and only if its spectrum is \( T_0 \) (resp. \( T_1 \)); see, e.g., [5]. In general, a GCR C*-algebra is a CCR if and only if its spectrum is \( T_1 \) ([11, Theorem 4]).

To end the paper we present another example as an evident to support our general conjecture that linear and zero product structures suffice to determine a C*-algebra.
Example 3.6 ([6]). Let $\mathcal{M}$ be a properly infinite $W^*$-algebra and $\theta$ a zero product preserving linear map from $\mathcal{M}$ onto a unital algebra $\mathcal{N}$. Then

$$\theta(a) = \theta(1)\Psi(a), \quad \text{for all } a \in \mathcal{M},$$

where $\theta(1)$ is an invertible element in the center of $\mathcal{N}$ and $\Psi$ is an algebra homomorphism from $\mathcal{M}$ onto $\mathcal{N}$. In particular, if $\mathcal{N}$ is a semi-simple Banach algebra then $\theta$ is automatically bounded, by, e.g., a result of Aupetit [3] which ensures that every surjective algebra homomorphism between semi-simple Banach algebras is bounded.

REFERENCES


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