NORM OF POSITIVE SUM PRESERVERS OF NONCOMMUTATIVE $L^p(\mathcal{M})$ SPACES

JUN ZHANG, MING-CHENG TSAI, AND NGAI-CHING WONG

Abstract. Let $1 < p < +\infty$. Let $L^p(\mathcal{M})$ and $L^p(\mathcal{N})$ be the noncommutative L^p-spaces associated to von Neumann algebras $\mathcal{M}$ and $\mathcal{N}$, respectively. Let $\varphi : L^p_+(\mathcal{M}) \to L^p_+(\mathcal{N})$ be a surjective map between positive elements preserving the norm of sum, i.e.,

$$\|\varphi(x) + \varphi(y)\|_p = \|x + y\|_p, \quad x, y \in L^p_+(\mathcal{M}).$$

We show that there is a Jordan $*$-isomorphism $J : \mathcal{M} \to \mathcal{N}$, and φ can be extended uniquely to a surjective real linear positive isometry from $L^p_{\text{sa}}(\mathcal{M})$ onto $L^p_{\text{sa}}(\mathcal{N})$. When $\mathcal{M}$ is approximately semifinite, especially semifinite or hyperfinite, $\varphi(R) = \Theta_*(R^p)^{1/p}$ for every $R \in L^p_+(\mathcal{M})$, where $\Theta = J^{-1}$ and $\Theta_* : L^1(\mathcal{M}) \, (\cong M_*) \to L^1(\mathcal{N}) \, (\cong N_*)$ is its predual map. In the case when $\mathcal{M}$ has a normal faithful semifinite trace $\tau_{\mathcal{M}}$ (and so does $\mathcal{N}$), $\varphi(x) = hJ(x)$ for every $x \in L^p_+(\mathcal{M}, \tau_{\mathcal{M}}) \cap M_+$, where $h^p = d(\tau_{\mathcal{M}} \circ \Theta)/d\tau_{\mathcal{N}}$ is the non-commutative Radon-Nikodym derivative of $\tau_{\mathcal{M}} \circ \Theta$ with respect to $\tau_{\mathcal{N}}$. We also provide a similar result when $p = +\infty$, and counter examples for the case $p = 1$.

1. Introduction

The celebrated Mazur-Ulam theorem [9] states that every surjective map $T : E \to F$ between normed spaces, preserving the norm of differences and fixing zero, extends to a real linear isometry from E onto F. One may ask what happens if T preserves the norm of sums instead of differences, i.e., if

$$\|Tx + Ty\| = \|x + y\|, \quad \forall x, y \in E.$$

It turns out to be easy, by noting that we have $T0 = 0$ and $T(-x) = -Tx$ automatically, and the Mazur-Ulam theorem applies.

It then arises the question when the domain and range of T are not the whole linear spaces; see, e.g., [1,4,5,11,15]. In [17], we propose the following open problem.

Problem 1.1. Let E, F be ordered Banach spaces with positive cones E_+, F_+, respectively. Let $T : E_+ \to F_+$ be a surjective map preserving the norm of sums. Can T be extended to a positive real linear isometry from E onto F?

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We answer Problem 1.1 affirmatively for the case when E, F are smooth Banach lattices, and L_p spaces when $p \in (1, \infty]$, while we also provide a counterexample for the case $p = 1$ in [17]. There are also positive answers for $C(X)$ spaces in [1], for von Neumann algebras in [10], and for general unital C^*-algebras in [2]. The same is true if one considers bijective maps between positive definite cones in unital C^*-algebras but equipped with a sort of Schatten p-norm [4] for $p \in (1, \infty]$.

In this paper, we give a positive answer for non-commutative $L_p(\mathcal{M})$-spaces in Theorem 3.4; see also Theorems 1.4 and 1.5 below. Note that a noncommutative $L^p(\mathcal{M})$ space is not a Banach lattice unless $\mathcal{M}$ is abelian. This prevents us from directly applying the technique developed in the abelian case in [17]. Anyway, let us recall our result for commutative L^p-spaces.

Theorem 1.2 ([17, Theorem 3.3])**.** *Let $\varphi : L^p_+(\Omega_1, \Sigma_1, \mu_1) \to L^p_+(\Omega_2, \Sigma_2, \mu_2)$ be a bijective map, where $1 < p \leq \infty$. Suppose*

$$\|x + y\|_p = \|\varphi(x) + \varphi(y)\|_p, \quad \forall x, y \in L^p_+(\Omega_1, \Sigma_1, \mu_1).$$

Then φ extends to a surjective positive linear isometry from $L^p(\Omega_1, \Sigma_1, \mu_1)$ onto $L^p(\Omega_2, \Sigma_2, \mu_2)$. More precisely, there exists a regular set isomorphism Ψ from Σ_1 onto Σ_2 inducing a bijective positive linear map $\psi : L^p(\Omega_1, \Sigma_1, \mu_1) \to L^p(\Omega_2, \Sigma_2, \mu_2)$, and a locally measurable function h on Ω_2 such that

$$\varphi(x) = h \cdot \psi(x), \quad \forall x \in L^p_+(\Omega_1, \Sigma_1, \mu_1). \tag{1.1}$$

When $1 < p < +\infty$, we have

$$\int_{\Psi(A)} |h(t)|^p d\mu_2 = \mu_1(A), \quad \textit{for each } \sigma\textit{-finite } A \in \Sigma_1.$$

In other words, $|h|^p = \dfrac{d(\mu_1 \circ \Psi^{-1})}{d\mu_2}$ is the Radon-Nikodym derivative of $\mu_1 \circ \Psi^{-1}$ with respect to μ_2. When $p = +\infty$, we have

$$h(y) = 1, \quad \textit{locally almost everywhere on } \Omega_2.$$

When the underlying measure spaces are localizable, $\mathcal{M} = L^\infty(\Omega_1, \Sigma_1, \mu_1)$ and $\mathcal{N} = L^\infty(\Omega_2, \Sigma_2, \mu_2)$ are commutative von Neumann algebras with predual spaces $L^1(\Omega_1, \Sigma_1, \mu_1)$ and $L^1(\Omega_2, \Sigma_2, \mu_2)$, respectively. In this case, the regular set isomorphism Ψ defining ψ in (1.1) can be thought of an orthomorphism between the projection lattices of $\mathcal{M}$ and $\mathcal{N}$. By Dye's Theorem [3], Ψ extends uniquely to a Jordan $*$-isomorphism $J : \mathcal{M} \to \mathcal{N}$. We simply have $\psi = J$ when $p = +\infty$. When $1 < p < +\infty$, let $\Theta = J^{-1}$ with the predual map $\Theta_* : L^1(\Omega_1, \Sigma_1, \mu_1) \to L^1(\Omega_2, \Sigma_2, \mu_2)$. Then we have $\psi(f) = \Theta_*(f^p)^{1/p}$ for all f in $L^p_+(\Omega_1, \Sigma_1, \mu_1)$.

We are going to provide a noncommutative version of Theorem 1.2. To this end, we need the following counter part result about norm of difference preservers recently developed in [8]. Set

$$L^p_+(\mathcal{M})^\beta_\alpha = \left\{S \in L^p_+(\mathcal{M}) : \alpha \leq \|S\|_p \leq \beta\right\}, \quad 0 \leq \alpha < \beta < +\infty.$$

Theorem 1.3 ([8, Theorem 1.3])**.** *Let $p \in [1, \infty]$, and $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras. Assume there is a metric preserving bijection $\Phi : L^p_+(\mathcal{M})^\beta_\alpha \to$*

$L^p_+(\mathcal{N})^\beta_\alpha$, *i.e.*,

$$\|\Phi(x) - \Phi(y)\|_p = \|x - y\|_p, \quad \forall x, y \in L^p_+(\mathcal{M})^\beta_\alpha.$$

(a) $\mathcal{M}$ *and* $\mathcal{N}$ *are* $*$*-isomorphic.*

(b) If $\mathcal{M} \not\cong \mathbb{C}$ *and* $\mathcal{M}$ *is approximately semifinite, then there is a unique Jordan* $*$*-isomorphism* $\Theta : \mathcal{N} \to \mathcal{M}$ *satisfying* $\Phi(R) = \Theta_*(R^p)^{1/p}$ *for any* $R \in L^p_+(\mathcal{M}, \tau_\mathcal{M})^\beta_\alpha$.

Here is the main result in this paper.

Theorem 1.4. *Let* $p \in (1, +\infty]$, *and* $\mathcal{M}$ *and* $\mathcal{N}$ *be two von Neumann algebras. Assume there is a surjective map* $\varphi : L^p_+(\mathcal{M}) \to L^p_+(\mathcal{N})$ *such that*

$$\|\varphi(x) + \varphi(y)\|_p = \|x + y\|_p, \quad \forall x, y \in L^p_+(\mathcal{M}).$$

(a) $\mathcal{M}$ *and* $\mathcal{N}$ *are* $*$*-isomorphic, and* φ *extends uniquely to a positive surjective real linear isometry* $\theta : L^p_{\mathrm{sa}}(\mathcal{M}) \to L^p_{\mathrm{sa}}(\mathcal{N})$.

(b) If $p = +\infty$ *then* φ *extends uniquely to a Jordan* $*$*-isomorphism* $J : \mathcal{M} \to \mathcal{N}$.

(c) If $1 < p < +\infty$ *and* $\mathcal{M}$ *is approximately semifinite, then there is a unique Jordan* $*$*-isomorphism* $\Theta : \mathcal{N} \to \mathcal{M}$ *satisfying* $\varphi(R) = \Theta_*(R^p)^{1/p}$ *for any* $R \in L^p_+(\mathcal{M})$.

In the abelian case, every function in $L^p_+(\mu)$ can be approximated in norm by functions from $L^\infty_+(\mu)$. However, one of the difficulties in studying noncommutative $L^p(\mathcal{M})$ space arises from the fact that $L^p(\mathcal{M}) \cap \mathcal{M} = \{0\}$ when $\mathcal{M}$ is not semifinite. If $\mathcal{M}$ has a faithful semifinite trace $\tau_\mathcal{M}$, nevertheless, there is a weak* dense two-sided self-adjoint ideal $S_\mathcal{M}$ of $\mathcal{M}$ embedded into the noncommutative $L^p(\mathcal{M}, \tau_\mathcal{M})$ space. In other words, the intersection $L^p_+(\mathcal{M}, \tau_\mathcal{M}) \cap \mathcal{M}_+$ is reasonably big to represent $\mathcal{M}$, as well as $L^p(\mathcal{M}, \tau_\mathcal{M})$. This motivates us to include the following result in this paper. We note that any one of $\mathcal{M}$ and $\mathcal{N}$ being semifinite suffices to ensure its conclusion due to Theorem 1.4(a).

Theorem 1.5. *Let* $1 < p \leq +\infty$. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be two semifinite von Neumann algebras with traces* $\tau_\mathcal{M}$ *and* $\tau_\mathcal{N}$, *respectively. Suppose that* $\varphi : L^p(\mathcal{M}, \tau_\mathcal{M}) \cap \mathcal{M}_+ \to L^p(\mathcal{N}, \tau_\mathcal{N}) \cap \mathcal{N}_+$ *is a surjective map satisfying that*

$$\|x + y\|_p = \|\varphi(x) + \varphi(y)\|_p, \quad \forall x, y \in L^p(\mathcal{M}, \tau_\mathcal{M}) \cap \mathcal{M}_+.$$

Then there exists uniquely a Jordan $*$*-isomorphism* $J : \mathcal{M} \to \mathcal{N}$ *such that*

$$\varphi(x) = J(x)h = hJ(x), \quad x \in L^p(\mathcal{M}, \tau_\mathcal{M}) \cap \mathcal{M}_+.$$

Here, $h^p = \dfrac{d\tau_\mathcal{M} \circ J^{-1}}{d\tau_N}$ *is the noncommutative Radon-Nikodym derivative of* $\tau_\mathcal{M} \circ J^{-1}$ *with respect to* τ_N *when* $1 < p < +\infty$; *and* $h = 1$ *if* $p = +\infty$.

In Section 2, we will give a brief description of the construction of noncommutative L^p-spaces. The proofs of Theorems 1.4 and 1.5 are given in Section 3. We note that, however, neither of Theorems 1.4 nor 1.5 holds when $p = 1$, as shown by counter examples. In Section 4, we will provide two concrete examples to demonstrate the case when $p = +\infty$.

2. Preliminaries

Let $\mathcal{M}$ be a von Neumann algebra, that is, a self-adjoint algebra of operators on a Hilbert space H that is closed in the weak operator topology. A *trace* on $\mathcal{M}$ is a nonnegative extended real-valued function τ defined on the positive part $\mathcal{M}_+$ of $\mathcal{M}$ which satisfies

(1) $\tau(x+y) = \tau(x) + \tau(y)$ for all $x, y \in \mathcal{M}_+$;
(2) $\tau(\lambda x) = \lambda\tau(x)$ for all $\lambda \geq 0$ and $x \in \mathcal{M}_+$;
(3) $\tau(xx^*) = \tau(x^*x)$ for all $x \in \mathcal{M}$.

If τ satisfies conditions (1) and (2) but not necessarily (3), then we call it a *weight*. We say that τ is *normal* if $\sup \tau(x_\alpha) = \tau(\sup x_\alpha)$ for any bounded increasing net $\{x_\alpha\}$ in $\mathcal{M}_+$, *semifinite* if for any nonzero $x \in \mathcal{M}_+$ there is a nonzero $y \in \mathcal{M}_+$ such that $y \leq x$ and $\tau(y) < +\infty$, and *faithful* if $\tau(x) = 0$ implies $x = 0$ for any $x \in \mathcal{M}_+$. If $\tau(1) < +\infty$, we say that τ is *finite*. A von Neumann algebra $\mathcal{M}$ is said to be *finite* (resp. *semifinite*) if it admits a normal finite (resp. semifinite) faithful trace.

Definition 2.1. A von Neumann algebra $\mathcal{M}$ is said to be *approximately semifinite* [14] if

- there is an increasing family $\{\mathcal{M}_i\}_{i\in\mathfrak{I}}$ of semifinite von Neumann subalgebras of $\mathcal{M}$ such that $\bigcup_{i\in\mathfrak{I}} \mathcal{M}_i$ is $\sigma(\mathcal{M}, \mathcal{M}_*)$-dense in $\mathcal{M}$, and
- there is a normal conditional expectation $E_i : \mathcal{M} \to \mathcal{M}_i$ with $E_i(1)$ being the identity of $\mathcal{M}_i$ such that $E_i \circ E_j = E_i$ whenever $i \leq j$ in $\mathfrak{I}$.

The class of approximately semifinite von Neumann algebras includes, in particular, all semifinite algebras, all hyperfinite algebras, and all type III_0-factors with separable preduals. See also [8] for more details.

We follow the construction of noncommutative L^p-spaces demonstrated in [13] and [16]. Let $\mathcal{M}$ denote a semifinite von Neumann algebra on a Hilbert space H with a given normal semifinite faithful trace $\tau_\mathcal{M}$. Let $S_\mathcal{M}$ be the subset of $\mathcal{M}$ of elements x of finite traces, i.e., $\tau_\mathcal{M}(|x|) < +\infty$, where $|x|$ denotes the operator $(x^*x)^{1/2}$. The set $S_\mathcal{M}$ is quite big, as it is a self-adjoint two sided ideal of $\mathcal{M}$ and dense in $\mathcal{M}$ in the strong operator topology. Moreover, it is closed under taking p powers, i.e., $|x|^p \in S_\mathcal{M}$ whenever $x \in S_\mathcal{M}$ and $0 < p < +\infty$.

For $x \in \mathcal{M}$ and $1 \leq p < +\infty$, let

$$\|x\|_p = \tau_\mathcal{M}(|x|^p)^{1/p}.$$

Then $\|\cdot\|_p$ defines a norm on $S_\mathcal{M}$. We call the norm completion of $S_\mathcal{M}$ the *noncommutative $L^p(\mathcal{M}, \tau_\mathcal{M})$ space*.

We can identify $L^\infty(\mathcal{M}, \tau_\mathcal{M})$ with $\mathcal{M}$ and $L^1(\mathcal{M}, \tau_\mathcal{M})$ with the predual $\mathcal{M}_*$ of $\mathcal{M}$. The positive cone $L^p_+(\mathcal{M}, \tau_\mathcal{M})$ of $L^p(\mathcal{M}, \tau_\mathcal{M})$ is the completion of $L^p(\mathcal{M}, \tau_\mathcal{M}) \cap \mathcal{M}_+$ under the norm $\|\cdot\|_p$. We will write $L^p(\mathcal{M})$ if the trace $\tau_\mathcal{M}$ is understood.

The situation when $\mathcal{M}$ is not semifinite is far more complicated. Let $\mathcal{M}$ be a general von Neumann algebra on a Hilbert space H, but not necessarily semifinite. We note that every von Neumann algebra has a normal faithful semifinite weight.

Fix a normal semifinite faithful weight ϕ on $\mathcal{M}$. Consider the modular automorphism group α corresponding to ϕ. There exists a normal faithful semifinite trace τ on the von Neumann algebra crossed product $\check{\mathcal{M}} := \mathcal{M}\bar{\rtimes}_\alpha\mathbb{R}$ satisfying some compatibility condition with ϕ. Denote by $L^0(\check{\mathcal{M}}, \tau)$ the completion of $\check{\mathcal{M}}$ under the vector topology defined by a neighborhood basis at 0 of the form

$$U(\epsilon, \delta) := \{x \in \check{\mathcal{M}} : \|xp\| \leq \epsilon \text{ and } \tau(1-p) \leq \delta, \text{ for a projection } p \in \check{\mathcal{M}}\}.$$

Then the *-algebra structure of $\check{\mathcal{M}}$ extends to a *-algebra structure of $L^0(\check{\mathcal{M}}, \tau)$.

Elements in $L^0(\check{\mathcal{M}}, \tau)$ can be regarded as closed densely defined operators on $L^2(\mathbb{R}; H)$. More precisely, let T be a densely defined closed operator on $L^2(\mathbb{R}; H)$ affiliated with $\check{\mathcal{M}}$, and $|T|$ be its absolute value with spectral projection-valued measure $E_{|T|}$. Then T corresponds uniquely to an element in $L^0(\check{\mathcal{M}}, \tau)$ if and only if $\tau\big(1 - E_{|T|}([0, \lambda])\big) < \infty$ when λ is large. Conversely, every element in $L^0(\check{\mathcal{M}}, \tau)$ arises from a closed operator in this way. Under this identification, the *-operation on $L^0(\check{\mathcal{M}}, \tau)$ coincides with the adjoint. The addition and the multiplication on $L^0(\check{\mathcal{M}}, \tau)$ are the closures of the corresponding operations for closed operators. Denote by $L^0_+(\check{\mathcal{M}}, \tau)$ the set of all positive self-adjoint operators in $L^0(\check{\mathcal{M}}, \tau)$. For x, y in $L^0(\check{\mathcal{M}}, \tau)$, we write $x \perp y$ if $|x||y| = 0$, i.e., the positive operators have orthogonal support projections.

The dual action $\hat{\alpha} : \mathbb{R} \to \mathrm{Aut}(\check{\mathcal{M}})$ extends to an action on $L^0(\check{\mathcal{M}}, \tau)$. For any $p \in [1, \infty]$, we set

$$L^p(\mathcal{M}) := \big\{T \in L^0(\check{\mathcal{M}}, \tau) : \hat{\alpha}_s(T) = e^{-s/p}T \text{ for all } s \in \mathbb{R}\big\}$$

(where, by convention, $e^{-s/\infty} = 1$). Then $L^\infty(\mathcal{M})$ coincides with the subalgebra $\mathcal{M}$ of $\check{\mathcal{M}} \subseteq L^0(\check{\mathcal{M}}, \tau)$. Moreover, if $T \in L^0(\check{\mathcal{M}}, \tau)$ and $T = u|T|$ is the polar decomposition, then $T \in L^p(\mathcal{M})$ if and only if $|T| \in L^p(\mathcal{M})$. The product of an element in $L^\infty(\mathcal{M})$ with an element in $L^p(\mathcal{M})$ (in whatever order) is again in $L^p(\mathcal{M})$. Hence, $L^p(\mathcal{M})$ is canonically an $\mathcal{M}$-bimodule. Let $L^p_{\mathrm{sa}}(\mathcal{M})$ denote the set of all self-adjoint operators in $L^p(\mathcal{M})$ and put $L^p_+(\mathcal{M}) := L^p(\mathcal{M}) \cap L^0_+(\check{\mathcal{M}}, \tau)$.

When $p \in (0, \infty)$, the Mazur map

$$S \mapsto S^{\frac{1}{p}} \qquad \big(S \in L^0_+(\check{\mathcal{M}}, \tau)\big)$$

restricts to a bijection from $L^1_+(\mathcal{M})$ onto $L^p_+(\mathcal{M})$. Elements in $L^p_+(\mathcal{M})$ are identified with $S^{\frac{1}{p}}$ for a unique element $S \in L^1_+(M)$. When $p \in (1, \infty)$, the function

$$\|T\|_p := \big\||T|^p\big\|_1^{1/p}$$

is a norm on $L^p(\mathcal{M})$, and $\big(L^p(\mathcal{M}), L^p_+(\mathcal{M})\big)$ becomes an ordered Banach space.

