METRIC PRESERVING BIJECTIONS BETWEEN POSITIVE SPHERICAL SHELLS OF NON-COMMUTATIVE L^p-SPACES

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Abstract. Let M be a von Neumann algebra and $p \in [1, \infty]$. Consider the non-commutative L^p-space $L^p(M)$ associated to M with canonical positive cone $L^p_+(M)$. By employing essentially normed space techniques, we extend our earlier result and show that for any $\epsilon \in (0, 1]$, the positive spherical shell

$$L^p_+(M)^1_{1-\epsilon} := \{T \in L^p_+(M) : 1 - \epsilon \leq \|T\| \leq 1\},$$

as a metric space, is a complete Jordan *-invariant for the von Neumann algebra M. In the particular case of $p = \infty$, we actually show that for any unital C^*-algebra A, the positive spherical shell $\{a \in A_+ : 1 - \epsilon \leq \|a\| \leq 1\}$ is a complete Jordan *-invariant for A.

Assume M is approximately semifinitie, which includes the case M is either a semifinite algebra, a hyper-finite algebra or a type III_0-factor with a separable predual. When $p \in (1, \infty)$ and $M \not\cong \mathbb{C}$ is approximately semifinite, a stronger conclusion follows: if $\Phi : L^p_+(M)^1_{1-\epsilon} \to L^p_+(N)^1_{1-\epsilon}$ is a metric preserving bijection, then there exists a Jordan *-isomorphism $\Theta : N \to M$ such that

$$\Phi(S^{\frac{1}{p}}) = \Theta_*(S)^{\frac{1}{p}} \quad \text{for all } S \in L^1_+(M)^1_{(1-\epsilon)^p}.$$

1. Introduction

In the literature, several partial structures of von Neumann algebras were shown to be complete Jordan *-invariants (see e.g. [4, Théorème 3.3], [6], [15, Theorem 2 and Corollary 5], [16, Theorem 4.5] and [30, Theorem 3]). Generalizing results in [26, 34, 35], D. Sherman showed in [27] that the metric space structure of the non-commutative L^p-space $L^p(M)$ is a complete Jordan *-invariant for the underlying von Neumann algebra M when $p \in [1, \infty] \setminus \{2\}$. Let us recall it clearly as follows.

Theorem 1.1 (Sherman [27]). *Let $p \in (1, \infty) \setminus \{2\}$, let M and N be two von Neumann algebras. If $T : L^p(M) \to L^p(N)$ is a bijective linear isometry, then there exists a Jordan *-isomorphism $J : M \to N$ and a unitary $w \in N$ such that $T(\varphi^{\frac{1}{p}}) = w(\varphi \circ J^{-1})^{\frac{1}{p}}$ for all $\varphi \in (M_*)_+$.*

The cases when $p = 1$ and $p = \infty$ are covered by the classical results of Kadison [14, 15], since $L^1(M) \cong M_*$ (where M_* is the predual of M) and $L^\infty(M) \cong M$. For a counter example for the exceptional case of $p = 2$, observe that the non-commutative L^2-space associated to the von Neumann algebra $\mathcal{B}(\ell^2)$ of bounded linear operators on the separable infinite dimensional Hilbert space ℓ^2 and the one associated to the commutative von Neumann algebra ℓ^∞ of bounded scalar sequences are both isometrically isomorphic to ℓ^2.

It is natural to ask whether it is possible to obtain a "smaller metric invariant". For example, motivated by the so-called Tingley's problem (see e.g., [5, 13, 33] and the references therein), the authors of [8] (respectively, [7]) showed that the unit sphere of $L^\infty(\mathcal{B}(H)) \cong \mathcal{B}(H)$ (respectively, $L^1(\mathcal{B}(H)) \cong \mathcal{B}(H)_*$)

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is a complete Jordan *-invariant for $\mathcal{B}(H)$. Moreover, it was shown in [31] that the unit sphere of $L^\infty(M) \cong M$ is a complete Jordan *-invariant for a finite von Neumann algebra M.

Along this line, we show in [20] that, for each $p \in [1, \infty]$, the contractive part $L^p_+(M)^1_0$ of the positive cone $L^p_+(M)$ of the non-commutative L^p-space is a complete Jordan *-invariant for the underlying von Neumann algebra M; namely, two von Neumann algebras M and N are Jordan *-isomorphic whenever there is a metric preserving bijection between $L^p_+(M)^1_0$ and $L^p_+(N)^1_0$. Note that one can include the case of $p = 2$ in this situation, since the positive cone of the non-commutative L^2-space encodes some information that cannot be recovered from merely the normed space structure.

Based on our earlier work [21], the first main result of the article is a further development along this direction. It shows that the positive spherical shell $L^p_+(M)^1_{1-\epsilon}$ is a complete Jordan *-invariant for the underlying von Neumann algebra for any $\epsilon \in (0, 1]$.

When E is a subset of a normed space X and $\alpha, \beta \in \mathbb{R}_+$ with $\alpha \leq \beta$, let us put

$$E^\beta_\alpha := \{x \in E : \alpha \leq \|x\| \leq \beta\}.$$

The precise statement of this first main result is the following.

Theorem 1.2. *Let $p \in [1, \infty]$, $\epsilon \in (0, 1]$, and M and N be two von Neumann algebras. If there is a metric preserving bijection $\Phi : L^p_+(M)^1_{1-\epsilon} \to L^p_+(N)^1_{1-\epsilon}$, then M and N are Jordan *-isomorphic.*

For $p = 1$, we have $L^1(M) \cong M_*$ and $L^1(N) \cong N_*$. Let $\mathcal{S}_M$ and $\mathcal{S}_N$ be the sets of normal states of M and N with proper support projections, respectively. We show that Φ restricts to a bijection from $\mathcal{S}_M$ onto $\mathcal{S}_N$, which preserves orthogonality. We then use a result of Dye in [6] to obtain the conclusion. In the case of $p = \infty$, we have $L^\infty(M) \cong M$ and $L^\infty(N) \cong N$, and the above theorem says $(M_+)^1_{1-\epsilon}$ is a complete Jordan *-invariant for the von Neumann algebra M. This assertion actually holds for unital C^*-algebras M and N, and it is proved via a generalization of the Mazur-Ulam theorem by Mankiewicz ([22]; see Proposition 4.3). For $p \in (1, \infty)$, we use a strict convexity argument to verify that Φ is "partially affine" and can be extended to a metric preserving bijection between the whole cones $L^p_+(M)$ and $L^p_+(N)$. Then we use results from [25] and [19] to finish the proof.

In line with Theorem 1.1, it is natural to ask whether the map Φ in Theorem 1.2 actually comes from a Jordan *-isomorphism. Although in the case of $p = \infty$, the precise answer to the above question is negative (see [20, Example 3.3]), we know from the argument of Theorem 4.4 that Φ extends to an isometric bijection, and hence is a Jordan *-isomorphism after translation and multiplication by a central symmetry. On the other hand, there is an evidence that the answer for the case of $p = 1$ could be positive. In fact, it was proved in [18] (see also [17]) that when $p = 1$ and M is of type I, then any isometric bijection from $L^1_+(M)^1_1$ onto $L^1_+(N)^1_1$ is defined by a Jordan *-isomorphism. Note that the arguments in [18] employ a lot of matrix function techniques and are very different from those in this article.

In order to tackle the above question for the case when $1 < p < \infty$, we will first show that the extension of Φ to the positive cones further extends to an isometric order isomorphism from $L^p_{\rm sa}(M)$ onto $L^p_{\rm sa}(N)$ (see Proposition 3.5). Note that a difficulty of this extension is that $L^p_+(M)$ may not contain any interior point of $L^p_{\rm sa}(M)$; otherwise, one could use a result of Mankiewicz (Proposition 4.3) to obtain this extension easily. On the other hand, to our best knowledge, it is not known if such a bijective isometry between the self-adjoint parts of non-commutative L^p-spaces has an isometric complexification (although it has to be the case if the strong version holds), and we cannot use Theorem 1.1 to obtain what we wanted.

Therefore, we will employ the concept of EP_1 as introduced by K. Watanabe [34] and D. Sherman [28], together with a result of Sourour [29] and Grein [10] concerning surjective isometries of vector-valued L^p-spaces (in the ordinary sense), to obtain the following second main theorem of the article. Note that a von Neumann algebra M with a nonzero type I_2 summand does not satisfy EP_1 (see Example 5.10), whilst approximately semifinite von Neumann algebras (see Definition A.5) without type I_2 summand satisfies EP_1 (see Proposition A.7). The class of approximately semifinite von Neumann algebras includes, in particular, all semifinite algebras, all hyperfinite algebras, and all type III_0-factors with separable preduals (see Remark 5.8(b)).

Theorem 1.3. *Let $p \in (1,\infty)$ and $\epsilon \in (0,1]$. Suppose that M and N are von Neumann algebras such that $M \ncong \mathbb{C}$ and M is approximately semifinite. If $\Phi : L^p_+(M)^1_{1-\epsilon} \to L^p_+(N)^1_{1-\epsilon}$ is a metric preserving surjection, there is a unique Jordan *-isomorphism $\Theta : N \to M$ satisfying $\Phi(R^{\frac{1}{p}}) = \Theta_*(R)^{\frac{1}{p}}$, for any $R \in L^1_+(M)^1_{(1-\epsilon)^p}$.*

Observe that in the case when $M = N = \mathbb{C}$, we have $L^p_+(M)^1_{1-\epsilon} = L^p_+(N)^1_{1-\epsilon} = [1-\epsilon, 1]$, and the induced metric is the Euclidean one: $d(x,y) = |x-y|$. The metric preserving bijection from $[1-\epsilon,1]$ to itself that sends x to $2-\epsilon-x$ cannot be extended to a linear map on $L^p_{\mathrm{sa}}(M)$. Therefore, we have an exception in this trivial case of $M \cong \mathbb{C}$.

It happens that all approximately semifinite algebras without a type I_2 summand satisfies EP_1. Therefore, we will consider the case when M is of type I_2 and the case when M satisfies EP_1, separately (and then combine the two cases together). For the benefit of the reader, some facts concerning the relation between EP_1 and approximately semifinite algebras will be recalled in the Appendix.

Theorems 1.2 and 1.3 concern with "closed" positive spherical shells. Of course, one can also consider the "open" positive spherical shells:

$$\big\{S \in L^p_+(M) : 1-\epsilon < \|S\| < 1\big\}.$$

Unlike the case of $p = \infty$ (in this case, $L^\infty(M)_+ = M_+$), the "open" positive spherical shells do not contain any open subset of $L^p_{\mathrm{sa}}(M)$ when $1 \le p < \infty$ (since $L^p_+(M)$ may not contain any open subset of $L^p_{\mathrm{sa}}(M)$; for example, $\ell^2_+ = L^2_+(\ell^\infty)$ does not contain any interior point of ℓ^2_{sa}). Thus, one cannot use the Mazur-Ulam-Mankiewicz theorem (see Proposition 4.3) to obtain a linear extension of a metric preserving bijection between "open" positive spherical shells. Nevertheless, the corresponding statements of both Theorems 1.2 and 1.3 for "open" positive spherical shells are also obtained.

Corollary 1.4. *Let $p \in [1,\infty]$ and $\epsilon \in (0,1]$. Suppose that there exists a metric preserving bijection*

$$\Phi : \big\{S \in L^p_+(M) : 1-\epsilon < \|S\| < 1\big\} \to \big\{T \in L^p_+(N) : 1-\epsilon < \|T\| < 1\big\}.$$

Then M and N are Jordan *-isomorphic. In the case when $p = \infty$, the map Φ can be extended to a Jordan *-isomorphism from M onto N after translation and multiplication by a central symmetry. Furthermore, if $p \in (1,\infty)$, $M \ncong \mathbb{C}$ and M is approximately semi-finite, then there is a Jordan *-isomorphism $\Theta : N \to M$ such that $\Phi(S^{\frac{1}{p}}) = \Theta_*(S)^{\frac{1}{p}}$.

Proof. Note that Φ can be extended to a metric preserving bijection between the metric completions of its domain and range, which coincide with the closed sets $L^p_+(M)^1_{1-\epsilon}$ and $L^p_+(N)^1_{1-\epsilon}$ of the Banach spaces $L^p(M)$ and $L^p(N)$, respectively. Thus the assertions follow from Theorems 1.2, 1.3 and 4.4. □

With a simple rescaling argument, we can also derive Theorems 1.2, 1.3 and 4.4 as well as Corollary 1.4 to hold when 1 and $1-\epsilon$ are replaced, respectively, by nonnegative numbers β and α satisfying $\alpha < \beta$. In other words, the existence of a metric preserving bijection $\Phi : L^p_+(M)^\beta_\alpha \to L^p_+(N)^\beta_\alpha$ guarantees similar conclusions in these results.

To end the introduction, we recall the link of our results to the Tingley's problem, which asks if every metric preserving bijection between the unit spheres of two Banach spaces extends to a linear isometry between the whole spaces (see, e.g., [5, 7, 8, 13, 33]). Since a linear map between non-commutative L^p-spaces is determined completely by its restriction to the positive sphere of the domain, one might expect that the "minimum" complete Jordan *-invariant for a von Neumann algebra M is $L^p_+(M)^1_1$. In this respect, we make the following conjecture.

Conjecture 1.5. Let M, N be von Neumann algebras, and let $p \in [1,\infty)$. If $\Psi : L^p_+(M)^1_1 \to L^p_+(N)^1_1$ is a metric preserving bijection, then there is a Jordan *-isomorphism $\Theta : N \to M$ satisfying $\Psi(R^{\frac{1}{p}}) = \Theta_*(R)^{\frac{1}{p}}$, for any $R \in L^1_+(M)^1_1$.

For $p = 1$, Conjecture 1.5 holds when M is commutative (see e.g., [17]), or more general, when M is of type I (see [18]). The case of $p > 1$ is basically unknown. Notice that one cannot use the solution for the Tingley's problem for Banach spaces and operator algebras (even if the full generality were obtained) to give a positive answer to the above conjecture (nor to prove Theorem 1.3). On the other hand, Theorems 1.2 and 1.3 suggest that Conjecture 1.5 has a positive answer. Furthermore, the methods provided in [12, 11] might be helpful, and we will explore into this and other possibilities in a future project.

2. