RANK PRESERVING IN INTEGRAL EXTENSIONS OF COMMUTATIVE $C^\ast$-ALGEBRAS

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Abstract. Let $A$, $B$ be two regular commutative unital Banach algebras such that $B$ is integral over $A$. In 2003, Dawson and Feinstein showed that the topological stable rank $\text{tsr}(B) = 1$ whenever $\text{tsr}(A) = 1$. In this note, we investigate whether we will have $\text{tsr}(A) = \text{tsr}(B)$ in general. For instance, when $A$ is a commutative unital $C^\ast$-algebra, we show that $\text{tsr}(A) \leq \text{tsr}(B)$, and the equality holds at least when the integral extension is separable. In general, $A$ and $B$ have the same Bass stable ranks $\text{Bsr}(A) = \text{Bsr}(B)$.

1. INTRODUCTION

Generalizing the concept of the covering dimension (see Section 2 for definitions), Rieffel [10] introduced the concept of the topological stable rank of a $C^\ast$-algebra in 1983. Motivated by the fact that closed subspaces have smaller covering dimensions, we may ask whether Banach subalgebras have smaller topological stable ranks. As an evidence, Dawson and Feinstein [5] showed in 2003 that the property having topological stable rank one is inherited by integral extensions of commutative Banach algebras. They first proved this for Arens-Hoffman extensions, and then extended the result to general integral extensions.

Let $A$ be a commutative unital normed algebra, and $\alpha(z)$ be a monic polynomial over $A$, that is $\alpha(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$ with $a_{n-1}, \ldots, a_0 \in A$. Here, we do not assume the irreducibility of $\alpha(z)$. No matter $\alpha(z)$ has a root in $A$ or not, we can enlarge $A$ to a commutative normed algebra $A_{\alpha}$, the so called Arens-Hoffman extension [1] of $A$ with respect to $\alpha(z)$, such that $\alpha(z)$ always has a new root in $A_{\alpha}$. As a simple ring extension, $A_{\alpha}$ is defined to be $A[z]/(\alpha(z))$. 
where \((\alpha(z))\) is the principal ideal in the polynomial ring \(A[z]\) generated by \(\alpha(z)\). The norm of \(A_\alpha\) is defined by

\[
\| \sum_{i=0}^{n-1} b_i z^i + (\alpha(z)) \| = \sum_{i=0}^{n-1} \| b_i \| t^i,
\]

where \(\sum_{i=0}^{n-1} b_i z^i \in A[z]\), and \(t\) is any but fixed positive number such that \(t^n \geq \sum_{i=0}^{n-1} \| a_i \| t^i\). Note that \(A_\alpha\) is complete whenever \(A\) is. Arens and Hoffman show that this extension is an isometry. Thus the algebraic and topological properties of \(A_\alpha\) should be closely tied with those of \(A\).

A natural question arises: Which topological and/or algebraic properties of \(A\) will be preserved to \(A_\alpha\) under the Arens-Hoffman construction. The readers can refer to Dawson’s paper [6, Table 2.2] for a list of partial answers to this question. Note that, in general, Arens-Hoffman extensions of a commutative unital \(C^*\)-algebra may not be a \(C^*\)-algebra. But from the table [6, Table 2.2], we know if the monic polynomial \(\alpha(z)\) is separable, that is, the discriminant (cf. [3]) of \(\alpha(z)\) is invertible in \(A\), then the Arens-Hoffman extension preserves these properties: semisimplicity, self-adjointness and supnorm closedness. Therefore, equipped with the supnorm (the Gelfand norm), \(A_\alpha\) becomes a \(C^*\)-algebra if \(\alpha\) is separable. It follows from the Stone-Weierstrass Theorem that \(A_\alpha \cong \mathcal{C}(\Sigma_{A_\alpha})\), where \(\Sigma_{A_\alpha}\) is the maximal ideal space of \(A_\alpha\). It follows from the open mapping theorem that the Arens-Hoffman norm and the supnorm are equivalent.

Dawson and Feinstein shows

**Theorem 1.** [5, Theorem 2.1]. Let \(A\) be a commutative unital Banach algebra and \(\alpha(z)\) a monic polynomial over \(A\). If \(\text{tsr}(A) = 1\), then \(\text{tsr}(A_\alpha) = 1\).

Using Theorem 1, they further show in [5] that if \(B\) is a commutative unital Banach algebra, which is an integral extension of \(A\) (i.e., every element in \(B\) is a root of a monic polynomial in \(A[z]\)), then \(\text{tsr}(A) = 1\) implies \(\text{tsr}(B) = 1\). It is interesting to ask the question

"whether the topological stable rank is inherited for Arens-Hoffman extensions, or generally, integral extensions of regular commutative unital Banach algebras, that is, whether we will always have \(\text{tsr}(A) = \text{tsr}(A_\alpha)\) and \(\text{tsr}(A) = \text{tsr}(B)\)."

We shall show in Section 3 that they have equal Bass stable ranks \(\text{Bsr}(A) = \text{Bsr}(A_\alpha) = \text{Bsr}(B)\). Assuming \(A\) is a commutative unital \(C^*\)-algebra, we shall have \(\text{tsr}(A) \leq \text{tsr}(A_\alpha)\), and \(\text{tsr}(A) = \text{tsr}(A_\alpha)\) if \(\alpha\) is separable. Moreover, \(\text{tsr}(A) \leq \text{tsr}(B)\), and \(\text{tsr}(A) = \text{tsr}(B)\) whenever the integral extension of \(A\) to \(B\) is separable.
More precisely, we will show that \( \dim \Sigma_A = \dim \Sigma_{A_n} = \dim \Sigma_B \) and the assertions follow.

Finally, we would like to point out that although it might be possible to derive our results, Theorems 11 and 12, by using arguments in [9, Chapter 9, Proposition 2.16], here we give a seemingly more direct and more elementary proof, modelling on the one in a paper of Deckard and Pearcy [7, Theorem 1]. We hope this approach would provide more insights into the problem.

2. Preliminaries

**Definition 2.** ([9, Chapter 3]). Let \( X \) be a topological space.

1. The order of a covering \( \{U_\lambda\}_{\lambda \in \Lambda} \) of \( X \) is the least integer \( n \) such that every point of \( X \) is contained in at most \( n + 1 \) elements of \( \{U_\lambda\}_{\lambda \in \Lambda} \). If no such integer exists, we say that the order is \( \infty \).

2. The covering dimension \( \dim X \) of \( X \) is the least integer \( n \) such that every finite open covering of \( X \) has an open refinement of order not exceeding \( n \). If no such integer exists, we say that the covering dimension \( \dim X \) is \( \infty \).

**Theorem 3.** ([9, Chapter 3]). Let \( X \) be a normal topological space.

1. If \( Y \) is a closed subspace of \( X \), then \( \dim Y \leq \dim X \).

2. If \( X = \bigcup_{i \in \mathbb{N}} A_i \) is a countable \( F_\sigma \)-set covering of \( X \), then \( \dim X = \sup_{i \in \mathbb{N}} \dim A_i \).

**Theorem 4.** ([9, Theorem 3.5.6]). Let \( A \) be a closed subset of a normal space \( X \). If \( \dim A \leq n \) and if \( \dim F \leq n \) for each closed subset \( F \) of \( X \) which does not meet \( A \), then \( \dim X \leq n \).

**Theorem 5.** ([9, Corollary 8.1.7]). If a space \( X \) is the inverse limit (i.e. the projective limit) of an inverse system \( \{X_\alpha, \pi_{\alpha\beta}\}_{\alpha, \beta \in \Omega} \) of nonempty compact Hausdorff spaces with \( \dim X_\alpha \leq n \) for each \( \alpha \) in \( \Omega \), then \( \dim X \leq n \).

Let \( A \) be a unital Banach algebra. We use the notation \( Lg_n(A) \) (resp. \( Rg_n(A) \)) to denote the set of \( n \)-tuples of elements of \( A \) which generates \( A \) as a left (resp. right) ideal, that is, \( A = \{ \sum_{i=1}^{n} a_i x_i : a_i \in A \} \) (resp. \( A = \{ \sum_{i=1}^{n} x_i a_i : a_i \in A \} \)) whenever \( (x_i) \in Lg_n(A) \) (resp. \( Rg_n(A) \)). In algebraic K-theory, the elements of \( Lg_n(A) \) (resp. \( Rg_n(A) \)) are traditionally called left (resp. right) unimodular rows.