It is known that $(L^p(\mathcal{M}), L^p_+(\mathcal{M}))$ is independent of the choice of the faithful semifinite weight ϕ up to an isometric order isomorphism (see, e.g., Theorem 37 and Corollary 38 in Chapter II of [16]). If $\mathcal{M}$ is semifinite with a faithful normal semifinite trace $\phi = \tau_{\mathcal{M}}$, then the above two constructions of noncommutative $L^p(\mathcal{M})$ space will be isometrically and order isomorphic to each other. In this paper, we usually write $L^p(\mathcal{M}, \tau_{\mathcal{M}})$ even when $\mathcal{M}$ is not semifinite; in this case, we refer to the Haagerup trace norm $\tau_{\mathcal{M}}(\cdot) = \|\cdot\|_1$ instead.

3. Norm of positive sum preservers

All noncommutative $L^p(\mathcal{M},\tau_{\mathcal{M}})$ spaces are uniformly convex and uniformly smooth with dual space $L^q(\mathcal{M},\tau_{\mathcal{M}})$ for $p,q\in(1,\infty)$ with $1/p+1/q=1$. In particular, the following result holds for general von Neumann algebra $\mathcal{M}$.

Lemma 3.1 ([7, Lemma 3.1])**.** *If $t\in\mathbb{R}\mapsto h(t)\in L^p_+(\mathcal{M},\tau_{\mathcal{M}}), 1<p<+\infty$, is differentiable (with respect to the L_p-norm) at $t=\alpha$ and $h(\alpha)\neq 0$, then $t\in\mathbb{R}\mapsto \tau_{\mathcal{M}}(h(t)^p)\in\mathbb{R}_+$ is differentiable at α and its derivative is*

$$\frac{d}{dt}\Big|_{t=\alpha}\tau_{\mathcal{M}}(h(t)^p)=p\tau_{\mathcal{M}}\left(h(\alpha)^{p-1}\frac{d}{dt}\Big|_{t=\alpha}h(t)\right). \tag{3.1}$$

While it always holds that

$$\|x\pm y\|_p^p=\|x\|_p^p+\|y\|_p^p \quad \text{whenever} \quad x,y\in L^p(\mathcal{M},\tau_{\mathcal{M}}) \text{ such that } x\perp y,$$

we also have a converse.

Lemma 3.2 ([7, Corollary 6.5]; see also [13, Proposition A.2])**.** *Let $\mathcal{M}$ be a von Neumann algebra and $1<p<+\infty$. For any $x,y\in L^p_+(\mathcal{M},\tau_{\mathcal{M}})$, we have*

$$\|x+y\|_p^p=\|x\|_p^p+\|y\|_p^p \quad \textit{if and only if} \quad xy=0.$$

Lemma 3.3. *Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras and $1<p<+\infty$. Suppose that $\varphi: L^p_+(\mathcal{M},\tau_{\mathcal{M}})\to L^p_+(\mathcal{N},\tau_{\mathcal{N}})$ is a surjective map satisfying that*

$$\|x+y\|_p=\|\varphi(x)+\varphi(y)\|_p. \tag{3.2}$$

Then we have

(1) *φ preserves orthogonality, that is $xy=0$ if and only if $\varphi(x)\varphi(y)=0$.*
(2) *φ is additive and nonnegative homogeneous, i.e.*
   (i) *$\varphi(y_1+y_2)=\varphi(y_1)+\varphi(y_2)$ for all $y_1,y_2\in L^p_+(\mathcal{M},\tau_{\mathcal{M}})$;*
   (ii) *$\varphi(\lambda y)=\lambda\varphi(y)$ for all $\lambda\geq 0$ and $y\in L^p_+(\mathcal{M},\tau_{\mathcal{M}})$.*

Proof. (1) Taking $x=y$ in equation (3.2), one has $\|x\|_p=\|\varphi(x)\|_p$. Hence, from Lemma 3.2,

$$\begin{aligned} xy=0 \quad &\Leftrightarrow \quad \|x+y\|_p^p=\|x\|_p^p+\|y\|_p^p \\ &\Leftrightarrow \quad \|\varphi(x)+\varphi(y)\|_p^p=\|\varphi(x)\|_p^p+\|\varphi(y)\|_p^p \\ &\Leftrightarrow \quad \varphi(x)\varphi(y)=0. \end{aligned}$$

(2) To see φ is nonnegative homogeneous, for $\lambda>0$ we observe that

$$\|\varphi(x)+\varphi(\lambda x)\|_p=\|x+\lambda x\|_p=\|x\|_p+\|\lambda x\|_p=\|\varphi(x)\|_p+\|\varphi(\lambda x)\|_p.$$

From the strictly convexity of $L^p(\mathcal{M},\tau_{\mathcal{M}})$, we have $\varphi(\lambda x)=\delta\varphi(x)$ for some $\delta>0$. Then $\lambda\|x\|_p=\|\varphi(\lambda x)\|_p=\|\delta\varphi(x)\|_p=\delta\|x\|_p$, we get $\delta=\lambda$, and thus $\varphi(\lambda x)=\lambda\varphi(x)$ for all x in $L^p_+(\mathcal{M},\tau_{\mathcal{M}})$ and for all $\lambda\geq 0$.

To see φ is additive, we observe again that $\|\varphi(x)+t\varphi(y)\|_p=\|\varphi(x)+\varphi(ty)\|_p=\|x+ty\|_p$ for all $t\geq 0$. Using Lemma 3.1 and setting $h(t)=x+ty$, we have

$$\frac{d\|x+ty\|_p^p}{dt}\Big|_{t=0^+}=\frac{d\tau_{\mathcal{M}}(h(t)^p)}{dt}\Big|_{t=0^+}=p\tau_{\mathcal{M}}(x^{p-1}y).$$

Hence, differentiating both sides of $\|x+ty\|_p^p = \|\varphi(x)+t\varphi(y)\|_p^p$ with respect to t at 0, we have

$$\tau_{\mathcal{M}}(x^{p-1}y) = \tau_{\mathcal{N}}(\varphi(x)^{p-1}\varphi(y)).$$

It follows

$$\begin{aligned}
&\tau_{\mathcal{N}}(\varphi(x)^{p-1}(\varphi(y_1+y_2)-\varphi(y_1)-\varphi(y_2))) \\
=\ &\tau_{\mathcal{N}}(\varphi(x)^{p-1}(\varphi(y_1+y_2))) - \tau_{\mathcal{N}}(\varphi(x)^{p-1}\varphi(y_1)) - \tau_{\mathcal{N}}(\varphi(x)^{p-1}\varphi(y_2)) \\
=\ &\tau_{\mathcal{M}}(x^{p-1}(y_1+y_2)) - \tau_{\mathcal{M}}(x^{p-1}y_1) - \tau_{\mathcal{M}}(x^{p-1}y_2) = 0.
\end{aligned}$$

Since φ is surjective, choosing $\varphi(x) = [\varphi(y_1+y_2)-\varphi(y_1)-\varphi(y_2)]^+$, the positive part of $\varphi(y_1+y_2)-\varphi(y_1)-\varphi(y_2)$, we get $\|[\varphi(y_1+y_2)-\varphi(y_1)-\varphi(y_2)]^+\|_p^p = 0$ since the positive part and the negative part of $\varphi(y_1+y_2)-\varphi(y_1)-\varphi(y_2)$ are orthogonal. Hence, the positive part of $\varphi(y_1+y_2)-\varphi(y_1)-\varphi(y_2)$ is 0. Similarly, the negative part is also 0, and therefore $\varphi(y_1+y_2) = \varphi(y_1)+\varphi(y_2)$. □

Theorem 3.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras and $1 < p < +\infty$. Suppose that $\varphi : L_+^p(\mathcal{M},\tau_{\mathcal{M}}) \to L_+^p(\mathcal{N},\tau_{\mathcal{N}})$ is a surjective map satisfying that*

$$\|x+y\|_p = \|\varphi(x)+\varphi(y)\|_p. \tag{3.3}$$

Then there exists a unique surjective complex linear map $\omega : L^p(\mathcal{M},\tau_{\mathcal{M}}) \to L^p(\mathcal{N},\tau_{\mathcal{N}})$ extending φ. Moreover, its restriction defines a surjective positive real linear isometry $\theta : L_{\mathrm{sa}}^p(\mathcal{M},\tau_{\mathcal{M}}) \to L_{\mathrm{sa}}^p(\mathcal{N},\tau_{\mathcal{N}})$.

Proof. Observe that for any $x,y \in L_+^p(\mathcal{M},\tau_{\mathcal{M}})$, we have $x-y = (x-y)^+ - (x-y)^-$, and thus $(x-y)^+ + y = (x-y)^- + x$. Since φ is additive by Lemma 3.3, we have $\varphi((x-y)^+) + \varphi(y) = \varphi((x-y)^-) + \varphi(x)$. This gives $\varphi(x)-\varphi(y) = \varphi((x-y)^+) - \varphi((x-y)^-)$. Since $(x-y)^+ \perp (x-y)^-$, we have $\varphi((x-y)^+) \perp \varphi((x-y)^-)$ by Lemma 3.3 again. It follows that

$$\begin{aligned}
\|\varphi(x)-\varphi(y)\|_p^p &= \|\varphi((x-y)^+) - \varphi((x-y)^-)\|_p^p = \|\varphi((x-y)^+) + \varphi((x-y)^-)\|_p^p \\
&= \|(x-y)^+ + (x-y)^-\|_p^p = \|x-y\|_p^p, \quad \forall x,y \in L_+^p(\mathcal{M},\tau_{\mathcal{M}}).
\end{aligned}$$

That is, φ preserves norm of differences.

For $x \in L_{\mathrm{sa}}^p(\mathcal{M},\tau_{\mathcal{M}})$, we define

$$\theta(x) = \varphi(x^+) - \varphi(x^-).$$

It follows from Lemma 3.3 that θ is well-defined and real linear. Moreover, $\theta(x) \perp \theta(y)$ if $x \perp y$. Furthermore,

$$\|\theta(x)\|_p = \|\varphi(x^+) - \varphi(x^-)\|_p = \|x^+ - x^-\|_p = \|x\|_p.$$

Thus, θ is a positive real linear isometry from $L^p(\mathcal{M},\tau_{\mathcal{M}})_{\mathrm{sa}}$ onto $L_{\mathrm{sa}}^p(\mathcal{N},\tau_{\mathcal{N}})$ extending φ.

For any $x \in L^p(\mathcal{M},\tau_{\mathcal{M}})$, we write $x = \dfrac{x+x^*}{2} + i\dfrac{x-x^*}{2i} := x_1 + ix_2$, where x_1, x_2 are self-adjoint elements in $L_{\mathrm{sa}}^p(\mathcal{M},\tau_{\mathcal{M}})$. Define

$$\omega(x_1+ix_2) = \theta(x_1) + i\theta(x_2).$$

It is easy to check that

$$\omega(x+y) = \omega(x) + \omega(y), \quad \omega(\lambda x) = \lambda\omega(x),$$

for all $x = x_1 + ix_2, y = y_1 + iy_2$ in $L^p(\mathcal{M}, \tau_{\mathcal{M}})$ and $\lambda = a + ib$ in $\mathbb{C}$. The uniqueness of θ and ω is plain. □

Note again that for any von Neumann algebra $\mathcal{M}$, we have $L^\infty(\mathcal{M}) \cong \mathcal{M}$ and $L^1(\mathcal{M}) \cong \mathcal{M}_*$.

Proof of Theorem 1.4. The case $p = +\infty$ can be derived from a result of Molnár [10, Theorem 2.7] which states that every surjective norm of sum preserver $\varphi : \mathcal{M}_+ \to \mathcal{N}_+$ extends uniquely to a Jordan $*$-isomorphism $J : \mathcal{M} \to \mathcal{N}$.

For the case $1 < p < +\infty$, by Theorem 3.4 we see in particular that φ extends to a bijection from the positive unit ball $L^1_+(\mathcal{M})^1_0$ of $L^1_+(\mathcal{M})$ onto the positive unit ball $L^1_+(\mathcal{N})^1_0$ of $L^1_+(\mathcal{N})$ such that $\|\varphi(x) - \varphi(y)\|_p = \|x - y\|_p$ for all x, y in $L^1_+(\mathcal{M})^1_0$. If $\mathcal{M}$ is not one-dimensional, then the assertions follow from Theorem 1.3.

Finally, when $\mathcal{M} = \mathcal{N} = \mathbb{C}$, we have

$$L^p(\mathcal{M}, \tau_{\mathcal{M}}) = L^p(\mathbb{C}, \mu) \quad \text{and} \quad L^p(\mathcal{N}, \tau_{\mathcal{N}}) = L^p(\mathbb{C}, \nu)$$

for some positive measures μ and ν on $\mathbb{C}$. The assertions follow from our previous results for the abelian case, namely, Theorem 1.2, and the discussion after it. □

When $p = 1$, we have a counter example in [17, Example 4.1]. There we have a norm of positive sum preserver of the commutative $\ell^1_n = L^1(\ell^\infty_n)$ space associated to the n-dimensional abelian von Neumann algebra ℓ^∞_n with $n \geq 2$, which is neither affine nor continuous. See also Example 3.8 for a noncommutative counter example.

Proof of Theorem 1.5. Arguing as in Lemma 3.3 and noticing that all operations are done inside the domain $L^p(\mathcal{M}, \tau_{\mathcal{M}}) \cap \mathcal{M}_+$ and range $L^p(\mathcal{N}, \tau_{\mathcal{N}}) \cap \mathcal{N}_+$ of φ, we have again the same conclusions there. More precisely, we have

(1) φ preserves orthogonality, that is, $xy = 0$ if and only if $\varphi(x)\varphi(y) = 0$;
(2) φ is additive and nonnegative homogeneous, that is,
  (i) $\varphi(x + y) = \varphi(x) + \varphi(y)$;
  (ii) $\varphi(\lambda y) = \lambda\varphi(y)$;
(3) φ preserves metric, that is, $\|\varphi(x) - \varphi(y)\|_p = \|x - y\|_p$;

where $x, y \in L^p(\mathcal{M}, \tau_{\mathcal{M}}) \cap \mathcal{M}_+$ and $\lambda \geq 0$.

We extend the domain of φ from $L^p(\mathcal{M}, \tau_{\mathcal{M}}) \cap \mathcal{M}_+$ to $L^p(\mathcal{M}, \tau_{\mathcal{M}}) \cap \mathcal{M}_{\text{sa}}$ by defining

$$\theta(x) = \varphi(x^+) - \varphi(x^-), \quad \forall x \in L^p(\mathcal{M}, \tau_{\mathcal{M}}) \cap \mathcal{M}_{\text{sa}}.$$

It follows from the fact $\theta(x^+) = \theta(x)^+$ that θ also preserves the metric, i.e.,

$$\begin{aligned}\|\theta(x) - \theta(y)\|_p^p &= \|\theta(x-y)^+ - \theta(x-y)^-\|_p^p = \|\theta(x-y)^+\|_p^p + \|\theta(x-y)^-\|_p^p \\ &= \|(x-y)^+\|_p^p + \|(x-y)^-\|_p^p = \|x-y\|_p^p,\end{aligned}$$

for all x, y in $L^p(\mathcal{M}, \tau_{\mathcal{M}}) \cap \mathcal{M}_{\text{sa}}$. In particular, φ can be extended to a surjective real linear isometry from $L^p_{\text{sa}}(\mathcal{M}, \tau_{\mathcal{M}})$ onto $L^p_{\text{sa}}(\mathcal{N}, \tau_{\mathcal{N}})$, and thus provides a surjective metric preserving map between their positive unit balls. Then Theorem 1.4 applies.

In particular, there is a unique Jordan $*$-isomorphism $\Theta : \mathcal{N} \to \mathcal{M}$ satisfying that $\varphi(R) = \Theta_*(R^p)^{1/p}$ for any $R \in L^p_+(\mathcal{M})$.

Let $J = \Theta^{-1} : \mathcal{M} \to \mathcal{N}$ and let $h = \left(\frac{d\tau_{\mathcal{M}} \circ \Theta}{d\tau_N}\right)^{1/p}$ be the $1/p$ th power of the non-commutative Radon-Nikodym derivative of $\tau_{\mathcal{M}} \circ \Theta$ with respect to $\tau_{\mathcal{N}}$ (see, e.g., [12, Theorem 5.12]). Note that the unbounded operator h is affiliated with $\mathcal{N}$, and commutes with all elements in $\mathcal{N}$. Then for all x in $L^p(\mathcal{M}, \tau_{\mathcal{M}}) \cap \mathcal{M}_+$ we have

$$\tau_{\mathcal{N}}((J(x)h)^p y) = \tau_{\mathcal{M}}(x^p \Theta(y)) = \tau_{\mathcal{N}}(\Theta_*(x^p)y) = \tau_{\mathcal{N}}(\varphi(x)^p y),$$

for all $y \in L^\infty(\mathcal{N}, \tau_{\mathcal{N}})_+ = \mathcal{N}_+$. Thus $\varphi(x) = J(x)h$ for all x in $L^p(\mathcal{M}, \tau_{\mathcal{M}}) \cap \mathcal{M}_+$ as asserted. □

Corollary 3.5. *Assume $\mathcal{M}$ is a factor and $1 < p < +\infty$. Suppose $\varphi : L^p(\mathcal{M}, \tau_{\mathcal{M}}) \cap \mathcal{M}_+ \to L^p(\mathcal{N}, \tau_{\mathcal{N}}) \cap \mathcal{N}_+$ is a surjective map satisfying that $\|x+y\|_p = \|\varphi(x)+\varphi(y)\|_p$ for all $x, y \in L^p(\mathcal{M}, \tau_{\mathcal{M}}) \cap \mathcal{M}_+$. Then there is a $*$-algebra isomorphism or anti-isomorphism J of $\mathcal{M}$ onto $\mathcal{N}$ and a positive scalar λ such that $\varphi = \lambda J$.*

Proof. The assertion follows from Theorem 1.5 and well-known facts on Jordan $*$-isomorphisms (cf. [6]). □

Corollary 3.6. *Let $\mathcal{M}$ be a finite factor with a normal faithful finite trace τ. Let φ be a transformation from $L^p(\mathcal{M}, \tau)$ onto itself satisfying that $\varphi(\mathcal{M}_+) = \mathcal{M}_+$ and $\|x + y\|_p = \|\varphi(x) + \varphi(y)\|_p$ for all $x, y \in \mathcal{M}_+$. Then the restriction of φ to $\mathcal{M}$ is either a $*$-algebra isomorphism or anti-isomorphism of $\mathcal{M}$.*

Corollary 3.7. *Let $\mathcal{M}$ be a type I factor with the canonical trace τ, and let φ be a transformation from $L^p(\mathcal{M}, \tau)$ onto itself satisfying that $\varphi(L^p_+(\mathcal{M}, \tau)) = L^p_+(\mathcal{M}, \tau)$ and $\|x + y\|_p = \|\varphi(x) + \varphi(y)\|_p$ for all $x, y \in \mathcal{M}_+$. Then there exists a $*$-algebra isomorphism or anti-isomorphism Φ of $\mathcal{M}$ such that $\varphi(x) = \Phi(x)$ for every $x \in L^p(\mathcal{M}, \tau)$.*

Example 3.8. For the case $p = 1$, Theorem 1.5 may not hold. For example, let $\varphi : L^1(\mathcal{M}, \tau_{\mathcal{M}}) \cap \mathcal{M}_+ \to L^1(\mathcal{M}, \tau_{\mathcal{M}}) \cap \mathcal{M}_+$ satisfy that $\varphi(x) = u_r x u_r^*$, where $\|x\|_1 = r$ and u_r is a randomly chosen unitary element in $\mathcal{M}$ associated with each $r \geq 0$. Clearly, φ is surjective. Moreover,

$$\begin{aligned}\|\varphi(x) + \varphi(y)\|_1 &= \tau_{\mathcal{M}}(\varphi(x)) + \tau_{\mathcal{M}}(\varphi(y)) = \tau_{\mathcal{M}}(u_{\|x\|} x u^*_{\|x\|}) + \tau_{\mathcal{M}}(u_{\|y\|} y u^*_{\|y\|}) \\ &= \tau_{\mathcal{M}}(x) + \tau_{\mathcal{M}}(y) = \tau_{\mathcal{M}}(x + y) = \|x + y\|_1.\end{aligned}$$

However, φ does not carry the form stated in Theorem 1.5.

4. Two examples when $p = +\infty$

In this section, two examples of norm of positive sum preservers are provided for the case $p = +\infty$. We verify the details by direct arguments.

Example 4.1. Consider the two dimensional abelian von Neumann algebra $\mathcal{M} = \mathcal{N} = \mathbb{C} \oplus_\infty \mathbb{C}$. Suppose $\varphi : \mathbb{R}_+ \oplus_\infty \mathbb{R}_+ \mapsto \mathbb{R}_+ \oplus_\infty \mathbb{R}_+$ is a map satisfying $\|x+y\|_\infty =$

$\|\varphi(x)+\varphi(y)\|_\infty$ for all x, y in $\mathbb{R}_+ \oplus_\infty \mathbb{R}_+$. We show directly that $\varphi(x) = Ux$ where U is a permutation; namely, φ assumes either the form

$$(x_1, y_1) \mapsto (x_1, y_1) \quad \text{or} \quad (x_1, y_1) \mapsto (y_1, x_1).$$

Proof. It is easy to see that $\varphi((0,0)) = (0,0)$ and $\|x\|_\infty = \|\varphi(x)\|_\infty$ for all x in $\mathbb{R}_+ \oplus_\infty \mathbb{R}_+$.

Claim 1: Either the case $\varphi((x_1, 0)) = (x_1, 0)$ and $\varphi((0, x_1)) = (0, x_1)$, or the case $\varphi((x_1, 0)) = (0, x_1)$ and $\varphi((0, x_1)) = (x_1, 0)$ holds for all $x_1 \geq 0$.

Suppose that $\varphi((x_1, 0)) = (x_1', y_1')$ and $\varphi((0, x_1)) = (x_2', y_2')$. We have

$$\max\{x_1', y_1'\} = \max\{x_2', y_2'\} = \max\{x_1' + x_2', y_1' + y_2'\} = x_1.$$

If $x_1' = x_1$, then $x_2' = 0$, $y_2' = x_1$ and $y_1' = 0$. The other case arises when $x_2' = x_1$.

Claim 2: $\varphi((x_1, x_1)) = (x_1, x_1)$ for all $x_1 \geq 0$.

Suppose that $\varphi((x_1, x_1)) = (x_1, y_1')$ in which $y_1' < x_1$. If $\varphi((x_1, 0)) = (x_1, 0)$ and $\varphi((0, x_1)) = (0, x_1)$, one gets $\|(x_1, x_1) + (0, 1)\|_\infty = \|(x_1, y_1') + (0, 1)\|_\infty$. Thus, $x_1 + 1 = y_1' + 1$, which is a contradiction. If $\varphi((x_1, 0)) = (0, x_1)$ and $\varphi((0, x_1)) = (x_1, 0)$, one gets $\|(x_1, x_1) + (1, 0)\|_\infty = \|(x_1, y_1') + (0, 1)\|_\infty$. This gives again the contradiction $x_1 + 1 = y_1' + 1$.

The same argument also removes the case $\varphi((x_1, x_1)) = (x_1', x_1)$ such that $x_1' < x_1$. Since $\|\varphi(x_1, x_1)\|_\infty = x_1$, we verify the claim.

Set

$$A = \{(x_1, y_1) : x_1 > 0, y_1 > 0, x_1 > y_1\}, \quad B = \{(x_1, y_1) : x_1 > 0, y_1 > 0, x_1 < y_1\}.$$

Claim 3: Either $\varphi(A) \subseteq A, \varphi(B) \subseteq B$, or $\varphi(A) \subseteq B, \varphi(B) \subseteq A$.

We prove that $\varphi(A) \subseteq A, \varphi(B) \subseteq B$ when the case $\varphi((x_1, 0)) = (x_1, 0)$ and $\varphi((0, x_1)) = (0, x_1)$ ever happens. Suppose on the contrary $\varphi(A) \nsubseteq A$, that is to say $\varphi((x_2, y_2)) = (x_2', y_2')$ for some $x_2 > y_2 > 0$ and $0 \leq x_2' \leq y_2'$. Then one has $y_2' = x_2$. It shows that $\|(x_2, y_2) + (0, x_2)\|_\infty = \|(x_2', x_2) + (0, x_2)\|_\infty$. Thus, $x_2 + y_2 = 2x_2$ which conflicts with $x_2 > y_2$. Similarly, $\varphi(B) \subseteq B$ is satisfied under this condition.

Analogously, we have $\varphi(A) \subseteq B, \varphi(B) \subseteq A$ when the case $\varphi((x_1, 0)) = (0, x_1)$ and $\varphi((0, x_1)) = (x_1, 0)$ ever holds.

Claim 4: Either $\varphi((x_1, y_1)) = (x_1, y_1)$ or $\varphi((x_1, y_1)) = (y_1, x_1)$ for all $(x_1, y_1) \in \mathbb{R}_+ \oplus_\infty \mathbb{R}_+$.

In the case $\varphi(A) \subseteq A$, we can assume that $\varphi((x_1, y_1)) = (x_1, y_1')$ where $x_1 > y_1, x_1 > y_1'$. It follows that $\|(x_1, y_1) + (0, x_1)\|_\infty = \|(x_1, y_1') + (0, x_1)\|_\infty$. Therefore, $y_1' = y_1$. Same argument can be used for the case $(x_1, y_1) \in B$. This shows that $\varphi((x_1, y_1)) = (x_1, y_1)$ for all $(x_1, y_1) \in \mathbb{R}_+ \oplus_\infty \mathbb{R}_+$.

On the other-hand, if $\varphi(A) \subseteq B, \varphi(B) \subseteq A$, similar arguments produce the other desired conclusion. □

Example 4.2. Consider the von Neumann algebra M_2 of 2×2 complex matrices with positive cone P_2. Suppose that $\varphi : P_2 \to P_2$ is a surjective map such that $\|A + B\|_\infty = \|\varphi(A) + \varphi(B)\|_\infty$ for any positive semidefinite matrices A, B in P_2.

We show directly that there exists a unitary matrix U such that φ assumes either the form
$$A \mapsto UAU^* \quad \text{or} \quad A \mapsto UA^{\mathrm{t}}U^*.$$

Proof. Fix $\lambda \geq 0$. Let $A = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$. Assume that there are unitary matrices U, V such that
$$\varphi(A) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^* \quad \text{and}$$
$$\varphi(B) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = V \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} V^*,$$
where $a_{11}, a_{22}, b_{11}, b_{22} \geq 0$, and $a_{21} = \overline{a_{12}}$ and $b_{21} = \overline{b_{12}}$ are complex conjugates.

As $\|\varphi(A)\|_\infty = \|\varphi(B)\|_\infty = \|\varphi(A) + \varphi(B)\|_\infty = \lambda$, computing traces we have
$$\lambda \leq a_{11} + a_{22} \leq 2\lambda, \ \lambda \leq b_{11} + b_{22} \leq 2\lambda, \quad \text{and} \quad \lambda \leq a_{11} + a_{22} + b_{11} + b_{22} \leq 2\lambda.$$
Hence, $\lambda_1 + \lambda_2 = a_{11} + a_{22} = \lambda$ and $\mu_1 + \mu_2 = b_{11} + b_{22} = \lambda$.

Since $\max\{\lambda_1, \lambda_2\} = \max\{\mu_1, \mu_2\} = \lambda$, it can be assumed that $\lambda_1 = \lambda$ and $\lambda_2 = 0$. Furthermore, set $\varphi(B) = U \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} U^*$, where $c_{11}, c_{22} \geq 0$ and $c_{21} = \overline{c_{12}}$. Hence, $\varphi(A) + \varphi(B) = U \begin{pmatrix} c_{11} + \lambda & c_{12} \\ c_{21} & c_{22} \end{pmatrix} U^*$ with
$$\|\varphi(A) + \varphi(B)\|_\infty = \frac{c_{11} + c_{22} + \lambda + \sqrt{(c_{11} + \lambda - c_{22})^2 + 4c_{12}c_{21}}}{2} = \lambda.$$
Since the trace of the matrix $\varphi(B)$ equals $c_{11} + c_{22} = \mu_1 + \mu_2 = \lambda$, we see that $c_{11} = c_{12} = c_{21} = 0$ and $c_{22} = \lambda$. Thus, there exists a unitary matrix U_λ such that
$$\varphi(\begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix}) = U_\lambda \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} U_\lambda^* \quad \text{and} \quad \varphi(\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}) = U_\lambda \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} U_\lambda^*.$$

Suppose that for another scalar $0 \leq \mu \leq \lambda$ and the matrix $D = \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix}$, we have $\varphi(D) = U_\lambda \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} U_\lambda^*$, where $d_{11}, d_{22} \geq 0$ and $d_{21} = \overline{d_{12}}$. Note that
$$\|\varphi(D)\|_\infty = \mu = \left\| \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \right\|_\infty,$$
and observe
$$\lambda + \mu = \left\| \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix} \right\|_\infty = \left\| \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \right\|_\infty.$$
The last sum of positive semi-definite matrices attains its norm $\lambda + \mu$ at the unit eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Consequently, $d_{11} = \mu$ and $d_{12} = d_{21} = 0$. Moreover, $0 \leq d_{22} \leq \mu$. On the other hand,
$$\max\{\lambda, \mu\} = \left\| \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} \mu & 0 \\ 0 & 0 \end{pmatrix} \right\|_\infty = \left\| \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix} + \begin{pmatrix} \mu & 0 \\ 0 & d_{22} \end{pmatrix} \right\|_\infty.$$

Hence $d_{22} = 0$ since $\mu \le \lambda$. Therefore,

$$\varphi(\begin{pmatrix}\mu & 0\\ 0 & 0\end{pmatrix}) = U_\lambda \begin{pmatrix}\mu & 0\\ 0 & 0\end{pmatrix} U_\lambda^*, \quad \text{whenever } 0 \le \mu \le \lambda.$$

Set $U = U_\lambda$ for a very large $\lambda > 0$. Then $\varphi(\begin{pmatrix}t & 0\\ 0 & 0\end{pmatrix}) = U \begin{pmatrix}t & 0\\ 0 & 0\end{pmatrix} U^*$ for any $t \in [0, \lambda]$. For any 2×2 positive semi-definite matrix $A = \begin{pmatrix}a & b\\ \bar{b} & c\end{pmatrix}$, let $\varphi(A) = U \begin{pmatrix}a' & b'\\ \bar{b}' & c'\end{pmatrix} U^*$. Hence,

$$\left\| \begin{pmatrix}t & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}a & b\\ \bar{b} & c\end{pmatrix} \right\|_\infty = \left\| \begin{pmatrix}t & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}a' & b'\\ \bar{b}' & c'\end{pmatrix} \right\|_\infty, \quad \forall t \in [0, \lambda].$$

It amounts to say that

$$t + a + c + \sqrt{(t+a-c)^2 + 4b\bar{b}} = t + a' + c' + \sqrt{(t+a'-c')^2 + 4b'\bar{b}'}. \tag{4.1}$$

Differentiating (4.1) with respect to t, we get

$$(t+a-c)^2((t+a'-c')^2 + 4b'\bar{b}') = (t+a'-c')^2((t+a-c)^2 + 4b\bar{b}),$$

or

$$b'\bar{b}'(t+a-c)^2 = b\bar{b}(t+a'-c')^2.$$

Comparing the coefficient of t^2, we get $b\bar{b} = b'\bar{b}'$.

In the case when $b = 0$, we have $b' = 0$. Put this into equation (4.1), we have $a = a'$ when t is chosen sufficiently large. Using the equation

$$\left\| \begin{pmatrix}0 & 0\\ 0 & t\end{pmatrix} + \begin{pmatrix}a & 0\\ 0 & c\end{pmatrix} \right\|_\infty = \left\| \begin{pmatrix}0 & 0\\ 0 & t\end{pmatrix} + \begin{pmatrix}a & 0\\ 0 & c'\end{pmatrix} \right\|_\infty,$$

we can also see $c = c'$.

On the other hand, $a - c = a' - c'$ when $b \neq 0$. In this case, the equation (4.1) ensures that $a + c = a' + c'$, and thus $a = a'$ and $c = c'$. Let $b' = e^{i\theta_b} b$ for some real scalar θ_b. It follows from the norm equality

$$\left\| \begin{pmatrix}a & b_1\\ \overline{b_1} & c\end{pmatrix} + \begin{pmatrix}a & b_2\\ \overline{b_2} & c\end{pmatrix} \right\|_\infty = \left\| \begin{pmatrix}a & e^{i\theta_{b_1}} b_1\\ e^{-i\theta_{b_1}}\overline{b_1} & c\end{pmatrix} + \begin{pmatrix}a & e^{i\theta_{b_2}} b_2\\ e^{-i\theta_{b_2}}\overline{b_2} & c\end{pmatrix} \right\|_\infty$$

that

$$\begin{aligned} & 2a + 2c + \sqrt{4(a-c)^2 + 4(b_1+b_2)(\overline{b_1}+\overline{b_2})} \\ = \; & 2a + 2c + \sqrt{4(a-c)^2 + 4(e^{i\theta_{b_1}} b_1 + e^{i\theta_{b_2}} b_2)(e^{-i\theta_{b_1}}\overline{b_1} + e^{-i\theta_{b_2}}\overline{b_2})}. \end{aligned}$$

It forces both $b_1\overline{b_2}$ and $e^{i(\theta_{b_1} - \theta_{b_2})} b_1\overline{b_2}$ have the same real parts.

Replacing U by the unitary $U \begin{pmatrix}1 & 0\\ 0 & e^{-i\theta_1}\end{pmatrix}$, we can assume $e^{i\theta_1} = 1$.

We discuss in two situations. The first case is for $e^{i\theta_i} = -1$, and we claim that $e^{i\theta_b} b = \bar{b}$ for all b in this situation. To this end, setting $(b_1, b_2) = (b, 1)$ and (b, i) respectively, we observe the real parts of complex numbers

$$\operatorname{Re} b = \operatorname{Re} e^{i\theta_b} b \quad \text{and} \quad \operatorname{Re}(-ib) = \operatorname{Re}(-ie^{i(\theta_b - \pi)} b) = \operatorname{Re}(ie^{i\theta_b} b).$$

It follows that $e^{i\theta_b} b = \bar{b}$ as claimed.

The second case is for $e^{i\theta_i} \neq -1$, and we claim that $e^{i\theta_b} = 1$ for all b in this situation. Setting $(b_1, b_2) = (b, 1)$ and (b, i) respectively, we observe the real parts of complex numbers

$$\operatorname{Re} b = \operatorname{Re} e^{i\theta_b} b \quad \text{and} \quad \operatorname{Re}(-ib) = \operatorname{Re}(-ie^{i(\theta_b - \theta_i)} b).$$

If $e^{i\theta_b} \neq 1$ then $e^{i\theta_b} b = \bar{b}$, and thus $\operatorname{Re}(-ib) = \operatorname{Re}(-ie^{-i\theta_i} \bar{b}) \neq -\operatorname{Re}(-i\bar{b})$. This contradiction shows that $e^{i\theta_b} = 1$ for all b as claimed.

Therefore, we have either

$$\varphi\left(\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}\right) = U \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} U^* \quad \text{or} \quad U \begin{pmatrix} a & \bar{b} \\ b & c \end{pmatrix} U^*.$$

That is to say, for any three 2×2 positive semi-definite matrices A_1, A_2, A_3, we can choose a large enough λ (depending on A_1, A_2, A_3) such that either

$$\varphi(A_j) = U_\lambda A_j U_\lambda^* \quad \text{or} \quad \varphi(A_j) = U_\lambda A_j^{\mathrm{t}} U_\lambda^*, \quad \forall j = 1, 2, 3.$$

This implies that φ is affine and preserves squares on the positive semi-definite cone P_2 of M_2. It then extends to a nonzero linear map from M_2 into M_2 sending projections to projections, and thus a Jordan homomorphism. The assertion then follows from known facts about Jordan $*$-homomorphisms of matrices. $\square$

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