PERTURBED ITERATIVE METHODS FOR A GENERAL FAMILY OF OPERATORS: CONVERGENCE THEORY AND APPLICATIONS

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Abstract. We study perturbations of a hybrid steepest descent method for locating common fixed points of an arbitrary pool $\{T_\lambda\}$ of nonexpansive mappings. The difficulty of handling a possibly uncountable family is settled down to dealing with a countable family of auxiliary mappings $\{S_n\}$ associated to $\{T_\lambda\}$, in the sense that the approximate fixed points of $\{S_n\}$ provide the common fixed points of $\{T_\lambda\}$. Algorithms with strong convergence for solving the associated variational inequality problems are presented. Applications to convex minimization problems and convex feasibility problems are provided, together with numerical examples for comparisons of our algorithms with the existing ones.

1. Introduction

The *convex minimization problem* [3,16,18,26] of a differentiable convex function ψ subject to a closed convex set C of a (real) Hilbert space $\mathcal{H}$ assumes the form:

$$\text{find } x^* \in C \quad \text{such that } \psi(x^*) = \min\{\psi(x) : x \in C\}. \tag{1.1}$$

It can be casted into the variational inequality problem:

$$\text{find } u \in C \quad \text{such that } \langle \nabla\psi(u), z - u\rangle \geq 0 \text{ for all } z \in C,$$

where $\nabla\psi : \mathcal{H} \to \mathcal{H}$ is the gradient of ψ. In general, for a nonlinear mapping $F : \mathcal{H} \to \mathcal{H}$, the *variational inequality problem* (in short, VIP) over a nonempty closed convex subset C of $\mathcal{H}$ is:

$$\text{find } u \in C \quad \text{such that } \langle F(u), z - u\rangle \geq 0 \text{ for all } z \in C. \tag{1.2}$$

The problem (1.2) is denoted by $\mathrm{VIP}_C(F, \mathcal{H})$. VIP is a popular research subject recently. See, e.g., [30].

It is well known that if F is η-strongly monotone and $L-$Lipschitz continuous, then for $\mu \in (0, 2\eta/L^2)$, the mapping $P_C(I - \mu F)$ is a contraction of $\mathcal{H}$ onto C, and hence $\mathrm{VIP}_C(F, \mathcal{H})$ has a unique solution $x^* \in C$, and the *projection gradient method*:

$$x_{n+1} = P_C(I - \mu F)x_n, \quad n \in \mathbb{N},$$

converges strongly to x^* (see [42, Theorem 46.C]).

The computation of the metric projection P_C of $\mathcal{H}$ onto C is not necessarily easy. To overcome this difficulty, when $C = \mathrm{Fix}(T)$ is the nonempty fixed point set of a nonexpansive mapping $T : \mathcal{H} \to \mathcal{H}$, Yamada introduced in [41, Theorem 3.3, p. 486] the following *hybrid steepest descent method* for solving $\mathrm{VIP}_{\mathrm{Fix}(T)}(F, \mathcal{H})$:

$$x_{n+1} = (I - \alpha_n \mu F)Tx_n, \quad n \in \mathbb{N}, \tag{1.3}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1]$. Yamada proved that the sequence $\{x_n\}$ defined by (1.3) converges strongly to a unique solution of $\mathrm{VIP}_{\mathrm{Fix}(T)}(F, \mathcal{H})$.

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In the case when

$$C = \bigcap_{i=1}^{N} \operatorname{Fix}(T_i) = \operatorname{Fix}(T_1 T_2 \cdots T_N) = \operatorname{Fix}(T_N T_1 \cdots T_{N-1}) = \cdots = \operatorname{Fix}(T_2 T_3 \cdots T_N T_1)$$

for a finite family $\{T_1, T_2, \cdots, T_N\}$ of nonexpansive mappings, Yamada [41] studied the strong convergence of the following algorithm:

$$x_{n+1} = (I - \alpha_n \mu F) T_{[n+1]} x_n, \quad n \in \mathbb{N}.$$

Here, $T_{[r]} = T_{r \bmod N}$ for $r \in \mathbb{N}$, and the sequence $\{\alpha_n\}$ satisfies the conditions:

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = +\infty, \quad \text{and} \ \ (\alpha_n - \alpha_{n+1})/\alpha_{n+1}^2 \to 0.$$

With a different set of assumptions, Xu and Kim established similar results in [40]. On the other hand, using an idea of Kuhfittig [23], Atsushiba and Takahashi [35] introduced W-mappings for computing common fixed points of a finite family of nonexpansive mappings. For other references and procedures for the finite case, one can read [25, 36].

The case when $C = \bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n)$ is the nonempty common fixed point set of a countably infinite family $\{T_n\}$ of nonexpansive mappings is also interesting. Auxiliary mappings S_n are constructed, for example, as the convex combinations of T_n and the identity mapping I of $\mathcal{H}$. Such auxiliary mappings play important roles in locating points in $\bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n)$. See, e.g., [6, 43]. A natural question aries for constructing such sequences $\{S_n\}$ of auxiliary mappings for a possibly uncountable family $\mathcal{T} = \{T_\lambda : \lambda \in \Lambda\}$.

In this paper, we investigate an inexact hybrid steepest descent-like method for solving the following general variational inequality problem:

$$\text{find } u \in \bigcap_{\lambda \in \Lambda} \operatorname{Fix}(T_\lambda) \quad \text{such that } \langle F(u), z - u \rangle \geq 0 \text{ for all } z \in \bigcap_{t \in \Lambda} \operatorname{Fix}(T_\lambda). \tag{1.4}$$

Here all T_λ are nonexpansive mappings such that $C = \bigcap_{\lambda \in \Lambda} \operatorname{Fix}(T_\lambda)$ is nonempty. Noticing that we might not be able to write the constrained set C as a countable intersection of closed convex sets. For example, if $\mathcal{T} = \{T_t : t \in [0, +\infty)\}$ is a semigroup of nonexpansive mappings, then $\bigcap_{t \geq 0} \operatorname{Fix}(T_t)$ might not reduce to a countable intersection. In many applications, the family $\mathcal{T}$ is even not necessarily a semigroup, e.g., the family of resolvents of a maximal monotone operator (see [28, 29, 31]). To respond to these difficulties, we develop a novel hybrid steepest descent-like method for computing the unique solution of the variational inequality problem (1.4). To the best of our knowledge, it is among the first inexact algorithm to tackle the case where the constrained set is not necessarily a countable intersection of fixed point sets of nonexpansive mappings.

Consider the quotient Banach space $\mathcal{H}^{(\infty)} := \ell_\infty(\mathcal{H})/c_0(\mathcal{H})$. The Hilbert space $\mathcal{H}$ embeds into $\mathcal{H}^{(\infty)}$ as the subspace arising from strong convergent sequences, i.e.,

$$\mathcal{H} = \{[h_n] \in \mathcal{H}^{(\infty)} : h_n \to h \text{ in norm for some } h \text{ in } \mathcal{H}.\}$$

By abusing notation, we write $h = [h_n]$ if $\{h_n\}$ converges to h strongly in $\mathcal{H}$. We also write $h \approx [h_n]$ (resp. $h \approx_w [h_n]$) if there is a subsequence $\{h_{n_k}\}$ converges to h strongly (resp. weakly). Any closed convex subset C of $\mathcal{H}$ embeds into $\mathcal{H}^{(\infty)}$ as a closed convex subset $C^{(\infty)}$ in a similar way.

For any nonexpansive mapping $T : C \to \mathcal{H}$, the *power map* $T^{(\infty)} : C^{(\infty)} \to \mathcal{H}^{(\infty)}$ sends $[h_n]$ to $[Th_n]$. In general, if $\{S_n\}$ is a sequence of mappings from C into $\mathcal{H}$ such that $\{S_n x_n\}$ is bounded whenever $\{x_n\}$ is bounded, and $\|S_n h_n - S_n k_n\| \to 0$ whenever $\|h_n - k_n\| \to 0$, then the *product map* $[S_n] : C^{(\infty)} \to \mathcal{H}^{(\infty)}$ sending $[h_n]$ to $[S_n h_n]$ is well-defined.

We will associate the family $\{T_\lambda\}$ an auxiliary sequence $\{S_n\}$ of mappings on C such that the product map $[S_n] : C^{(\infty)} \to \mathcal{H}^{(\infty)}$ is well-defined. Our proposed algorithms will produce a bounded sequence $\{x_n\}$ in C such that $\|S_n x_n - x_n\| \to 0$. It amounts to saying that $[S_n][x_n] = [x_n]$, i.e., $[x_n]$ is a fixed point of the product map $[S_n]$ in $C^{(\infty)}$.

Our first technical assumption, the so-called Property ($\mathscr{A}$) (see [28]), says that every fixed point of $[S_n]$ is a fixed point of the power map $T_\lambda^{(\infty)}$ for each λ. In other words,

$$\mathrm{Fix}([S_n]) \subseteq \bigcap_\lambda \mathrm{Fix}(T_\lambda^{(\infty)}).$$

Since T_λ is continuous, any strong cluster point h of $\{h_n\}$ with $\|T_\lambda h_n - h_n\| \to 0$, will be a fixed point of T_λ. Hence, Property ($\mathscr{A}$) ensures that

$$h \approx [h_n] \in \mathrm{Fix}([S_n]) \quad \Longrightarrow \quad h \in \bigcap_\lambda \mathrm{Fix}(T_\lambda).$$

In general, the bounded sequence $\{x_n\}$ will have weak sequential cluster points z instead. Our second technical assumption, the so-called demiclosedness principle (see [2,9]), says that all maps $I - T_\lambda$ are demiclosed at zero; namely, if h_n converges to h weakly and $\|T_\lambda h_n - h_n\| \to 0$, then $T_\lambda h = h$. Together with Property ($\mathscr{A}$), we have

$$h \approx_w [h_n] \in \mathrm{Fix}([S_n]) \quad \Longrightarrow \quad h \in \bigcap_\lambda \mathrm{Fix}(T_\lambda).$$

We refer this as the demiclosedness of the family $\{I - T_\lambda\}$ with respect to the sequence $\{I - S_n\}$.

There are several important applications we can verify that both Property ($\mathscr{A}$) and the demiclosedness principle hold, and thus our algorithms apply. We will also implement some mild assumptions to ensure that $\{x_n\}$ converges strongly to the unique solution we are looking for.

In Section 2, we summarize some known concepts and results. In Section 3, we present Property ($\mathscr{A}$) and the demiclosedness principle. In Section 4, a general hybrid steepest descent-like method generated by nearly nonexpansive mappings is presented. We complete our task of locating common fixed points of a family of arbitrary many mappings in Section 5. In particular, an analysis on the stability under perturbations of our projection methods for solving convex feasibility problems is provided. Finally, applications to convex optimization problems are presented in Section 6, together with several numerical examples, which compare the convergent rates of our proposed algorithms with the established ones in literature.

2. Preliminaries

Throughout this paper, the underlying field is the real numbers $\mathbb{R}$. Let C be a nonempty subset of a Banach space X and $T : C \to X$ a mapping. Denote by

$$\mathrm{Fix}(T) = \{x \in C : Tx = x\}$$

the fixed point set of T. For a finite constant $L \geq 0$, we call T an *L-Lipschitz* mapping if

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

An L-Lipschitz mapping T is called *nonexpansive* if $L = 1$, and a *contraction* if $L < 1$. The fixed point set $\mathrm{Fix}(T)$ of a nonexpansive mapping $T\colon C \to C$ is closed and convex when C is a closed convex subset of a strictly convex Banach space, and it is nonempty when C is a bounded closed convex subset of a uniformly convex Banach space (see, e.g., [21, Lemma 3.4 and Theorem 7.2]).

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and identity operator I. Let C be a nonempty closed and convex subset of $\mathcal{H}$. We use P_C to denote the metric projection from $\mathcal{H}$ onto C; namely, $P_C(x)$ is the unique point in C such that

$$\|x - P_C(x)\| = d(x, C) := \inf\{\|x - z\| : z \in C\}.$$

See [1] for the properties of metric projections. We say that a nonlinear operator $T : C \to \mathcal{H}$ is

(i) *firmly nonexpansive* if $\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2$ for all $x, y \in C$;
(ii) *α-averaged*, for a constant $\alpha \in (0, 1)$, if there exists a nonexpansive mapping $S : C \to \mathcal{H}$ such that $T = (1 - \alpha)I + \alpha S$;
(iii) *η-strongly monotone*, for a constant $\eta > 0$, if

$$\langle Tx - Ty, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C;$$

(iv) *β-inverse strongly monotone*, for a constant $\beta > 0$, if

$$\langle Tx - Ty, x - y\rangle \geq \beta\|Tx - Ty\|^2, \quad \forall x, y \in C.$$

Remark 2.1. Every firmly nonexpansive mapping is a 1/2-averaged mapping. For a mapping $T : C \to C$, one has

- T is η-strongly monotone if and only if T^{-1} is $\frac{1}{\eta}$-inverse strongly monotone,
- T is nonexpansive if and only if $I - T$ is $\frac{1}{2}$-inverse strongly monotone, and
- T is α-averaged if and only if $I - T$ is $\frac{1}{2\alpha}$-inverse strongly monotone.

Moreover, if T is β-inverse strongly monotone and $\gamma > 0$, then γT is $\frac{\beta}{\gamma}$-inverse strongly monotone. On the other hand, if T is L-Lipschitz and η-strongly monotone then T is η/L^2-inverse strongly monotone. See, e.g., [39].

Lemma 2.2 (see [41])**.** *Let $\mathcal{H}$ be a Hilbert space and $F : \mathcal{H} \to \mathcal{H}$ an η-strongly monotone and $L-$Lipschitz continuous mapping. Then, for each $\lambda \in (0, 1)$ and fixed $\mu \in (0, 2\eta/L^2)$, the mapping $I - \lambda\mu F$ is a contraction with Lipschitz constant $1 - \tau\lambda$, where $\tau = \sqrt{1 - \mu(2\eta - \mu L^2)}$.*

Lemma 2.3 (see [38, Proposition 6.1] and [4])**.** *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Assume $\{w_i\}$ is a finite or a countable sequence of positive scalars summing up to 1, and assume all $T_i : C \to C$ are nonexpansive mappings with nonempty common fixed point set $\bigcap_i \mathrm{Fix}(T_i)$. Let $T = \sum_i w_i T_i$. Then T is nonexpansive from C into itself, and $\mathrm{Fix}(T) = \bigcap_i \mathrm{Fix}(T_i)$.*

Lemma 2.4 (Bruck [5])**.** *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and $T : C \to C$ a nonexpansive mapping. Define*

$$T_n = \frac{1}{n}(I + T + T^2 + \cdots + T^{n-1}), \quad n \in \mathbb{N}. \tag{2.1}$$

Then $\lim_{n\to\infty}(\sup_{x\in C} \|T_n x - TT_n x\|) = 0$.

Lemma 2.5 (see [32])**.** *Let C be a nonempty closed convex bounded subset of a Hilbert space $\mathcal{H}$ and let $S, T : C \to C$ be nonexpansive mappings such that $ST = TS$. For each $n \in \mathbb{N}$, define*

$$R_n x = \frac{1}{n(n+1)} \sum_{k=0}^{n-1} \sum_{i+j=k} S^i T^j x, \quad x \in C.$$

Then $\lim_{n\to\infty}(\sup_{x\in C} \|R_n x - TR_n x\|) = 0$ and $\lim_{n\to\infty}(\sup_{x\in C} \|R_n x - SR_n x\|) = 0$.

Lemma 2.6 (see [13, 27, 33])**.** *Let C be a nonempty closed convex subset of a uniformly convex Banach space X, and D a closed convex bounded subset of C. Let $\mathscr{F} = \{T_s : s \in \mathbb{R}^+\}$ be a strongly continuous semigroup of nonexpansive mappings from C into itself with $\bigcap_{s>0} \mathrm{Fix}(T_s) \neq \emptyset$. Define $\sigma_h : C \to C$ by*

$$\sigma_h(x) := \frac{1}{h}\int_0^h T_s x \, ds, \quad \text{for all } x \in C \text{ and } h > 0. \tag{2.2}$$

Then $\lim_{t\to+\infty} \sup_{x\in D} \|\sigma_t(x) - T_h\sigma_t(x)\| = 0$ for all $h > 0$.

Lemma 2.7 (see [34])**.** *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Set*

$$x_{n+1} = (1 - \beta_n)x_n + \beta_n z_n, \quad \forall n \in \mathbb{N}.$$

Suppose $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.8 (see [24])**.** *Let all $a_n \geq 0$, $c_n \geq 0$, $b_n \in \mathbb{R}$, and $\alpha_n \in (0, 1]$ such that $\sum_{n=1}^{\infty} c_n < +\infty$. Suppose that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + b_n + c_n, \quad \forall n \in \mathbb{N}.$$

(a) If $K := \limsup_{n\to\infty}(b_n/\alpha_n) < +\infty$*, then*

$$a_{n+1} \le \delta_n a_1 + (1-\delta_n)K + \sum_{j=1}^{n} c_j, \quad \forall n \in \mathbb{N},$$

where $\delta_n = \Pi_{j=1}^{n}(1-\alpha_j)$*, and hence* $\{a_n\}$ *is bounded.*

(b) If $\limsup_{n\to\infty}(b_n/\alpha_n) \le 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = +\infty$ *then* $\{a_n\}_{n=1}^{\infty}$ *converges to zero.*

3. Property ($\mathscr{A}$) and demiclosedness principle

3.1. **Powers of Banach spaces and mappings.** Let X be a Banach space. Let

$$\ell_\infty(X) = \{(x_n) : x_n \in X, n = 1, 2, \dots, \text{ with } \sup_n \|x_n\| < +\infty\},$$

and

$$c_0(X) = \{(x_n) : x_n \in X, n = 1, 2, \dots, \text{ with } \lim_{n\to\infty} \|x_n\| = 0\}.$$

The *power space* of X is defined to be the Banach quotient space

$$X^{(\infty)} = \ell_\infty(X)/c_0(X),$$

equipped with the norm $\|[x_n]\| = \limsup_{n\to\infty} \|x_n\|$.

Two elements $[x_n], [y_n]$ in $X^{(\infty)}$ coincide exactly when $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Elements x in X can be identified with any $[x_n]$ in $X^{(\infty)}$ with $\lim_{n\to\infty} \|x_n - x\| = 0$. In this setting, we write $x = [x_n]$. In the case when there is a subsequence $\{x_{n_k}\}$ converges to some x strongly or weakly, we write $x \approx [x_n]$ or $x \approx_w [x_n]$ accordingly.

If C is a nonempty subset of X, then we write $C^{(\infty)}$ for the subset of $X^{(\infty)}$ consisting of elements $[x_n]$ such that all $x_n \in C$. If $T : C \to X$ is a nonexpansive mapping, then we can define the *power map* $T^{(\infty)} : C^{(\infty)} \to X^{(\infty)}$ of T by $T^{(\infty)}([x_n]) = [Tx_n]$ for all $[x_n]$ in $X^{(\infty)}$.

Following [28], we say that a sequence $\{T_n\}$ of mappings from C into X is *nearly nonexpansive with respect to a positive null sequence* $\{a_n\}$ if

$$\|T_n x - T_n y\| \le \|x - y\| + a_n, \quad \forall x, y \in C, \forall n \in \mathbb{N}.$$

Suppose, in addition, there is a point z in C such that $\{T_n z\}$ is bounded. Then $\{T_n x_n\}$ is bounded whenever $\{x_n\}$ is bounded. Moreover,

$$\|T_n x_n - T_n y_n\| \le \|x_n - y_n\| + a_n \to 0, \quad \text{as } n \to \infty,$$

whenever $\|x_n - y_n\| \to 0$. In other words,

$$[x_n] = [y_n] \text{ in } C^{(\infty)} \quad \Longrightarrow \quad [T_n x_n] = [T_n y_n] \text{ in } X^{(\infty)}.$$

Therefore, we can define the *product map* $[T_n] : C^{(\infty)} \to X^{(\infty)}$ by $[T_n]([x_n]) = [T_n x_n]$.

Let $\{x_n\}$ be a bounded approximate fixed point sequence of T in C, i.e., $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. It amounts to saying that $[x_n]$ is the fixed point of the power map $T^{(\infty)}$ in $C^{(\infty)}$. Similarly, if $\lim_{n\to\infty} \|T_n x_n - x_n\| = 0$, then $[T_n]([x_n]) = [T_n x_n] = [x_n]$; namely, $[x_n]$ is a fixed point of the product map $[T_n]$ in $C^{(\infty)}$. For the power map $T^{(\infty)}$ and all $x, x_n \in C$, we also have

$$x \approx [x_n] \in \mathrm{Fix}(T^{(\infty)}) \quad \Longrightarrow \quad x \in \mathrm{Fix}(T).$$

Let C be a nonempty subset of a Banach space X. Let E be a nonempty subset of C and let $S, T : C \to X$ be mappings. Following [30], we define the *deviation* between S and T on E by

$$\mathscr{D}_E(S, T) = \sup\{\|Sx - Tx\| : x \in E\}.$$

In [37], the notions of boundedly uniform convergence and boundedly uniform sequential convergence of a sequence of mappings on Banach spaces are introduced.

Definition 3.1. Let T, T_n be mappings from a nonempty subset C into a Banach space X for $n \in \mathbb{N}$. We say that

(i) $\{T_n\}$ *boundedly uniformly converges* to T if $\lim_{n\to\infty} \mathscr{D}_B(T_n, T) = 0$ for each bounded subset B of C,
(ii) $\{T_n\}$ *boundedly uniformly sequentially converges* to T if $\lim_{n\to\infty} \|T_n x_n - T x_n\| = 0$ for each bounded sequence $\{x_n\}$ in C.

Proposition 3.2. *Let C be a nonempty subset of a Banach space X. Let $T, T_n : C \to X$ be mappings for $n \in \mathbb{N}$. The following are equivalent to each other.*

(a) $\{T_n\}$ boundedly uniformly converges to T.
(b) $\{T_n\}$ boundedly uniformly sequentially converges to T.

If, in addition, the power map $T^{(\infty)}$ and the product map $[T_n]$ are both well-defined from $C^{(\infty)}$ into $X^{(\infty)}$, then they are also equivalent to

(c) $T^{(\infty)} = [T_n]$.

Proof. The implications (a) $\implies$ (b) $\iff$ (c) are plain. Suppose now that $\{T_n\}$ boundedly uniformly sequentially converges to T, but $\{T_n\}$ does not uniformly converges to T on some bounded subset B of C. Therefore, there is an $\epsilon > 0$, an increasing sequence $1 \le n_1 < n_2 < \cdots < n_k < \cdots$ of indices, and a sequence $\{x_k\}$ from the bounded set B such that $\|T_{n_k} x_k - T x_k\| > \epsilon$. Fix any x_0 in C and construct a bounded sequence $\{y_n\}$ from C by setting $y_{n_k} = x_{n_k}$ for $k = 1, 2, \ldots$ and $y_m = x_0$ elsewhere. Therefore, $T^{(\infty)}([y_n]) \neq [T_n]([y_n])$ as $\|T_n y_n - T y_n\|$ does not converge to zero. $\square$

3.2. **Property ($\mathscr{A}$).** Property ($\mathscr{A}$) plays a key role in the study of common fixed points of a family of mappings (see [14, 28, 29, 31]).

Definition 3.3 (see [28])**.** Let C be a nonempty subset of a Banach space X. Let $\mathcal{T} = \{T_\lambda : \lambda \in \Lambda\}$ be a family of mappings from C into X and let $\{S_n\}$ be a sequence of mappings from C into X. We say that the family $\mathcal{T}$ has *property ($\mathscr{A}$) with respect to the sequence* $\{S_n\}$ if the following holds: for any bounded sequence $\{x_n\}$ in C we have

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0 \quad \implies \quad \lim_{n\to\infty} \|x_n - T_\lambda x_n\| = 0, \qquad \forall \lambda \in \Lambda. \tag{3.1}$$

In Definition 3.3, if the power maps $T_\lambda^{(\infty)}$ and the product map $[S_n]$ are all well-defined from $C^{(\infty)}$ into $X^{(\infty)}$, then condition (3.1) can be restated as

$$\mathrm{Fix}([S_n]) \subseteq \bigcap_{\lambda\in\Lambda} \mathrm{Fix}(T_\lambda^{(\infty)}). \tag{3.2}$$

Example 3.4. (a) In Lemma 2.4, T and all T_n are nonexpansive, and have a common fixed point. The conclusion can be stated as

$$T^{(\infty)}[T_n] = [T_n].$$

It follows that

$$\mathrm{Fix}([T_n]) \subseteq \mathrm{Fix}(T^{(\infty)}).$$

Thus every nonexpansive mapping $T : C \to C$ has property ($\mathscr{A}$) with respect to the sequence $\{T_n\}$ defined by (2.1).

(b) In Lemma 2.5, S, T and all R_n are nonexpansive, and have a common fixed point. The conclusions read

$$S^{(\infty)}[R_n] = [R_n] \quad \text{and} \quad T^{(\infty)}[R_n] = [R_n].$$

Therefore,

$$\mathrm{Fix}([R_n]) \subseteq \mathrm{Fix}(S^{(\infty)}) \quad \text{and} \quad \mathrm{Fix}([R_n]) \subseteq \mathrm{Fix}(T^{(\infty)}).$$

Consequently, the commuting family $\{S, T\}$ of nonexpansive mappings has property ($\mathscr{A}$) with respect to the sequence $\{R_n\}$.

We provide here some more examples.

Definition 3.5 (see [31])**.** Let C be a nonempty subset of a Banach space X. Let $\mathcal{S}$ denote an unbounded subset of $\mathbb{R}^+ := [0, +\infty)$. Let $\mathcal{T} = \{T_s : s \in \mathcal{S}\}$ be a family of mappings from C into itself. The family $\mathcal{T}$ is said to be *uniformly asymptotically regular on* C if

$$\lim_{s\in\mathcal{S}, s\to+\infty} (\sup_{x\in\tilde{C}} \|T_s x - T_h T_s x\|) = 0 \quad \text{for all } h \in \mathcal{S} \quad \text{and all bounded subset } \tilde{C} \text{ of } C.$$

Example 3.6. Let C be a nonempty subset of a Banach space X. Let $\mathcal{T} = \{T_t : t \in \mathbb{R}^+\}$ be a uniformly asymptotically regular nonexpansive semigroup on C with a common fixed point. Let $\{t_n\}$ be a sequence in $(0, +\infty)$ such that $\lim_{n\to\infty} t_n = +\infty$. Then $\mathcal{T}$ has property $(\mathscr{A})$ with respect to the sequence $\{T_{t_n}\}$.

Proof. With Proposition 3.2 we see that

$$T_t^{(\infty)}[T_{t_n}] = [T_{t_n}], \quad \forall t \in \mathbb{R}^+.$$

It follows that

$$\mathrm{Fix}([T_{t_n}]) \subseteq \mathrm{Fix}(T_t^{(\infty)}), \quad \forall t \in \mathbb{R}^+.$$

The assertion follows. □

Example 3.7. Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $\mathscr{F} = \{T_s : s \in \mathbb{R}^+\}$ a strongly continuous semigroup of nonexpansive mappings from C into itself with $\bigcap_{s>0} \mathrm{Fix}(T_s) \neq \emptyset$. For $h > 0$, define $\sigma_h : C \to C$ by (2.2) and let $\{t_n\}$ be a sequence in $(0, +\infty)$ such that $\lim_{n\to\infty} t_n = +\infty$. Then the semigroup $\mathscr{F}$ has property $(\mathscr{A})$ with respect to the sequence $\{\sigma_{t_n}\}$.

Proof. From Lemma 2.6, we have

$$T_t^{(\infty)}[\sigma_{t_n}] = [\sigma_{t_n}],$$

and thus

$$\mathrm{Fix}([\sigma_{t_n}]) \subseteq \mathrm{Fix}(T_t^{(\infty)}), \quad \forall t > 0.$$

The assertion follows. □

3.3. **Demiclosedness principle.** We write $x_n \to x$ and $x_n \rightharpoonup x$ to indicate the strong and the weak convergence of a sequence $\{x_n\}$ to x in a Banach space, respectively. Let C be a nonempty weakly closed subset of a Banach space X with the identity operator I. A mapping $T \colon C \to X$ is called *demiclosed at* 0 if

$$Tx_n \to 0 \text{ and } x_n \rightharpoonup x \quad \Longrightarrow \quad Tx = 0. \tag{3.3}$$

Lemma 3.8 (see [21, Theorem 10.4])**.** *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. If $T \colon C \to X$ is a nonexpansive mapping, then $I - T$ is demiclosed at* 0.

Recall that a Banach space X is said to have the *Opial condition* if for any weak convergent sequence $x_n \rightharpoonup x$ in X, we have

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \neq x.$$

We underline that lim sup in this definition can be replaced by lim inf, and also that every Hilbert space satisfies the Opial condition, see, e.g., [1]. Note that the uniform convexity and the Opial condition are independent (see, e.g., [21, p. 107]). The following supplements Lemma 3.8. For a proof, see, e.g., [21, Theorem 10.3].

Lemma 3.9. *Let X be a Banach space satisfying the Opial condition, let C be a nonempty weakly closed subset of X, and let $T : C \to X$ a nonexpansive mapping. Then $I - T$ is demiclosed at* 0.

We note that both Lemmas 3.8 and 3.9 ensure that $Tx = x$ if x is only a weak sequential cluster point of $\{x_n\}$ in condition (3.3). Thus we have

$$x \approx_w [x_n] \in \mathrm{Fix}(T^{(\infty)}) \quad \Longrightarrow \quad x \in \mathrm{Fix}(T). \tag{3.4}$$

Motivated by Cegielski [9], and Aoyama, Kimura, and Kohsaka [2], we present the following relative demiclosedness for two families of mappings.

Definition 3.10. Let C be a nonempty weakly closed subset of a Banach space X. Let $\{T_\lambda : \lambda \in \Lambda\}$ and $\{S_n : n \in \mathbb{N}\}$ be two families of mappings from C into X. We say that $\{I - T_\lambda\}$ is *demiclosed at zero with respect to* $\{I - S_n\}$ if

$$(I - S_n)x_n \to 0 \text{ and for some subsequence } x_{n_k} \rightharpoonup x \quad \Longrightarrow \quad (I - T_\lambda)x = 0 \text{ for all } \lambda \in \Lambda. \tag{3.5}$$

Proposition 3.11. *Let C be a nonempty weakly closed subset of a Banach space X. Let $\mathcal{T} = \{T_\lambda : \lambda \in \Lambda\}$ and $\{S_n : n \in \mathbb{N}\}$ be two families of mappings from C into X. Assume that*

- *$\mathcal{T}$ has property ($\mathscr{A}$) with respect to $\{S_n\}$, and*
- *$I - T_\lambda$ is demiclosed at 0 for each $\lambda \in \Lambda$.*

Then the family $\{I - T_\lambda\}$ is demiclosed at zero with respect to $\{I - S_n\}$.

Proof. Suppose that $\{x_n\}$ is a sequence in C such that $\|x_n - S_n x_n\| \to 0$. Assume a subsequence $x_{n_k} \rightharpoonup x$ for some $x \in X$. By the property ($\mathscr{A}$) assumption, we have $\lim_{n\to\infty} \|x_n - T_\lambda x_n\| = 0$, and thus $\lim_{k\to\infty} \|x_{n_k} - T_\lambda x_{n_k}\| = 0$ for all $\lambda \in \Lambda$. Then the demiclosedness assumption on $I - T_\lambda$ ensures that $(I - T_\lambda)x = 0$. Therefore, $x \in \operatorname{Fix}(\mathcal{T})$. □

Again, if the product map $[S_n]$ is well-defined on $C^{(\infty)}$, then the condition (3.5) in Definition 3.10 can be stated as

$$x \approx_w [x_n] \in \operatorname{Fix}([S_n]) \quad \Longrightarrow \quad x \in \bigcap_{\lambda\in\Lambda} \operatorname{Fix}(T_\lambda). \tag{3.6}$$

It is then plain that Proposition 3.11 follows from (3.2) and (3.4).

Combining Example 3.7 and Lemma 3.8, we obtain

Proposition 3.12. *Let C be a nonempty closed convex subset of a uniformly convex Banach space X. Let $\mathscr{F} = \{T_s : s \in \mathbb{R}^+\}$ be a strongly continuous semigroup of nonexpansive mappings from C into itself with $\bigcap_{s>0} \operatorname{Fix}(T_s) \neq \emptyset$. For $h > 0$, define $\sigma_h : C \to C$ by (2.2) and let $\{t_n\}$ be a sequence in $(0, +\infty)$ such that $\lim_{n\to\infty} t_n = +\infty$. Then the family $\{I - T_s : s \in \mathbb{R}^+\}$ is demiclosed at zero with respect to $\{I - \sigma_{t_n}\}$.*

Proposition 3.13. *Let X be a strictly convex Banach space satisfying the Opial condition. Let C be a nonempty closed convex subset of X and $\{T_i\}_{i=1}^N$ a finite family of nonexpansive mappings from C into itself such that $\bigcap_{i=}^N \operatorname{Fix}(T_i) \neq \emptyset$. For $i \in \{1, 2, \cdots, N\}$, let $w_i \in (0, 1)$ such that $\sum_{i=1}^N w_i = 1$ and let $\{\widetilde{S}_{n,i}\}$ be a sequence of mappings from C into itself satisfying the following assumption:*

(A0) $\lim_{n\to\infty} \|\widetilde{S}_{n,i}(z_n) - T_i(z_n)\| = 0$ *for every bounded sequence $\{z_n\}$ in C.*

Define $\widetilde{S}_n = \sum_{i=1}^N w_i \widetilde{S}_{n,i}$ for all $n \in \mathbb{N}$. Then the family $\{I - T_i\}_{i=1}^N$ is demiclosed at zero with respect to $\{I - \widetilde{S}_n\}$.

Proof. Define $S = \sum_{i=1}^N w_i T_i$. Note that S is nonexpansive and $\operatorname{Fix}(S) = \bigcap_{i=1}^N \operatorname{Fix}(T_i)$ by Lemma 2.3. Suppose that $\{z_n\}$ is a sequence in C such that $z_n - \widetilde{S}_n(z_n) \to 0$. One can see, from the assumption (A0), that

$$\|\widetilde{S}_n(z_n) - S(z_n)\| \leq \sum_{i=1}^N w_i \|\widetilde{S}_{n,i}(z_n) - T_i(z_n)\| \to 0 \text{ as } n \to \infty.$$

Hence $\|z_n - S(z_n)\| \to 0$ as $n \to \infty$. It follows from Lemma 3.9 that $I - S$ is demiclosed at 0. Consequently,

$$z \in \operatorname{Fix}(S) = \bigcap_{i=1}^N \operatorname{Fix}(T_i)$$

for every weak sequential cluster point z of $\{z_n\}$. □

The following technical result will be used when we discuss the convergence of our algorithms.

Theorem 3.14. *Let C be a nonempty closed convex subset of a strictly convex Banach X. Let $\{w_n\}$ be a summable sequence in $(0,+\infty)$ with $w = \sum_{n=1}^{\infty} w_n$. Let $\{T_n\}$ be a sequence of nonexpansive mappings from C into itself with $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$. Define $S_n : C \to C$ by*

$$S_n = \frac{1}{w_1 + w_2 + \cdots + w_n}(w_1 T_1 + w_2 T_2 + \cdots + w_n T_n), \quad n \in \mathbb{N}. \tag{3.7}$$

Then the following hold.

(a) $\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.
(b) For each $x \in C$, and $p \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$, we have

$$\|S_{n+1}(x) - S_n(x)\| \leq 2\frac{w_{n+1}}{w_1}(\|x-p\| + \|p\|), \tag{3.8}$$

and thus $\{S_n(x)\}$ converges strongly to

$$S(x) := \lim_{n\to\infty} S_n(x) = \frac{1}{w}\sum_{i=1}^{\infty} w_i T_i(x). \tag{3.9}$$

(c) The mapping $S : C \to C$ is a nonexpansive mapping with $\mathrm{Fix}(S) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$.
(d) $\{S_n\}$ boundedly uniformly converges to S; namely, $[S_n] = S^{(\infty)}$ on $C^{(\infty)}$.

Proof. (a) Clearly, we have $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \subseteq \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$. By Lemma 2.3, we have $\mathrm{Fix}(S_n) = \bigcap_{k=1}^{n} \mathrm{Fix}(T_k)$ for all $n \in \mathbb{N}$. Thus, $\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) \subseteq \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$.

(b) Let $x \in C$ and $p \in \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$. Consider the partial sums $t_n = \sum_{i=1}^{n} w_i$. We have

$$\left\|\sum_{i=1}^{n} w_i T_i(x)\right\| = \sum_{i=1}^{n} w_i \|T_i(x) - p + p\| \leq t_n(\|x-p\| + \|p\|)$$

and

$$\|T_{n+1}(x)\| \leq \|T_{n+1}(x) - p\| + \|p\| \leq \|x-p\| + \|p\|.$$

Hence

$$\begin{aligned}
\|S_{n+1}(x) - S_n(x)\| &= \left\|\frac{1}{t_{n+1}}\left(\sum_{i=1}^{n} w_i T_i(x) + w_{n+1}T_{n+1}(x)\right) - \frac{1}{t_n}\sum_{i=1}^{n} w_i T_i(x)\right\| \\
&\leq \left|\frac{1}{t_{n+1}} - \frac{1}{t_n}\right| \left\|\sum_{i=1}^{n} w_i T_i(x)\right\| + \frac{w_{n+1}}{t_{n+1}}\|T_{n+1}(x)\| \\
&\leq \left|\frac{1}{t_{n+1}} - \frac{1}{t_n}\right| t_n(\|x-p\| + \|p\|) + \frac{w_{n+1}}{t_{n+1}}(\|x-p\| + \|p\|) \\
&= 2\frac{w_{n+1}}{w_1}(\|x-p\| + \|p\|).
\end{aligned}$$

It follows that $\sum_{n=1}^{\infty} \|S_{n+1}x - S_n x\| < +\infty$. Therefore, $S_n(x)$ converges strongly to an element $S(x)$ in C. Clearly, $S(x) = \frac{1}{w}\sum_{i=1}^{\infty} w_i T_i(x)$, $x \in C$.

(c) The assertion follows from Lemma 2.3.