In this note, we investigate two concepts of dimensions of Banach algebras. The first one is the Bass stable rank introduced by Bass [2].

**Definition 6.** ([2, Section 1.4]). The Bass stable rank \( \text{Bsr}(A) \) of a unital Banach algebra \( A \) is defined to be the least integer \( n \) such that for any \( (a_1, a_2, \ldots, a_n, a_{n+1}) \) in \( Lg_{n+1}(A) \) there is a \( (b_1, b_2, \ldots, b_n) \) in \( A^n \) with

\[
(a_1 + b_1 a_{n+1}, a_2 + b_2 a_{n+1}, \ldots, a_n + b_n a_{n+1}) \in Lg_n(A).
\]
If there is no such \( n \) then set \( Bsr(A) = \infty \).

The second one is the topological stable rank introduced by Rieffel [10].

**Definition 7.** ([10, Definition 1.4]). By the left (resp. right) topological stable rank of a unital Banach algebra \( A \), denoted by \( ltsr(A) \) (resp. \( rtsr(A) \)), we mean the least integer \( n \) such that \( Lg_n(A) \) (resp. \( Rg_n(A) \)) is dense in \( A^n \) (in the product topology). If no such integer exists we set \( ltsr(A) = \infty \) (resp. \( rtsr(A) = \infty \)). If \( A \) does not have an identity, then its topological stable rank is defined to be that of the Banach algebra \( \tilde{A} \) obtained from \( A \) by adjoining an identity.

For a commutative Banach algebra, it is clear that the left and right topological stable ranks are equal, and so is the case of a Banach algebra with a continuous involution. On the other hand, the left topological stable rank of a unital Banach algebra is 1 if and only if its right topological stable rank is 1. By definition, a unital Banach algebra has topological stable rank one if and only if it has a dense invertible group. We write \( tsr(A) \) for \( ltsr(A) \) in case \( ltsr(A) = rtsr(A) \).

**Theorem 8.** ([4], [10]). If \( X \) is a compact Hausdorff space, then
\[
\text{tsr}(C(X)) = Bsr(C(X)) = \left\lceil \frac{\dim X}{2} \right\rceil + 1.
\]
In general, for a regular commutative unital Banach algebra \( A \), we have
\[
\text{tsr}(A) \geq Bsr(A) = \left\lceil \frac{\dim \Sigma_A}{2} \right\rceil + 1,
\]
where \( \Sigma_A \) is the maximal ideal space of \( A \), and \( \lceil r \rceil \) is the greatest integral part of a real number \( r \).

We also need the following two results to prove our theorems.

**Lemma 9.** ([7, Lemma 2.2]). Suppose \( X \) is any topological space. Let
\[
P(x, z) = z^n + a_{n-1}(x)z^{n-1} + \cdots + a_1(x)z + a_0(x),
\]
where all \( a_i \in C(X) \). Fix an \( x_0 \) in \( X \) and let \( z_0 \) be a root of multiplicity \( \mu \) of the polynomial \( P(x_0, z) \) in \( z \in \mathbb{C} \). If \( \delta > 0 \) so that \( P(x_0, z) \) has no root other than \( z_0 \) satisfying \( |z - z_0| < \delta \), then there is a neighborhood \( U \) of \( x_0 \) such that for all \( x \) in \( U \), the polynomial \( P(x, z) \) has exactly \( \mu \) roots (counting multiplicities) satisfying \( |z - z_0| < \delta \).

**Theorem 10.** ([8, Theorem 3.3]). Let \( A \) be a commutative Banach algebra and \( B \) an integral extension of \( A \). If \( \pi^{-1}(h) \) is an infinite subset of \( \Sigma_B \) for some \( h \in \Sigma_A \), then \( B \) is incomplete under any norm.
3. The Results

**Theorem 11.** Let $A$ be a commutative unital normed algebra with maximal ideal space $\Sigma_A$, and let

$$\alpha(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$$

be a monic polynomial over $A$ with $n > 1$. Let $A_{\alpha}$ be the Arens-Hoffman extension of $A$ with respect to $\alpha(z)$. Then

$$\dim \Sigma_A = \dim \Sigma_{A_{\alpha}}.$$

**Proof.** As mentioned in the paper of Arens and Hoffman [1], the maximal ideal space $\Sigma_{A_{\alpha}}$ of $A_{\alpha}$ is contained in $\Sigma_A \times \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. More precisely, $\Sigma_{A_{\alpha}}$ consists exactly of all $(h, z)$ such that

$$(h, z)(\alpha) = z^n + a_{n-1}(h)z^{n-1} + \cdots + a_1(h)z + a_0(h) = 0.$$ 

We define $P : \Sigma_A \times \mathbb{C} \to \mathbb{C}$ by

$$P(h, z) = z^n + a_{n-1}(h)z^{n-1} + \cdots + a_1(h)z + a_0(h)$$

and $\pi : \Sigma_{A_{\alpha}} \to \Sigma_A$ by

$$\pi(h, z) = h \quad \text{or} \quad \pi(\tilde{h}) = \tilde{h}|_A, \quad \text{where } \tilde{h} = (h, z) \in \Sigma_{A_{\alpha}}.$$ 

Note that for every $h_0$ in $\Sigma_A$, there is a $\delta > 0$ such that the closed balls $B(z_i, \delta)$ in $\mathbb{C}$ are pairwise disjoint, where $z_i$ are distinct roots of the complex polynomial $P(h_0, z)$. From Lemma 9, there is a compact neighborhood $V_{h_0}$ of $h_0$ such that $V_{h_0} = \pi(V_i)$ where $V_i = (V_{h_0} \times B(z_i, \delta)) \cap \Sigma_{A_{\alpha}}$ are pairwise disjoint compact neighborhoods of $(h_0, z_i)$ and $\pi^{-1}(V_{h_0}) = \cup V_i$. Note that for every $h'$ in $V_{h_0}$, the complex polynomial $P(h', z)$ has exactly $k$ roots (counting multiplicities) in the open balls $B(z_i, \delta)$ as $z_i$ is a root of $P(h_0, z)$ of multiplicity $k$.

**Claim.** For every point $(h_0, z_0)$ in $\Sigma_{A_{\alpha}}$, there are $\sigma$-compact neighborhoods $U_{h_0}$ of $h_0$ in $\Sigma_A$ and $U_{(h_0, z_0)}$ of $(h_0, z_0)$ in $\Sigma_{A_{\alpha}}$ such that

$$\dim U_{h_0} = \dim U_{(h_0, z_0)}.$$ 

We prove the claim by induction on the multiplicity of the root $z_0$ of the complex polynomial $P(h_0, z)$.

Let $(h_0, z_0)$ be in $\Sigma_{A_{\alpha}}$ with $z_0$ being a root of $P(h_0, z)$ of multiplicity 1. By Lemma 9 with $\delta > 0$ mentioned above, $\pi|_{V_0} : V_0 \to V_{h_0}$ is a homeomorphism from the compact neighborhood $V_0$ of $(h_0, z_0)$ onto the compact neighborhood $V_{h_0}$ of $h_0$. Set $\tilde{U}_{(h_0, z_0)} = V_0$ and $U_{h_0} = V_{h_0}$. Then $\dim U_{h_0} = \dim U_{(h_0, z_0)}$. 

Now suppose that for each \((h_0, z_0)\) in \(\Sigma_{A_0}\) with \(z_0\) being a root of \(P(h_0, z)\) of multiplicity less than \(k\), there are \(\sigma\)-compact neighborhoods \(U_{h_0}\) of \(h_0\) in \(\Sigma_A\) and \(U_{(h_0, z_0)}\) of \((h_0, z_0)\) in \(\Sigma_{A_0}\) such that \(\dim U_{h_0} = \dim U_{(h_0, z_0)}\). Let \((h_0, z_0)\) \(\in \Sigma_{A_0}\) with \(z_0\) being a root of \(P(h_0, z)\) of multiplicity \(k\). By Lemma 9 with \(\delta > 0\) mentioned above, there is a compact neighborhood \(V_{h_0}\) of \(h_0\) such that for \(h^\prime\) in \(V_{h_0}\), the complex polynomial \(P(h^\prime, z)\) has exactly \(k\) roots (counting multiplicities) in the open ball \(B(z_0, \delta)\). Furthermore, we can assume that for \(h^\prime\) in \(V_{h_0}\), the complex polynomial \(P(h^\prime, z)\) has no root satisfying \(|z - z_0| = \delta\). Let \(P_m \subset V_{h_0}\) be the set of points \(h^\prime\) for which \(P(h^\prime, z)\) has two distinct roots with distance not less than \(\frac{1}{m}\) and they lie in the open ball \(B(z_0, \delta)\). Let \(P = \bigcup_{m=1}^{\infty} P_m\).

For \(h^\prime\) in \(P\), the complex polynomial \(P(h^\prime, z)\) has two distinct roots in the open ball \(B(z_0, \delta)\). By Lemma 9, \(h^\prime\) has a neighborhood such that for every point \(h\) in this neighborhood, \(P(h, z)\) has two distinct roots in the open ball \(B(z_0, \delta)\). Hence \(P\) is relatively open in \(V_{h_0}\) and thus \(V_{h_0} \setminus P\) is relatively closed in \(V_{h_0}\). For every \(h^\prime\) in \(V_{h_0} \setminus P\), the polynomial \(P(h^\prime, z)\) has exactly one root of multiplicity \(k\) in the open ball \(B(z_0, \delta)\). As argued for the case of multiplicity one above, we can prove that 
\[
\dim (V_{h_0} \setminus P) = \dim \pi^{-1}(V_{h_0} \setminus P) \cap V_0.
\]

We next show that \(P_m\) is relatively closed in \(V_{h_0}\) for \(m = 1, 2, \ldots\). If \(P_m\) is nonempty, let \(\{h_\lambda\}\) be a net in \(P_m\) such that \(h_\lambda \to h_1\) for some \(h_1\) in \(V_{h_0}\), and for each \(\lambda\), let \(z_{1\lambda}\) and \(z_{2\lambda}\) be roots of \(P(h_\lambda, z)\) satisfying \(|z_{1\lambda} - z_{2\lambda}| \geq \frac{1}{m}\) and \(|z_{1\lambda} - z_0| < \delta\). Choose a subnet \(\{h_\sigma\}\) of \(\{h_\lambda\}\) with the property that the nets \(\{z_{1\sigma}\}\) and \(\{z_{2\sigma}\}\) converge to some \(z_1\) and \(z_2\), respectively. Then, since \(P(h, z)\) is jointly continuous in \(h\) and \(z\), \(P(h_1, z_1) = P(h_1, z_2) = 0\), and \(|z_1 - z_2| \geq \frac{1}{m}\). By our previous arrangements we can not have \(|z_i - z_0| = \delta\). So \(|z_i - z_0| < \delta\) for \(i = 1, 2\). Thus \(h_1 \in P_m\). and consequently, \(P_m\) is relatively closed and hence compact.

On the other hand,
\[
P'_m := \pi^{-1}(P_m) \cap V_0
\]
is also compact for each \(m\). For each point \((h^\prime, z^\prime)\) in \(P'_m\), we have \(z^\prime\) is a root of \(P(h^\prime, z)\) of multiplicity less than \(k\). By the induction hypothesis, there are \(\sigma\)-compact neighborhoods \(U_{h^\prime}\) of \(h^\prime\) in \(\Sigma_A\) and \(U_{(h^\prime, z^\prime)}\) of \((h^\prime, z^\prime)\) in \(\Sigma_{A_0}\) such that 
\[
\dim U_{h^\prime} = \dim U_{(h^\prime, z^\prime)}.
\]
Consider the covering \(\{U_{(h, z)} : (h, z) \in P'_m\}\) of \(P'_m\). By the compactness of \(P'_m\), it has a finite subcovering \(\{U_{(h_i, z_i)} : (h_i, z_i) \in P'_m\}_{i=1}^{n}\). Let \(\tilde{P}'_m = \bigcup_{i=1}^{n} U_{(h_i, z_i)}\) and \(\tilde{P}_m = \bigcup_{i=1}^{n} U_{h_i}\). Note that \(\tilde{P}'_m\) and \(\tilde{P}_m\) are \(\sigma\)-compact and 
\[
\dim \tilde{P}_m = \dim \tilde{P}'_m.
\]

Let 
\[
U_{h_0} = (V_{h_0} \setminus P) \cup \bigcup_{m=1}^{\infty} P_m = V_{h_0} \cup \bigcup_{m=1}^{\infty} \tilde{P}_m
\]
and
\[ U_{(h_0, z_0)} = (\pi^{-1}(V_{h_0} \setminus \mathcal{P}) \cap V_0) \cup \bigcup_{m=1}^{\infty} \mathcal{P}'_m = V_0 \cup \bigcup_{m=1}^{\infty} \mathcal{P}'_m. \]

Note that \( U_{h_0} \) and \( U_{(h_0, z_0)} \) are \( \sigma \)-compact neighborhoods of \( h_0 \) and \( (h_0, z_0) \), respectively. By the Sum Theorem again, we have \( \dim U_{h_0} = \dim U_{(h_0, z_0)} \), and we have verified the claim.

Finally, using the compactness of \( \Sigma_A \) and \( \Sigma_{A_n} \), we see that \( \Sigma_{A_n} \) and \( \Sigma_A \) have finite \( \sigma \)-compact neighborhood covering \( \{ U_{(h_i, z_i)} \}_{i=1}^{n} \) and \( \{ U_{h_i} \}_{i=1}^{n} \), respectively. Since \( \dim U_{h_i} = \dim U_{(h_i, z_i)} \) for \( i = 1, \ldots, n \), the Sum Theorem furnishes that \( \dim \Sigma_A = \dim \Sigma_{A_n} \).

**Theorem 12.** Let \( A \) and \( B \) be commutative unital Banach algebras such that \( B \) is an integral extension of \( A \). Then
\[ \dim \Sigma_A = \dim \Sigma_B. \]

**Proof.** Similar to the case of the Arens-Hoffman extension, we can embed \( \Sigma_B \) into the product space \( \Sigma_A \times \mathbb{C}^{B \setminus A} \) as the subset consisting of points \( (h, \{ h_b \}_{b \in B \setminus A}) \) such that \( h(\alpha)(h_b) = 0 \) whenever \( \alpha(b) = 0 \), for all \( b \in B \setminus A \) and for all \( \alpha \) in \( A[z] \).

In addition, for each \( b \in B \setminus A \), fix a minimal monic polynomial \( \alpha_b \) in \( A[z] \) such that \( \alpha_b(0) = 0 \). Let \( \Lambda = \{ \alpha_b : b \in B \setminus A \} \) and \( \mathcal{F} \) be the collection of all finite subsets of \( \Lambda \). Then \( \Sigma_B \) is the limit of the inverse system \( \{ \Sigma_{A_n}, \pi_{\alpha, \beta} \}_{\alpha, \beta \in \mathcal{F}} \). By Theorems 5 and 11, we have \( \dim \Sigma_B \leq \sup_i \dim \Sigma_{A_n} = \dim \Sigma_A \).

Now we prove the reverse inequality. By Theorem 10, we have \( \Sigma_A = \bigcup_{i=1}^{\infty} U_i \) where \( U_i \) is the collection of \( h \) in \( \Sigma_A \) which has at most \( i \) extensions in \( \Sigma_B \). Every \( U_i \) is closed in \( \Sigma_A \) and thus compact. Note that
\[ U_1 \cong \pi^{-1}(U_1), \]
where \( \pi : \Sigma_B \to \Sigma_A \) is defined by \( \pi(h, z) = h \). Thus, \( \dim U_1 = \dim \pi^{-1}(U_1) \leq \dim \Sigma_B \) by Theorem 3(1). Suppose that \( \dim U_{i-1} \leq \dim \Sigma_B \). For \( h \in U_i \setminus U_{i-1} \), by definition \( h \) has exactly \( i \) extensions \( h_1, \ldots, h_i \) to \( B \). Choose a point \( b \in B \setminus A \) such that \( h_j(b) \neq h_k(b) \), \( \forall 1 \leq j < k \leq i \). By Lemma 9, \( h \) has a compact neighborhood \( U_h \) such that \( U_h \cap U_{i-1} = \emptyset \) and \( h \) has an extension belonging to \( U_h \times \mathcal{F}(h_j(b), \delta) \times \mathbb{C}^{B \setminus \{ A \cup \{ b \} \}} \) for \( 1 \leq j \leq i \). Then
\[ U_h \cap U_i \cong \pi^{-1}(U_h) \cap (U_h \times \mathcal{F}(h_j(b), \delta) \times \mathbb{C}^{B \setminus \{ A \cup \{ b \} \}}) \]

It follows from Theorem 3(1) that \( \dim U_h \cap U_i \leq \dim \Sigma_B \).

Let \( V \) be a closed subset of \( U_i \) such that \( V \cap U_{i-1} = \emptyset \). By the above arguments, every point \( h \) in \( V \) has a compact neighborhood \( U_h \) such that \( \dim U_h \cap V \leq \dim \Sigma_B \).
dim \( U_h \cap U_i \leq \dim \Sigma_B \). By the compactness of \( V \), \( V = \bigcup_{m=1}^n U_{h_m} \cap V \). Thus \( \dim V = \max \dim U_{h_m} \cap V \leq \dim \Sigma_B \). By Theorem 4, \( \dim U_i \leq \dim \Sigma_B \). By the induction hypothesis and Theorem 3(2), we have \( \dim \Sigma_A = \sup \{ \dim U_i \} \leq \dim \Sigma_B \).

**Theorem 13.** Let \( A \) be a regular commutative unital Banach algebra.

(1) Let \( \alpha(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \) be a monic polynomial over \( A \) with \( n > 1 \). Then the Bass stable ranks satisfy

\[
\text{Bsr}(A) = \text{Bsr}(A_{\alpha}).
\]

(2) Let \( B \) be a regular integral extension of \( A \). Then the Bass stable ranks satisfy

\[
\text{Bsr}(A) = \text{Bsr}(B).
\]

**Proof.**

(1) Since \( A \) is a regular commutative Banach algebra, \( A_{\alpha} \) is also regular (see, e.g. [6]). So case (1) reduces to case (2).

(2) By Theorem 8 and Theorem 12, we have

\[
\text{Bsr}(A) = \left[ \frac{\dim \Sigma_A}{2} \right] + 1 = \left[ \frac{\dim \Sigma_B}{2} \right] + 1 = \text{Bsr}(B).
\]

**Theorem 14.** Let \( A \) be a commutative unital \( C^* \)-algebra.

(1) Let \( \alpha(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \) be a monic polynomial over \( A \) with \( n > 1 \). Then the topological stable ranks satisfy

\[
\text{tsr}(A) \leq \text{tsr}(A_{\alpha}).
\]

They are equal if \( \alpha(z) \) is separable.

(2) Let \( B \) be a regular integral extension of \( A \). Then

\[
\text{tsr}(A) \leq \text{tsr}(B).
\]

If, in addition, every element in \( B \) is a root of a separable monic polynomial over \( A \), then

\[
\text{tsr}(A) = \text{tsr}(B).
\]

**Proof.**

(1) By Theorems 8 and 11, we have \( \text{tsr}(A) \leq \text{tsr}(A_{\alpha}) \). If \( \alpha(z) \) is separable, \( A_{\alpha} \) is a \( C^* \)-algebra in the supnorm. Thus, \( \text{tsr}(A) = \text{tsr}(A_{\alpha}) \).
(2) By Theorem 8 and Theorem 12, we have
\[ \text{tsr}(A) = \left[ \frac{\dim \Sigma_A}{2} \right] + 1 = \left[ \frac{\dim \Sigma_B}{2} \right] + 1 \leq \text{tsr}(B). \]
Suppose that \( \text{tsr}(A) = n \). Let \((a_1, a_2, \ldots, a_n) \in B^n\), and \(a_i\) be a root of a separable monic polynomial \(\alpha_i(z)\) in \(A[z]\). Consider the Arens-Hoffman extension \(A_{\alpha_1\ldots\alpha_n}\) of \(A\) and let \(x_i\) be the root of \(\alpha_i(z)\) in the Arens-Hoffman construction \(A_{\alpha_1\ldots\alpha_n}\). Then there is a continuous homomorphism \(\phi\) from \(A_{\alpha_1\ldots\alpha_n}\) into \(B\) with respect to the Arens-Hoffman norm, which fixes \(A\) and maps \(x_i\) to \(a_i\). By (1), we have \(\text{tsr}(A_{\alpha_1\ldots\alpha_n}) = n\). Thus \((x_1, \ldots, x_n)\) can be approximated by elements in \(L_{\alpha_n}(A_{\alpha_1\ldots\alpha_n})\). By the continuity of \(\phi\), \((a_1, \ldots, a_n)\) can be approximated by elements in \(L_{\alpha_n}(B)\). This shows that \(\text{tsr}(B) = n\).

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