(d) Let B be a bounded set in C. Then, for $x \in B$, we have

$$\begin{aligned}
\|S_n(x) - S(x)\| &= \left\|\left(\frac{1}{t_n} - \frac{1}{w}\right)\sum_{i=1}^{n} w_i T_i x - \sum_{i=n+1}^{\infty} \frac{w_i}{w} T_i x\right\| \\
&\leq \left(\frac{1}{t_n} - \frac{1}{w}\right)\sum_{i=1}^{n} w_i \|T_i x\| + \frac{1}{w}\sum_{i=n+1}^{\infty} w_i \|T_i x\| \\
&\leq \left(\frac{1}{t_n} - \frac{1}{w}\right)\sum_{i=1}^{n} w_i R + \frac{1}{w}\sum_{i=n+1}^{\infty} w_i R = 2\frac{w - t_n}{w} R
\end{aligned}$$

for some constant $R > 0$. Thus, $\lim_{n\to\infty} \mathscr{D}_B(S_n, S) = 0$. □

Proposition 3.15. *Let X be a strictly convex Banach space satisfying the Opial condition. Let C be a nonempty closed convex subset of X and $\{T_n\}$ a sequence of nonexpansive mappings from C into itself with $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$. Let $\{w_n\}$ be a summable sequence in $(0, +\infty)$. Let $\{S_n\}$ be a sequence of mappings from C into itself as defined by (3.7). Let $\{\widetilde{S}_n\}$ be a sequence of mappings from C into itself such that the following assumption holds:*

(A0$'$) $\lim_{n\to\infty} \|\widetilde{S}_n(z_n) - S_n(z_n)\| = 0$ *for all bounded sequence $\{z_n\}$ in C.*

Then the family $\{I - T_m\}$ is demiclosed at zero with respect to $\{I - \widetilde{S}_n\}$.

Proof. Let $t_n := \sum_{i=1}^{n} w_i \to w := \sum_{n=1}^{\infty} w_n < +\infty$. Suppose that $\{z_n\}$ is a bounded sequence in C such that $z_n - \widetilde{S}_n(z_n) \to 0$ and $z_{n_k} \rightharpoonup z \in C$ for a subsequence $\{z_{n_k}\}$. Define S by (3.9). Note that S is nonexpansive and $\mathrm{Fix}(S) = \bigcap_{m=1}^{\infty} \mathrm{Fix}(T_m)$ by Theorem 3.14(c). We now show that $z \in \mathrm{Fix}(S)$. Theorem 3.14(d) shows that $\{S_n\}$ boundedly uniformly converges to S. Thus, from the assumption (A0$'$), we have

$$\|\widetilde{S}_n(z_n) - S(z_n)\| \leq \|\widetilde{S}_n(z_n) - S_n(z_n)\| + \|S_n(z_n) - S(z_n)\| \to 0 \text{ as } n \to \infty.$$

Since $z_n - \widetilde{S}_n(z_n) \to 0$, which implies that $\|z_n - S(z_n)\| \to 0$ as $n \to \infty$. Therefore, from Lemma 3.9, we see that $z \in \mathrm{Fix}(S)$. □

4. A hybrid steepest descent-like method and its properties

Let D be a nonempty closed convex subset of a real Hilbert $\mathcal{H}$ and let $F : \mathcal{H} \to \mathcal{H}$ be an η-strongly monotone and L-Lipschitz continuous operator. It follows from [42, Theorem 46.C] that the variational inequality problem $\mathrm{VIP}_D(F, \mathcal{H})$ has a unique solution $x^* \in D$. In the following, we assume that $\mu \in (0, 2\eta/L^2)$ and τ is a constant given in Lemma 2.2. We introduce a general hybrid steepest descent-like method involving a nearly nonexpansive sequence $\{\widetilde{S}_n\}$ with respect to a positive null sequence $\{a_n\}$ for locating the unique solution $x^* \in D$ of $\mathrm{VIP}_D(F, \mathcal{H})$.

Algorithm 4.1 (**Hybrid steepest descent-like method**).
Initialization: Select an arbitrary starting points $x_1 \in \mathcal{H}$.
Iterative step: Given the current iterate x_n, calculate the next iterate as follows:

$$\begin{cases} y_n = \theta_n x_n + (1 - \theta_n)\widetilde{S}_n(x_n), \\ x_{n+1} = (I - \alpha_n \mu F) y_n \quad \text{for all } n \in \mathbb{N}; \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ and $\{\theta_n\}$ are sequences in $(0, 1]$.

Note that the constrained set D is arbitrary and the mappings $\widetilde{S}_n$ involved in Algorithm 4.1 are not necessarily continuous. Therefore, many iterative algorithms can be derived from Algorithm 4.1 due to its generality. We consider the following assumptions:

(A1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = +\infty$;

(A2) $0 < a \leq \theta_n \leq b < 1$ for all $n \in \mathbb{N}$;

(A3*) $\lim_{n\to\infty} \frac{\|\widetilde{S}_n(x^*) - x^*\| + a_n}{\alpha_n} = 0$, and thus $\frac{\|\widetilde{S}_n(x^*) - x^*\| + a_n}{\alpha_n} \leq K$ for some constant $K > 0$ and for all $n \in \mathbb{N}$.

Some basic properties of iterates of Algorithm 4.1 are presented below.

Proposition 4.2. *Let $\{x_n\}$ be generated by (4.1). Assume (A1), (A2) and (A3*) hold.*

(a) *$\{x_n\}$ is bounded with the following estimate:*

$$\|x_{n+1} - x^*\| \leq \max\{\|x_1 - x^*\|, (\mu\|F(x^*)\| + K)/\tau\} \textit{ for all } n \in \mathbb{N}. \tag{4.2}$$

In particular, $\{x_n\}$ *is in the closed ball* $B_r[x^*] := \{x \in \mathcal{H} : \|x - x^*\| \le r\}$ *and* $\{y_n\}$ *is in the closed ball* $B_{r+K}[x^*]$*, where*

$$\max\{\|x_1 - x^*\|, (\mu\|F(x^*)\| + K)/\tau\} \le r < +\infty.$$

(b) *Suppose, in addition, that*

(A4) $\|\widetilde{S}_{n+1}(x_n) - \widetilde{S}_n(x_n)\| \to 0.$

Then $\|x_{n+1} - x_n\| \to 0$ *and* $\|x_n - \widetilde{S}_n(x_n)\| \to 0$ *as* $n \to \infty$.

Proof. (a) Observe that

$$\begin{aligned}
\|y_n - x^*\| &\le \theta_n\|x_n - x^*\| + (1-\theta_n)\|\widetilde{S}_n(x_n) - x^*\| \\
&\le \theta_n\|x_n - x^*\| + (1-\theta_n)(\|\widetilde{S}_n(x_n) - \widetilde{S}_n(x^*)\| + \|\widetilde{S}_n(x^*) - x^*\|) \\
&\le \|x_n - x^*\| + \|\widetilde{S}_n(x^*) - x^*\| + a_n. \qquad (4.3)
\end{aligned}$$

From Lemma 2.2 and (4.3), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|y_n - \alpha_n\mu F(y_n) - [x^* - \alpha_n\mu F(x^*)] + [x^* - \alpha_n\mu F(x^*)] - x^*\| \\
&\le (1-\alpha_n\tau)\|y_n - x^*\| + \alpha_n\mu\|F(x^*)\| \\
&\le (1-\alpha_n\tau)(\|x_n - x^*\| + \|\widetilde{S}_n(x^*) - x^*\| + a_n) + \alpha_n\mu\|F(x^*)\| \\
&\le (1-\alpha_n\tau)\|x_n - x^*\| + \|\widetilde{S}_n(x^*) - x^*\| + \alpha_n\mu\|F(x^*)\| + a_n \\
&\le (1-\alpha_n\tau)\|x_n - x^*\| + \alpha_n\tau\left(\frac{\mu}{\tau}\|F(x^*)\| + \frac{\|\widetilde{S}_n(x^*) - x^*\| + a_n}{\alpha_n\tau}\right) \\
&\le \max\{\|x_n - x^*\|, (\mu\|F(x^*)\| + K)/\tau\} \\
&\le \max\{\|x_1 - x^*\|, (\mu\|F(x^*)\| + K)/\tau\} \le r.
\end{aligned}$$

From (4.3), we also have

$$\|y_n - x^*\| \le \|x_n - x^*\| + \alpha_n K \le r + K \text{ for all } n \in \mathbb{N}.$$

(b) Set

$$z_n = \widetilde{S}_n(x_n) - \frac{1}{1-\theta_n}\alpha_n\mu F(y_n).$$

From (4.1), we have

$$x_{n+1} = \theta_n x_n + (1-\theta_n)z_n.$$

Note that

$$\begin{aligned}
z_{n+1} - z_n &= \widetilde{S}_{n+1}(x_{n+1}) - \widetilde{S}_n(x_n) + \frac{\alpha_n}{1-\theta_n}\mu F(y_n) - \frac{\alpha_{n+1}}{1-\theta_{n+1}}\mu F(y_{n+1}) \\
&= \widetilde{S}_{n+1}(x_{n+1}) - \widetilde{S}_{n+1}(x_n) + \widetilde{S}_{n+1}(x_n) - \widetilde{S}_n(x_n) \\
&\quad + \frac{\alpha_n}{1-\theta_n}\mu F(y_n) - \frac{\alpha_{n+1}}{1-\theta_{n+1}}\mu F(y_{n+1}).
\end{aligned}$$

Hence

$$\begin{aligned}
\|z_{n+1} - z_n\| &\le \|\widetilde{S}_{n+1}(x_{n+1}) - \widetilde{S}_{n+1}(x_n)\| + \|\widetilde{S}_{n+1}(x_n) - \widetilde{S}_n(x_n)\| \\
&\quad + \frac{\alpha_n}{1-\theta_n}\mu\|F(y_n)\| + \frac{\alpha_{n+1}}{1-\theta_{n+1}}\|\mu F(y_{n+1})\| \\
&\le \|x_{n+1} - x_n\| + \|\widetilde{S}_{n+1}(x_n) - \widetilde{S}_n(x_n)\| + a_{n+1} \\
&\quad + \frac{\alpha_n}{1-\theta_n}\mu\|F(y_n)\| + \frac{\alpha_{n+1}}{1-\theta_{n+1}}\|\mu F(y_{n+1})\|.
\end{aligned}$$

Since $\alpha_n \to 0$, $\{y_n\}$ is bounded, F is Lipschitz continuous, and $\|\widetilde{S}_{n+1}(x_n) - \widetilde{S}_n(x_n)\| \to 0$, we obtain from (A2) that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

From Lemma 2.7, it follows that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{4.4}$$

From (4.1) and (4.4), we have

$$\|x_{n+1} - y_n\| = \alpha_n \mu \|F(y_n)\| \to 0$$

and

$$\|x_{n+1} - x_n\| = (1 - \theta_n)\|z_n - x_n\| \to 0, \quad \text{as } n \to \infty.$$

It follows that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Therefore,

$$(1 - b)\|x_n - \widetilde{S}_n(x_n)\| \leq (1 - \theta_n)\|x_n - \widetilde{S}_n(x_n)\| = \|x_n - y_n\| \to 0 \text{ as } n \to \infty.$$

□

Under the stated assumptions, Proposition 4.2 shows that the orbit $\{x_n\}$ in $\mathcal{H}$ generated by x_1 and defined by (4.1) is an approximating fixed point sequence of the nearly nonexpansive sequence $\{\widetilde{S}_n\}$; namely,

$$[\widetilde{S}_n][x_n] = [x_n] \quad \text{in } D^{(\infty)}.$$

5. Convergence of the Algorithm

Theorem 5.1. *Let $\mathcal{T} = \{T_\lambda : \lambda \in \Lambda\}$ be a family of mappings from $\mathcal{H}$ into itself such that $\mathrm{Fix}(\mathcal{T})$ is nonempty, closed and convex. Let $F : \mathcal{H} \to \mathcal{H}$ be an η-strongly monotone and L-Lipschitz continuous operator such that $\mathrm{VIP}_{\mathrm{Fix}(\mathcal{T})}(F, \mathcal{H})$ has a unique solution x^*. Let $\{\widetilde{S}_n\}$ be a nearly nonexpansive sequence of mappings from $\mathcal{H}$ into itself with respect to a positive null sequence $\{a_n\}$. Let $\{x_n\}$ be the orbit in $\mathcal{H}$ generated by $x_1 \in \mathcal{H}$ and defined by (4.1). Assume that (A1), (A2) together with the following assumption (A3) hold.*

(A3) $\lim_{n\to\infty} \frac{\|\widetilde{S}_n(z) - z\| + a_n}{\alpha_n} = 0$*; and thus* $\frac{\|\widetilde{S}_n(z) - z\| + a_n}{\alpha_n} \leq K$
for some constant $K \geq 0$, and for all $z \in \mathrm{Fix}(\mathcal{T})$ and $n \in \mathbb{N}$.

Then the following hold.

(a) *$\{x_n\}$ is bounded with the estimate given by (4.2).*

(b) *If $\{I - T_\lambda : \lambda \in \Lambda\}$ is demiclosed at zero with respect to $\{I - \widetilde{S}_n\}$ and the assumption (A4) holds, then $\{x_n\}$ converges strongly to x^* with the following error estimate:*

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \quad &\leq \quad (1 - \alpha_n\tau)\|x_n - x^*\|^2 + \mu^2\alpha_n^2\|F(x^*)\|^2 + 2rL\mu^2\alpha_n^2 \ \|F(x^*)\| \\ &\quad +2\mu\alpha_n\langle F(x^*), x^* - y_n\rangle \quad \textit{for all } n \in \mathbb{N},\end{aligned} \tag{5.1}$$

where $\max\left\{\|x_1 - x^*\|, \frac{\mu}{\tau}\|F(x^*)\| + \frac{K}{\tau}\right\} \leq r < +\infty.$

Proof. (a) It follows from Proposition 4.2(a).

(b) From the assumption (A4) and Proposition 4.2(b), we have $\|x_n - \widetilde{S}_n(x_n)\| \to 0$. Below, we show that $x_n \to x^*$ as $n \to \infty$.

Assume that there exists a subsequence $\{y_{n_k}\}_{k=1}^\infty$ of $\{y_n\}_{n=1}^\infty$ such that

$$\limsup_{n\to\infty} \langle F(x^*), x^* - y_n\rangle = \lim_{k\to\infty} \langle F(x^*), x^* - y_{n_k}\rangle.$$

Since $\{x_n\}$ is bounded, without loss of generality, we can assume that $\{x_{n_k}\}$ converges weakly to $z \in \mathcal{H}$. Since $\{I - T_\lambda : \lambda \in \Lambda\}$ is demiclosed at zero with respect to $\{I - \widetilde{S}_n\}$, by (3.6) we have $z \in \mathrm{Fix}(\mathcal{T})$. Observe that

$x^* \in \mathrm{Fix}(\mathcal{T})$ is the unique solution of the variational inequality problem $\mathrm{VIP}_{\mathrm{Fix}(\mathcal{T})}(F, \mathcal{H})$ and $\|x_n - y_n\| \to 0$. Hence

$$\limsup_{n\to\infty} \langle F(x^*), x^* - y_n\rangle = \langle F(x^*), x^* - z\rangle \le 0. \tag{5.2}$$

From (4.3) and the assumption (A3), we obtain

$$\begin{aligned}
\|y_n - x^*\|^2 &\le (\|x_n - x^*\| + \|\widetilde{S}_n(x^*) - x^*\| + a_n)^2 \\
&\le \|x_n - x^*\|^2 + 2(\|\widetilde{S}_n(x^*) - x^*\| + a_n)\|x_n - x^*\| + (\|\widetilde{S}_n(x^*) - x^*\| + a_n)^2 \\
&\le \|x_n - x^*\|^2 + 2r(\|\widetilde{S}_n(x^*) - x^*\| + a_n) + \alpha_n K(\|\widetilde{S}_n(x^*) - x^*\| + a_n) \\
&\le \|x_n - x^*\|^2 + (2r + K)(\|\widetilde{S}_n(x^*) - x^*\| + a_n).
\end{aligned}$$

As in Proposition 4.2(a), we see that $\|y_n - x^*\| \le r + K$. Since F is L-Lipschitz,

$$\|F(y_n) - F(x^*)\| \le L\|y_n - x^*\| \le L(r + K).$$

In virtue of (4.1) and Lemma 2.2, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|(I - \alpha_n\mu\mathcal{F})(y_n) - (I - \alpha_n\mu F)(x^*) - \alpha_n\mu F(x^*)\|^2 \\
&= \|(I - \alpha_n\mu\mathcal{F})(y_n) - (I - \alpha_n\mu F)(x^*)\|^2 + \alpha_n^2\mu^2\|F(x^*)\|^2 \\
&\quad + 2\alpha_n\mu\langle F(x^*), (I - \alpha_n\mu\mathcal{F})(x^*) - (I - \alpha_n\mu\mathcal{F})(y_n)\rangle \\
&\le (1 - \alpha_n\tau)\|y_n - x^*\|^2 + \alpha_n^2\mu^2\|F(x^*)\|^2 \\
&\quad + 2\mu\alpha_n\langle F(x^*), x^* - y_n\rangle + 2\mu^2\alpha_n^2\|F(y_n) - F(x^*)\|\ \|F(x^*)\| \\
&\le (1 - \alpha_n\tau)(\|x_n - x^*\|^2 + (2r + K)(\|\widetilde{S}_n(x^*) - x^*\| + a_n)) + \alpha_n^2\mu^2\|F(x^*)\|^2 \\
&\quad + 2\mu\alpha_n\langle F(x^*), x^* - y_n\rangle + 2\mu^2\alpha_n^2\|F(y_n) - F(x^*)\|\ \|F(x^*)\| \\
&\le (1 - \alpha_n\tau)\|x_n - x^*\|^2 + (2r + K)(\|\widetilde{S}_n(x^*) - x^*\| + a_n) \\
&\quad + \alpha_n^2\mu^2\|F(x^*)\|^2 + 2L(r + K)\mu^2\alpha_n^2\ \|F(x^*)\| + 2\alpha_n\mu\langle F(x^*), x^* - y_n\rangle.
\end{aligned}$$

Note that $\frac{\|\widetilde{S}_n(x^*) - x^*\| + a_n}{\alpha_n} \to 0$. Hence, by (5.2) and Lemma 2.8, we conclude that $x_n \to x^*$ as $n \to \infty$. □

Theorem 5.1 is a general result in nature. Many significant real-world problems are modeled as a *general convex feasibility problem*:

$$\text{find a point } x^* \in \bigcap_{\lambda\in\Lambda} C_\lambda,$$

where $\{C_t : \lambda \in \Lambda\}$ is a collection of nonempty closed convex subsets of a real Hilbert space $\mathcal{H}$. Projection methods are effective iterative techniques for solving the general convex feasibility problem (see [3,8,12,15,19]). In [17], Combettes studied the weak convergence of the *approximate parallel projection method* (APPM):

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n(w_1S_{n,1} + w_2S_{n,2} + \cdots + w_NS_{n,N})(x_n) \quad \text{for all } n \in \mathbb{N}, \tag{5.3}$$

where $S_{n,i}(x) = P_{C_i}(x) + e_{n,i}$. We note that $e_{n,i}$ stands for the error made in computing the metric projection P_{C_i} at the nth iteration. Later, Dan *et. al.* [7] studied the convergence of the following projection method for solving a convex feasibility problem in $\mathbb{R}^d$:

$$x_{n+1} = (1 - \lambda_n)x_n + \lambda_n\sum_{i=1}^{N} w_iP_{C_i}(x_n + \beta_nv_n) \quad \text{for all } n \in \mathbb{N}.$$

In fact, this is just a special case of the amalgamated projection method studied in [7]. They proved that the algorithms converge to solutions of a consistent convex feasibility problem, and that their convergence is stable under summable perturbations. On the other hand, Flam [20] studied the weak convergence of the following algorithm:

$$x_{n+1} \in T_nx_n + \beta_nB_1[0] \quad \text{for all } n \in \mathbb{N}, \tag{5.4}$$

where all T_n are nonexpansive self-mappings on $\mathcal{H}$ having a common fixed point, and the (nonnegative) error bounds β_n are summable.

Theorem 5.2. *Let $\mathcal{T} = \{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings from $\mathcal{H}$ into itself such that $\mathrm{Fix}(\mathcal{T}) = \bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$. Let $F : \mathcal{H} \to \mathcal{H}$ be an η-strongly monotone and L-Lipschitz continuous operator such that $\mathrm{VIP}_{\mathrm{Fix}(\mathcal{T})}(F, \mathcal{H})$ has a unique solution x^*. For $i \in \{1, 2, \cdots, N\}$, let $w_i \in (0, 1)$ such that $\sum_{i=1}^N w_i = 1$, and let $\{\widetilde{S}_{n,i}\}$ be a sequence of mappings from $\mathcal{H}$ into itself such that $\{\widetilde{S}_{n,i}\}$ is nearly nonexpansive with respect to a positive null sequence $\{a_{n,i}\}$, and the assumption (A0) holds. Let $\{x_n\}$ be the orbit in $\mathcal{H}$ generated by $x_1 \in \mathcal{H}$ and defined by*

$$\begin{cases} y_n = \theta_n x_n + (1 - \theta_n)(w_1\widetilde{S}_{n,1} + w_2\widetilde{S}_{n,2} + \cdots + w_N\widetilde{S}_{n,N})(x_n), \\ x_{n+1} = (I - \alpha_n \mu F)y_n, \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\theta_n\}$ are sequences in $(0, 1]$ satisfying (A1), (A2) and

(A3′) $\lim_{n\to\infty} \frac{\|\widetilde{S}_{n,i}(z)-z\|+a_{n,i}}{\alpha_n} = 0$*; and thus* $\frac{\|\widetilde{S}_{n,i}(z)-z\|+a_{n,i}}{\alpha_n} \leq K$
for some constant $K \geq 0$, and for all $z \in \bigcap_{j=1}^N \mathrm{Fix}(T_j)$, $i \in \{1, 2, \cdots, N\}$, and $n \in \mathbb{N}$.

Then the following hold.

(a) *$\{x_n\}$ is bounded with the estimate given by (4.2).*
(b) *$\{x_n\}$ converges strongly to x^* with the error estimate (5.1).*

Proof. (a) Define $\widetilde{S}_n = w_1\widetilde{S}_{n,1} + w_2\widetilde{S}_{n,2} + \cdots + w_N\widetilde{S}_{n,N}$. Then $\{\widetilde{S}_n\}$ is nearly nonexpansive with respect to $\{a_n\}$, where $a_n = \sum_{i=1}^N a_{n,i}w_i$. For $z \in \bigcap_{i=1}^N \mathrm{Fix}(T_i)$, we have

$$\frac{1}{\alpha_n}(\|\widetilde{S}_n(z) - z\| + a_n) \leq \frac{1}{\alpha_n}\sum_{i=1}^N w_i(\|\widetilde{S}_{n,i}(z) - z\| + a_{n,i}) \leq K.$$

Hence, from the assumption (A3′), we have $\frac{1}{\alpha_n}(\|\widetilde{S}_n(z)-z\|+a_n) \to 0$. Thus, assumption (A3) holds. Therefore, part (a) follows from Proposition 4.2(a).

(b) In order to apply Theorem 5.1(b), we show that

(i) the assumption (A4) holds,
(ii) the family $\{I - T_i\}_{i=1}^N$ is demiclosed at zero with respect to $\{I - \widetilde{S}_n\}$.

From the assumption (A0), we have

$$\|\widetilde{S}_{n+1,i}(x_n) - \widetilde{S}_{n,i}(x_n)\| \leq \|\widetilde{S}_{n+1,i}(x_n) - T_i(x_n)\| + \|T_i(x_n) - \widetilde{S}_{n,i}(x_n)\| \to 0 \text{ as } n \to \infty$$

for all $i \in \{1, 2, \cdots, N\}$. Hence,

$$\|\widetilde{S}_{n+1}(x_n) - \widetilde{S}_n(x_n)\| \leq \sum_{i=1}^N w_i\|\widetilde{S}_{n+1,i}(x_n) - \widetilde{S}_{n,i}(x_n)\| \to 0 \text{ as } n \to \infty.$$

Thus, the assumption (A4) holds.

Since the Hilbert space $\mathcal{H}$ is strictly convex and satisfies the Opial condition, Proposition 3.13 implies that the finite family $\{I - T_i\}_{i=1}^N$ is demiclosed at zero with respect to $\{I - \widetilde{S}_n\}$. Therefore, the assertion follows from Theorem 5.1(b). ☐

Theorem 5.3. *Let $\mathcal{T} = \{T_n\}$ be a sequence of nonexpansive mappings from a Hilbert space $\mathcal{H}$ into itself such that $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$. Let $F : \mathcal{H} \to \mathcal{H}$ be an η-strongly monotone and $L-$Lipschitz continuous operator such that $\mathrm{VIP}_{\mathrm{Fix}(\mathcal{T})}(F, \mathcal{H}))$ has a unique solution x^*. Let $\{w_n\}$ be a summable sequence in $(0, +\infty)$. Let $\{S_n\}$ be a sequence of mappings from $\mathcal{H}$ into itself defined by (3.7). Let $\{\widetilde{S}_n\}$ be a sequence of mappings from $\mathcal{H}$ into itself such that $\{\widetilde{S}_n\}$ is nearly nonexpansive with respect to a positive null sequence $\{a_n\}$ and the assumption* (A0′)

holds. Let $\{x_n\}$ be the orbit in $\mathcal{H}$ generated by $x_1 \in \mathcal{H}$ defined by (4.1). Assume that the assumptions (A1), (A2) and (A3) hold. Then we have

(a) *$\{x_n\}$ is bounded with the estimate given by (4.2);*
(b) *$\{x_n\}$ converges strongly to x^* with the error estimate (5.1).*

Proof. Part (a) follows from Proposition 4.2(a).

For part (b), from (3.8) we have

$$\begin{aligned}\|\widetilde{S}_{n+1}(x_n) - \widetilde{S}_n(x_n)\| &\le \|\widetilde{S}_{n+1}(x_n) - S_{n+1}(x_n)\| + \|S_{n+1}(x_n) - S_n(x_n)\| + \|S_n(x_n) - \widetilde{S}_n(x_n)\| \\ &\le \|\widetilde{S}_{n+1}(x_n) - S_{n+1}(x_n)\| + 2\frac{w_{n+1}}{w_1}(\|x_n - x^*\| + \|x^*\|) + \|S_n(x_n) - \widetilde{S}_n(x_n)\|.\end{aligned}$$

Since $w_n \to 0$, it follows from the assumption (A0$'$) that $\|\widetilde{S}_{n+1}(x_n) - \widetilde{S}_n(x_n)\| \to 0$. Thus, the assumption (A4) holds.

Finally, since the Hilbert space $\mathcal{H}$ is strictly convex and satisfies the Opial condition, Proposition 3.15 shows that the finite family $\{I - T_n\}$ is demiclosed at zero with respect to $\{I - \widetilde{S}_n\}$. Therefore, the assertion follows from Theorem 5.1(b). □

Corollary 5.4. *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$ and let $F : \mathcal{H} \to \mathcal{H}$ be an η-strongly monotone and L-Lipschitz continuous operator such that $\mathrm{VIP}_{\mathrm{Fix}(T)}(F, \mathcal{H})$ has a unique solution x^*. Let $\{\widetilde{S}_n\}$ be a sequence of nonexpansive mappings from $\mathcal{H}$ into itself such that $\{\widetilde{S}_n\}$ boundedly uniformly sequentially converges to T. Let $\{x_n\}$ be the orbit in $\mathcal{H}$ generated by $x_1 \in \mathcal{H}$ and defined by*

$$\begin{cases} y_n = \theta_n x_n + (1 - \theta_n)\widetilde{S}_n(x_n), \\ x_{n+1} = (I - \alpha_n \mu F)y_n, \quad \text{for all } n \in \mathbb{N}, \end{cases} \tag{5.5}$$

where $\{\alpha_n\}$ and $\{\theta_n\}$ are sequences in $(0, 1]$ satisfying the assumptions (A1), (A2) and

(A3$''$) $\lim_{n\to\infty} \frac{\|\widetilde{S}_n(z)-z\|}{\alpha_n} = 0$*; and thus* $\frac{\|\widetilde{S}_n(z)-z\|}{\alpha_n} \le K$
for some constant $K \ge 0$, and for all $z \in \mathrm{Fix}(T)$ and $n \in \mathbb{N}$.

Then the following hold:

(a) *$\{x_n\}$ is bounded with the estimate given by (4.2).*
(b) *$\{x_n\}$ converges strongly to x^* with the error estimate (5.1).*

The proof of the following proposition is straightforward.

Proposition 5.5. *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$. Let $\{\beta_n\}$ be a sequence in $\mathbb{R}^+$ with $\sum_{n=1}^{\infty} \beta_n < +\infty$ and $\{v_n\}$ a bounded sequence in $\mathcal{H}$ with $\|v_n\| \le M < +\infty$ for all $n \in \mathbb{N}$. Define*

$$\widetilde{S}_n(x) = T(x + \beta_n v_n), \quad \forall x \in \mathcal{H}, \forall n \in \mathbb{N}. \tag{5.6}$$

Then the following hold:

(a) *All $\widetilde{S}_n$ are nonexpansive mappings.*
(b) *For $x \in \mathcal{H}$, we have*

$$\|\widetilde{S}_n x - Tx\| = \|T(x + \beta_n v_n) - Tx\| \le M\beta_n, \quad \text{for all } n \in \mathbb{N}.$$

(c) *$\{\widetilde{S}_n\}$ uniformly converges to T on $\mathcal{H}$.*

Theorem 5.6. *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping such that $\mathrm{Fix}(T) \neq \emptyset$, and let $F : \mathcal{H} \to \mathcal{H}$ be an η-strongly monotone and L-Lipschitz continuous operator such that $\mathrm{VIP}_{\mathrm{Fix}(T)}(F, \mathcal{H})$ has a unique solution x^*.*

Let $\{\beta_n\}$ be a sequence in $(0,+\infty)$ and $\{v_n\}$ a bounded sequence in $\mathcal{H}$ with $\|v_n\| \le M < +\infty$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ be the orbit in $\mathcal{H}$ generated by $x_1 \in \mathcal{H}$ and defined by

$$\begin{cases} y_n = \theta_n x_n + (1-\theta_n)T(x_n + \beta_n v_n), \\ x_{n+1} = (I - \alpha_n \mu F)y_n, \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\theta_n\}$ are sequences in $(0,1]$ satisfying the assumptions (A1) and (A2). Assume that $\lim_{n\to\infty} \frac{\beta_n}{\alpha_n} = 0$, and thus $\frac{\beta_n}{\alpha_n}M \le K$ for some constant $K > 0$ and all $n \in \mathbb{N}$. Then the following hold.

(a) *$\{x_n\}$ is bounded with the estimate given by (4.2).*
(b) *$\{x_n\}$ converges strongly to x^* with the error estimate (5.1).*

Proof. Let $\{\widetilde{S}_n\}$ be a sequence of nonexpansive mappings from $\mathcal{H}$ into itself defined by (5.6). For $z \in \mathrm{Fix}(T)$, from Proposition 5.5(b), we have

$$\frac{\|\widetilde{S}_n z - z\|}{\alpha_n} \le \frac{\beta_n}{\alpha_n} M, \quad \text{for all } n \in \mathbb{N}.$$

Hence, $\lim_{n\to\infty} \frac{\|\widetilde{S}_n(z) - z\|}{\alpha_n} = 0$. Thus, the condition (A3$''$) of Corollary 5.4 holds, and the assertions follow. ☐

Remark 5.7. (i) Our perturbation technique in Theorem 5.6 is slightly different from Lopez, Martin and Xu [22].

(ii) In Corollary 5.4, if the sequence $\{\widetilde{S}_n\}$ is defined by $\widetilde{S}_n(x) = Tx + u_n$, where $\{u_n\}$ is a sequence in $\mathcal{H}$, then the condition (A3$''$) reduces to $\lim_{n\to\infty} \frac{\|u_n\|}{\alpha_n} = 0$ and the iteration process (5.5) reduces to

$$x_{n+1} = (I - \alpha_n \mu F)(Tx_n + u_n) \quad \text{for all } n \in \mathbb{N}.$$

Corollary 5.8. *Let T be an α-averaged nonexpansive mapping from $\mathcal{H}$ into itself such that $\mathrm{Fix}(T) \ne \emptyset$. Let $F : \mathcal{H} \to \mathcal{H}$ be an η-strongly monotone and L-Lipschitz continuous operator such that $\mathrm{VIP}_{\mathrm{Fix}(T)}(F, \mathcal{H})$ has a unique solution x^*. Let $\{x_n\}$ be the orbit in $\mathcal{H}$ generated by $x_1 \in \mathcal{H}$ and defined by*

$$x_{n+1} = (I - \alpha_n \mu F)T(x_n) \quad \text{for all } n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying the assumption (A1). Then the following hold:

(a) *$\{x_n\}$ is bounded with the following estimate:*

$$\|x_{n+1} - x^*\| \le r := \max\{\|x_1 - x^*\|, \frac{\mu}{\tau}\|F(x^*)\|\} \text{ for all } n \in \mathbb{N}. \tag{5.7}$$

(b) *$\{x_n\}$ converges strongly to x^* with the following error estimate:*

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \quad \le \quad & (1 - \alpha_n \tau)\|x_n - x^*\|^2 + \mu^2 \alpha_n^2 \|F(x^*)\|^2 + 2rL\mu^2\alpha_n^2 \ \|F(x^*)\| \\ & + 2\mu\alpha_n \langle F(x^*), x^* - y_n \rangle \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{5.8}$$

Buong and Duong [6] introduced the following explicit iterative algorithm for solving (1.4) when $\mathcal{T} = \{T_i\}_{i=1}^N$:

$$x_{n+1} = (I - \beta_n^0)x_n + \beta_n^0 (I - \alpha_n \mu F) T_N^n T_{N-1}^n \cdots T_1^n(x_n), \quad \text{for all } n \in \mathbb{N}, \tag{5.9}$$

where $T_i^n = (1-\beta_n^i)I + \beta_n^i T_i$, $\beta_n^i \in (\alpha, \beta) \subset (0,1)$ for $i \in \{1, 2, \cdots, N\}$, and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the conditions:

$$\alpha_n \to 0, \quad \sum_{n=1}^{\infty} \alpha_n = +\infty \quad \text{and} \quad |\beta_n^i - \beta_{n+1}^i| \to 0, \qquad \forall i = 1, 2, \ldots, N.$$

Later, Zhou and Wang [43] proved a strong convergence theorem for an explicit iterative method seemingly simpler than (5.9), defined by

$$x_{n+1} = (I - \alpha_n \mu F) T_N^n T_{N-1}^n \cdots T_1^n(x_n), \quad \text{for all } n \in \mathbb{N}. \tag{5.10}$$

An immediate consequence of Theorem 5.2 is the following:

Corollary 5.9. *Let $\mathcal{T} = \{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings from $\mathcal{H}$ into itself such that $\mathrm{Fix}(\mathcal{T}) = \bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$. Let $F : \mathcal{H} \to \mathcal{H}$ be an η-strongly monotone and L-Lipschitz continuous operator such that $\mathrm{VIP}_{\mathrm{Fix}(\mathcal{T})}(F, \mathcal{H})$ has a unique solution x^*. Let $w_i \in (0,1)$ such that $\sum_{i=1}^N w_i = 1$. Let $\{e_{n,i}\}$ be a sequence in $\mathcal{H}$ and let $\{S_{n,i}\}$ be a sequence of mappings from $\mathcal{H}$ into itself defined by $S_{n,i}(x) = T_i x + e_{n,i}$, $x \in \mathcal{H}$, for $i = 1, 2, \dots, N$. Let $\{x_n\}$ be the orbit in $\mathcal{H}$ generated by $x_1 \in \mathcal{H}$ and defined by*

$$\begin{cases} y_n &= (1-\theta_n)x_n + \theta_n \sum_{i=1}^N w_i S_{n,i}(x_n), \\ x_{n+1} &= (I - \alpha_n \mu F) y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$ and $\{\theta_n\}$ are sequences in $(0,1]$ satisfying the assumptions (A1) and (A2), and that $\lim_{n\to\infty} \|e_{n,i}\|/\alpha_n = 0$ for $i = 1, 2, \dots, N$. Then $\{x_n\}$ converges strongly to x^*.

Theorem 5.10. *Let $\mathcal{T} = \{T_i\}_{i=1}^N$ be a finite family of firmly nonexpansive mappings from $\mathcal{H}$ into itself such that $\mathrm{Fix}(\mathcal{T}) = \bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$. Let $F : \mathcal{H} \to \mathcal{H}$ be an η-strongly monotone and L-Lipschitz continuous operator such that $\mathrm{VIP}_{\mathrm{Fix}(\mathcal{T})}(F, \mathcal{H})$ has a unique solution x^*. Let $w_i \in (0,1)$ such that $\sum_{i=1}^N w_i = 1$. Let $\{x_n\}$ be the orbit in $\mathcal{H}$ generated by $x_1 \in \mathcal{H}$ and defined by*

$$x_{n+1} = (I - \alpha_n \mu F) \sum_{i=1}^N w_i T_i(x_n) \quad \textit{for all } n \in \mathbb{N}, \tag{5.11}$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying the assumption (A1). Then the following hold:

(a) *$\{x_n\}$ is bounded with the estimate given by (5.7).*
(b) *$\{x_n\}$ converges strongly to x^* with the error estimate (5.8).*

Proof. Note that each T_i is firmly nonexpansive. By Remark 2.1, there exists a nonexpansive mapping $G_i : \mathcal{H} \to \mathcal{H}$ such that $T_i = \frac{1}{2}(I + G_i)$, and hence $\mathrm{Fix}(T_i) = \mathrm{Fix}(G_i)$ for $i \in \{1, 2, \cdots, N\}$. For $x \in \mathcal{H}$, we have $\sum_{i=1}^N w_i T_i(x) = \frac{1}{2}x + \frac{1}{2}\sum_{i=1}^N w_i G_i(x)$. Hence (5.11) reduces to

$$\begin{cases} y_n &= \frac{1}{2}x_n + \frac{1}{2}\sum_{i=1}^N w_i \widetilde{S}_{n,i}(x_n), \\ x_{n+1} &= (I - \alpha_n \mu F) y_n \quad \text{for all } n \in \mathbb{N}, \end{cases}$$

where $\widetilde{S}_{n,i} = G_i$. Observe that (A0) and (A3′) holds. Hence the assertion follows from Theorem 5.2. ☐

Theorem 5.11. *Let $\mathcal{T} = \{T_n\}$ be a sequence of firmly nonexpansive mappings from $\mathcal{H}$ into itself such that $\mathrm{Fix}(\mathcal{T}) \neq \emptyset$. Let $F : \mathcal{H} \to \mathcal{H}$ be an η-strongly monotone and $L-$Lipschitz continuous operator such that $\mathrm{VIP}_{\mathrm{Fix}(\mathcal{T})}(F, \mathcal{H})$ has a unique solution x^*. Let $\{w_n\}$ be a summable sequence in $(0, +\infty)$. Let $\{S_n\}$ be a sequence of mappings from $\mathcal{H}$ into itself defined by (3.7). Let $\{x_n\}$ be the orbit in $\mathcal{H}$ generated by $x_1 \in \mathcal{H}$ and defined by*

$$x_{n+1} = (I - \alpha_n \mu F) S_n(x_n), \quad \textit{for all } n \in \mathbb{N}, \tag{5.12}$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$ satisfying the assumption (A1). Then the following hold:

(a) *$\{x_n\}$ is bounded with the following estimate:*

$$\|x_{n+1} - x^*\| \leq r := \max\{\|x_1 - x^*\|, \frac{\mu}{\tau}\|F(x^*)\|\} \textit{ for all } n \in \mathbb{N}.$$

(b) *$\{x_n\}$ converges strongly to x^* with the following error estimate:*

$$\begin{aligned} \|x_{n+1} - x^*\|^2 \quad &\leq \quad (1 - \alpha_n \tau)\|x_n - x^*\|^2 + \mu^2 \alpha_n^2 \|F(x^*)\|^2 + 2rL\mu^2\alpha_n^2 \ \|F(x^*)\| \\ &\quad + 2\mu\alpha_n \langle F(x^*), x^* - y_n \rangle \textit{ for all } n \in \mathbb{N}. \end{aligned}$$

Proof. As in Theorem 5.10, there exists a nonexpansive mapping $G_n : \mathcal{H} \to \mathcal{H}$ such that $T_n = \frac{1}{2}(I + G_n)$ for each $n \in \mathbb{N}$. Hence $S_n(x) = \frac{1}{2}x + \frac{1}{2}\sum_{i=1}^n w_i G_i(x)$ for $x \in \mathcal{H}$ and $n \in \mathbb{N}$. Thus, (5.12) reduces to

$$\begin{cases} y_n &= \frac{1}{2}x_n + \frac{1}{2}\sum_{i=1}^n w_i G_i(x_n), \\ x_{n+1} &= (I - \alpha_n \mu F) y_n, \quad \text{for all } n \in \mathbb{N}. \end{cases}$$

The assertion follows from Theorem 5.3. □

Remark 5.12. (i) Our algorithms (5.11) and (5.12) are based on the simple convex combinations.
(ii) Algorithm (5.11) has strong convergence in an infinite dimensional Hilbert space, while the realistic approximate parallel projection method (5.3) has weak convergence for P_{C_i}'s under some different assumptions.

(iii) Theorem 5.11 guarantees the strong convergence of (5.12) to a common fixed point of the family $\mathcal{T} = \{T_n\}$ of firmly nonexpansive mappings, while Flam's method (5.4) has weak convergence under some different assumptions.

6. Applications and numerical examples

To demonstrate the effectiveness, performance, and convergence of algorithms (5.11) and (5.12), we discuss convex feasibly problems and multi-set split feasibility problems.

6.1. **Convex feasibly problems.**

Problem 6.1. Consider the following optimization problem:

$$\text{find } x^* \in C = \bigcap_{\lambda\in\Lambda} C_\lambda \text{ such that } \psi(x^*) = \min\{\psi(x) : x \in C\}, \tag{6.1}$$

where $\psi : \mathbb{R}^d \to \mathbb{R}$ is a differentiable convex function and C_λ is a nonempty closed convex set in $\mathbb{R}^d$ for each $\lambda \in \Lambda$.

Problem 6.1 can be casted into the variational inequality problem over $C = \bigcap_{\lambda\in\Lambda} C_\lambda$:

$$\text{find } u \in C \text{ such that } \langle \nabla\psi(u), z - u\rangle \geq 0 \text{ for all } z \in C,$$

where $\nabla\psi : \mathbb{R}^d \to \mathbb{R}^d$ is the gradient of ψ. If ψ is strongly convex, then our algorithms can be applied by setting $T_\lambda = P_{C_\lambda}$ for all $\lambda \in \Lambda$. For $\Lambda = \{1, 2..., N\}$, starting from $x_1 \in \mathbb{R}^d$, Algorithm (5.11) reads

$$\begin{cases} y_n &= (w_1 P_{C_1} + w_2 P_{C_2} + \cdots + w_N P_{C_N})(x_n), \\ x_{n+1} &= (I - \alpha_n \mu F) y_n, \quad \forall n \in \mathbb{N}. \end{cases} \tag{6.2}$$

Example 6.2. Consider $d = 2$,

$$\begin{aligned} C_1 &= \{(u, v) : -3 \leq u \leq 3; -1 \leq v \leq 1\}, \\ C_2 &= \{(u, v) : -1 \leq u \leq 1; -2 \leq v \leq 2\}, \text{ and} \\ C &= C_1 \cap C_2 = \{(u, v) : -1 \leq u, v \leq 1\}. \end{aligned}$$

Consider the map $F(x) = \psi'(x)$, where $\psi(x) = \|x\|^2/2$. One can see that F is 1-Lipschitz continuous and 1-strongly monotone. Algorithm (5.11) requires us to choose $\mu \in (0, 2)$. Theorem 5.10 guarantees that Algorithm (6.2) with an initial point $x_1 \in \mathbb{R}^d$ converges to the solution of Problem 6.1 with $\Lambda = \{1, 2\}$. Set $w_1 = w_2 = 1/2$ and $\alpha_n := 1/(n + 1)^a$, where $a \in (0, 1]$.

(1) For the initial point $x_1 = (1, -3)$ and $a = 1$, Figure 1 shows that $\psi(x_n)$ for $n = 1, 2, \ldots, 10000$ given in Algorithm (6.2) has faster convergence when $\mu = 1.8$ than the case when $\mu = 0.6$, 0.75, or 1.2. We observe that Algorithm (6.2) has faster convergence to the minimum value 0 when $\mu = 0.75$, $a = 0.8$ than the case when $\mu = 0.75$, $a = 0.2$, 0.4, or 0.6.

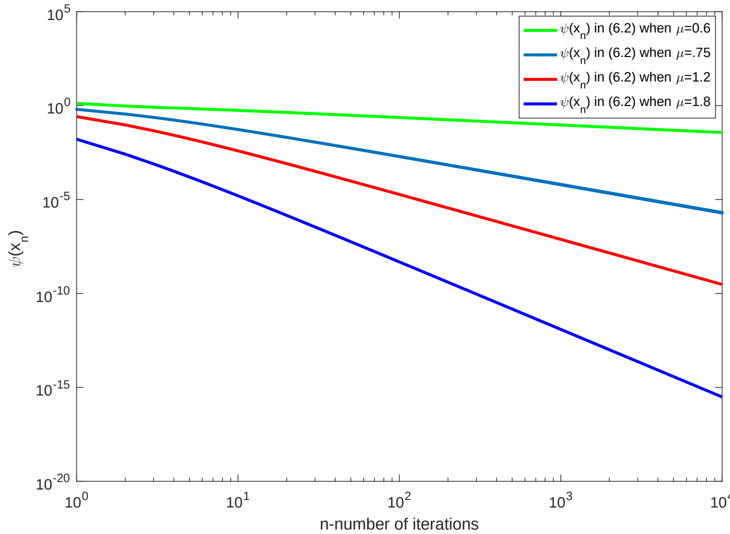

FIGURE 1. Plots of the values of the objective function $\psi(x_n)$ $(n = 1, 2, \ldots, 10000)$ in Example 6.2 when $\mu = 0.6, 0.75, 1.2, 1.8$, $a = 1$.

(2) Comparison of the convergence rates among Algorithm (5.9) of Buong and Duong [6], Algorithm (5.10) of Zhou and Wang [43] and our Algorithm (6.2) are shown in Figure 2, where we plot $\psi(x_n)$ for $n = 1, 2, \ldots, 1000$, $\mu = 0.75$ and $a = 1$.

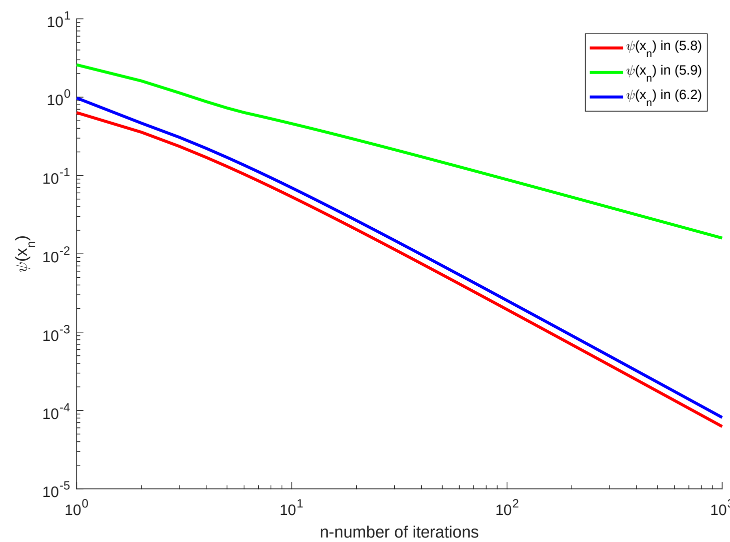

FIGURE 2. Plots of $\psi(x_n)$ $(n = 1, 2, \ldots, 100)$ given by Algorithm (5.9) of Buong and Duong [6], Algorithm (5.10) of Zhou and Wang [43], and our Algorithm (6.2) in Example 6.2.

(3) With different choices of the initial point x_1, Figure 3 plots the values of $\psi(x_n)$ for $n = 1, 2, \ldots, 1000$, generated in Algorithms (5.9), (5.10) and (6.2), when $\mu = 0.75$. It shows that Algorithm (6.2) is more stable than the other two for different choices of x_1.

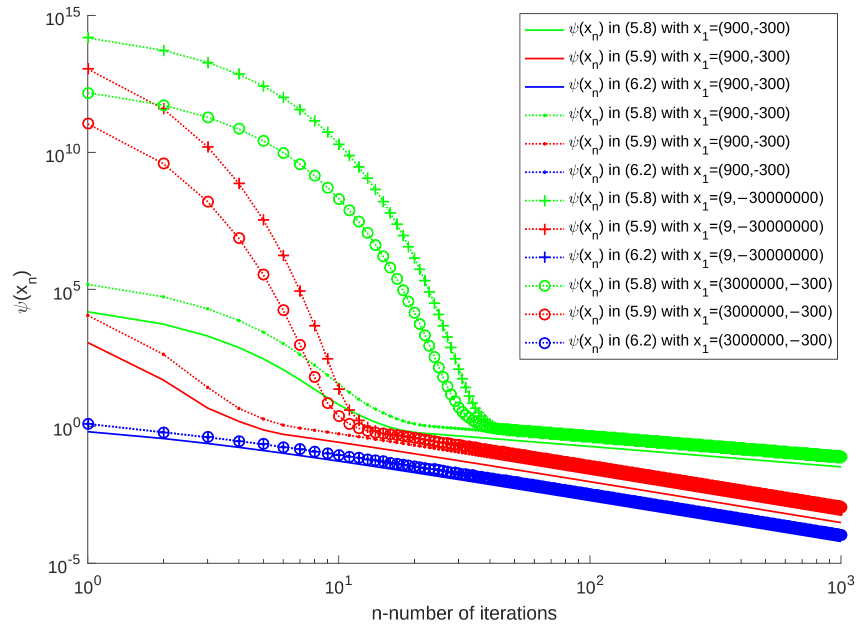

FIGURE 3. The stability of Algorithm (5.9) of Buong and Duong [6], Algorithm (5.10) of Zhou and Wang [43], and our Algorithm (6.2), with respect to the initial value x_1 in Example 6.2.

We consider next the mathematical programming problem (1.1) with $\Lambda = \mathbb{N}$. Starting from $x_1 \in \mathbb{R}^d$, Algorithm 5.12 takes the form:

$$\begin{cases} y_n &= \frac{1}{t_n}(w_1 P_{C_1} + w_2 P_{C_2} + \cdots + w_n P_{C_n})(x_n), \\ x_{n+1} &= (I - \alpha_n \mu F) y_n, \quad \text{for all } n \in \mathbb{N}. \end{cases} \tag{6.3}$$

Example 6.3. We consider the case $d = 2$,

$$C_i = B_1[a_i] \quad \text{with} \quad a_i = (1 + 1/i, 0) \in \mathbb{R}^2, \quad i \in \mathbb{N}.$$

Consider the objective function $\psi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\psi(x) = (u-1)^2 + (v-2)^2, \quad \forall x = (u,v) \in \mathbb{R}^2.$$

Here, $F(x) = (2u-2, 2v-4) \in \mathbb{R}^2$ for $x = (u,v) \in \mathbb{R}^2$.

One can see that F is 2-Lipschitz continuous and 2-strongly monotone. Therefore, for $\mu \in (0,1)$, Theorem 5.11 guarantees that Algorithm (6.3) with an initial point $x_1 \in \mathbb{R}^d$ converges to the solution of Problem 6.1. Set $w_i := 1/2^i$ for $i \in \mathbb{N}$ and $\alpha_n := 1/(n+2)$. With the starting point $x_1 = (3,3)$, the computational results of Algorithm (6.3) are presented in Figure 4, which plots the value of $\psi(x_n)$ $(n = 1, 2, ..., 3000)$ when $\mu = 1/10, 1/30, 1/60$, respectively.

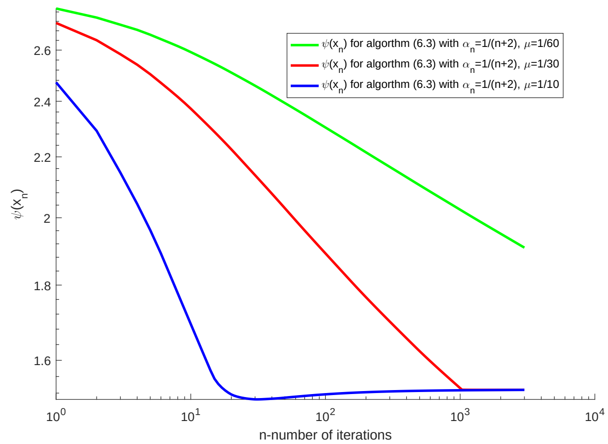

FIGURE 4. Plots of $\psi(x_n)$ $(n = 1, 2, ..., 3000)$ in Example 6.3 when $\alpha_n = 1/(n + 2)$ and $\mu = 1/10, 1/30, 1/60$.

Example 6.4. We consider the case $d = 64$. Consider the map

$$F(x) = \psi'(x), \quad \text{where } \psi(x) = \|x\|^2/2.$$

The mapping F is 1-Lipschitz continuous and 1-strongly monotone. Let

$$C_1 = B_1[0] \quad \text{and} \quad C_2 = \{x \in \mathbb{R}^{64} : \|x - (1, 1, 0 \ldots, 0)\| \leq 1\}.$$

To perform the exact computation of $P_{C_1 \cap C_2}$ is not easy, and hence the implementation of (1.3) with $T = P_{C_1 \cap C_2}$ is not easy. However, for $\mu \in (0, 2)$, the parallel Algorithm (6.2) can be applied for finding the solution of problem (1.1). We take

$$w_1 = w_2 = 1/2, \quad \mu = 1/100 \quad \text{and} \quad \alpha_n = 1/(n + 1)^a \text{ for some } a \in (0, 1].$$

With the starting point $x_1 = (-0.5, -0.5, \cdots, -0.5) \in \mathbb{R}^{64}$, the computational results of Algorithm (6.2) are presented in Figure 5. It plots the values of $\psi(x_n)$ $(n = 1, 2, ..., 10000)$ when $a = 1/1000, 1/100, 1/10$ and 1, respectively. We note that, as shown in Figure 5, the values of $\psi(x_n)$ $(n = 1, 2, ..., 10000)$ coincide when $a = 1/100$ and $a = 1/1000$.

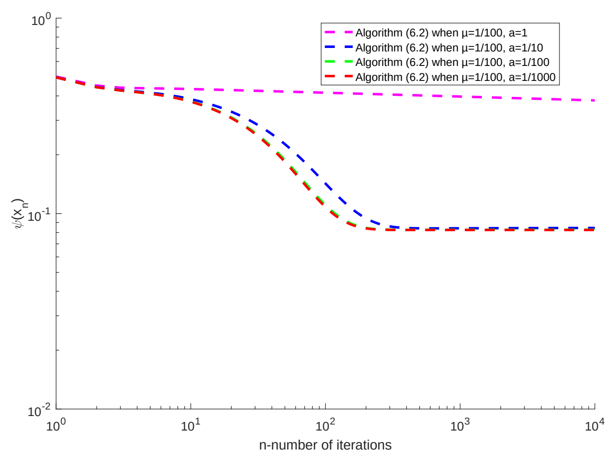

FIGURE 5. Plots of $\psi(x_n)$ $(n = 1, 2, ..., 10000)$ in Example 6.4 when $a = 1/1000, 1/100, 1/10$ and 1.

6.2. **Multiple-set split feasibility problems.** The multiple-set spilt feasibility problem (MSFP) [11] can be expressed as follows. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be real Hilbert spaces and let p and q be two natural numbers. For each $i \in \Lambda_p := \{1, 2, \cdots, p\}$, let C_i be a nonempty closed convex subset of $\mathcal{H}_1$. For each $j \in \Lambda_q := \{1, 2, \cdots, q\}$, let Q_j be a nonempty closed convex subset of $\mathcal{H}_2$, and let $A_j : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The multiple-set split feasibility problem MSSFP(Λ_p, Λ_q) is

$$\text{find } x^* \in \mathcal{H}_1 \text{ such that } \quad x^* \in C := \bigcap_{i \in \Lambda_p} C_i \text{ and } A_j x^* \in Q_j, \quad \forall j \in \Lambda_q.$$

The MSSFP(Λ_1, Λ_1) reduces to the split feasibility problem discussed in [10].

For $i \in \Lambda_p$, let $w_i \in (0, 1)$ such that $\sum_{i=1}^p w_i = 1$, and for $j \in \Lambda_q$, let $s_j \in (0, +\infty)$. Let $B : \mathcal{H}_1 \to \mathcal{H}_1$ be the gradient ∇f of the convex and continuously differentiable function $f : \mathcal{H}_1 \to \mathbb{R}$ defined by

$$f(x) = \sum_{j=1}^{q} s_j f_j(x), \quad \forall x \in \mathcal{H}_1, \tag{6.4}$$

where

$$f_j(x) := \frac{1}{2}\|A_j x - P_{Q_j} A_j x\|^2, \quad \forall x \in \mathcal{H}_1,$$

with

$$\nabla f_j = A_j^*(I - P_{Q_j})A_j.$$

Hence

$$B = \sum_{j=1}^{q} s_j \nabla f_j.$$

For $\gamma \in (0, +\infty)$, define

$$T_i = P_{C_i}(I - \gamma B).$$

Observe that

$$\begin{aligned} x^* \in \bigcap_{i=1}^{p} \mathrm{Fix}(P_{C_i}(I - \gamma B)) \quad &\Longleftrightarrow \quad x^* = P_{C_i}(I - \gamma B)x^* \text{ for all } i \in \Lambda_p \\ &\Longleftrightarrow \quad x^* \in \bigcap_{i=1}^{p} C_i \text{ and } 0 = A_j^*(I - P_{Q_j})A_j(x^*) \quad \text{for all } j \in \Lambda_q. \\ &\Longleftrightarrow \quad x^* \in \bigcap_{i=1}^{p} C_i \text{ and } A_j x^* \in Q_j \quad \text{for all } j \in \Lambda_q. \end{aligned}$$

Being the gradient of a convex and continuously differentiable real function, B is 1-strongly monotone. Moreover, B is also $\mathcal{L}$-Lipschitz, where $\mathcal{L} = \sum_{j=1}^q s_j \|A_j\|^2$. It follows that B is $\frac{1}{\mathcal{L}}$−inverse strongly monotone. Hence, for $\gamma \in (0, +\infty)$, the mapping γB is $\frac{1}{\gamma\mathcal{L}}$-inverse strongly monotone. If $\frac{1}{\gamma\mathcal{L}} \in (1/2, +\infty)$, then $I - \gamma B$ is $\frac{\gamma\mathcal{L}}{2}$-averaged and hence T_i is $(2 + \mathcal{L}\gamma)/4$-averaged. This shows that, for $\gamma \in (0, 2/\mathcal{L})$, the map $\sum_{i=1}^p w_i T_i$ is $(2 + \mathcal{L}\gamma)/4$-averaged. Thus, MSSFP$(\Lambda_p, \Lambda_q)$ is the common fixed point problem of the finite family $\{T_i : i \in \Lambda_p\}$ of $(2 + \mathcal{L}\gamma)/4$-averaged mappings.

Problem 6.5. Let $\mathcal{H}_1 = \mathbb{R}^{d_1}$ and $\mathcal{H}_2 = \mathbb{R}^{d_2}$. Consider the following optimization problem:

$$\text{find } x^* \in \bigcap_{i \in \Lambda_p} \mathrm{Fix}(T_i) \text{ such that } \quad \psi(x^*) = \min\{\psi(x) : x \in \bigcap_{i \in \Lambda_p} \mathrm{Fix}(T_i)\}, \tag{6.5}$$

where $\psi : \mathbb{R}^{d_1} \to \mathbb{R}$ is a differentiable convex function and $T_i := P_{C_i}(I - \gamma B)$, $\forall i \in \Lambda_p$.

We now consider MSSFP(Λ_p, Λ_q) by employing our parallel iterative technique. One can derive different exact and inexact algorithms from Section 5 for solving MSSFP(Λ_p, Λ_q) and MSSFP$(\mathbb{N}, \mathbb{N})$. For illustration, from Corollary 5.8, we have

Theorem 6.6. *Assume that the solution set $\Gamma_{p,q}$ of* $\mathrm{MSSFP}(\Lambda_p, \Lambda_q)$ *is nonempty. Let $F : \mathcal{H}_1 \to \mathcal{H}_1$ be an η-strongly monotone and L-Lipschitz continuous operator such that $\mathrm{VIP}_{\Gamma_{p,q}}(F, \mathcal{H}_1)$ has a unique solution x^*. Assume that $\gamma \in (0, 2/\mathcal{L})$. Let $\{x_n\}$ be the orbit in $\mathcal{H}_1$ generated by $x_1 \in \mathcal{H}_1$ and defined by*

$$x_{n+1} = (I - \alpha_n \mu F) \sum_{i=1}^{p} w_i T_i(x_n) \text{ for all } n \in \mathbb{N}, \tag{6.6}$$

where $\{\alpha_n\}$ is a sequences in $(0,1]$ satisfying the assumption (A1) and $T_i := P_{C_i}(I - \gamma B)$.

(a) *$\{x_n\}$ converges strongly to x^*.*
(b) *$\{x_n\}$ is in the closed ball $B_r[x^*]$ provided that* $\max\left\{\|x_1 - x^*\|, \frac{\mu}{\tau}\|F(x^*)\|\right\} \leq r < +\infty$.

If ψ is strongly convex in Problem 6.5, then Algorithm (6.6) can be applied.

Example 6.7. We consider $d_1 = 64$, $d_2 = 75$, $p = 30$ and $q = 50$. Let

$$e_1^1 = [1, 1, \cdots, 1]^T \in \mathbb{R}^{64} \quad \text{and} \quad e_1^2 = [1, 1, \cdots, 1]^T \in \mathbb{R}^{75}.$$

For $i \in \Lambda_{30}$, let $C_i = B_{r_i}[a_i]$ be a closed ball in $\mathbb{R}^{64}$, where the center a_i is randomly chosen from the 64-dimensional order box $[-15e_1^1, 15e_1^1]$ and the radius r_i is randomly chosen from the scalar interval $[50, 70]$. For $j \in \Lambda_{50}$, let $Q_j = \{y \in \mathbb{R}^{75} : L_j \leq y \leq U_j\}$, where L_j is a vector in $\mathbb{R}^{75}$ randomly chosen from the 75-dimensional box $[-20e_1^2, 20e_1^2]$, and U_j is another vector in $\mathbb{R}^{75}$ randomly chosen from the 75-dimensional box $[40e_1^2, 90e_1^2]$. Consider the map $F(x) = \psi'(x)$, where $\psi(x) = \|x\|^2/2$.

Since F is 1-Lipschitz continuous and 1-strongly monotone, for $\mu \in (0,1)$, Corollary 5.8 guarantees that Algorithm (6.6) with an initial point $x_1 \in \mathbb{R}^{64}$ converges to the solution of Problem 6.5. Here the objective functional f is defined by (6.4). With the starting point $x_1 = 1000e_1^1 \in \mathbb{R}^{64}$, the computational results of Algorithm (6.6) with $\mu = 0.75$ are presented in Figure 6, which plots the values of $f(x_n)$ $(n = 1, 2, \ldots, 1000)$ among the four cases in which all $0 < s_i < 100$, $0 < s_i < 1$, $s_i = 1$, and $s_i \in (0,1)$ with $\sum_{j=1}^{50} s_j = 1$, respectively. We observe that Algorithm (6.6) has better performance when $\{s_1, \cdots, s_{50}\} \subset (0,1)$ with $\sum_{j=1}^{50} s_j = 1$.

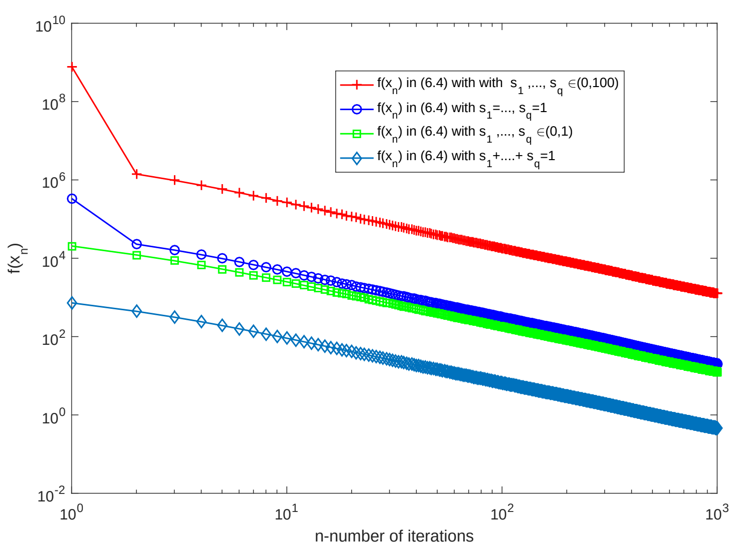

FIGURE 6. The plots of $f(x_n)$ $(n = 1, 2, \ldots, 1000)$ in Example 6.7 when $\{s_1, \cdots, s_{50}\} \subset (0, 100)$, $\{s_1, \cdots, s_{50}\} \subset (0, 1)$, $\{s_1, s_2, \ldots, s_{50}\} = \{1, 1 \ldots, 1\}$, and $\{s_1, \ldots, s_{50}\} \subset (0, 1)$ with $\sum_{j=1}^{50} s_j = 1$.

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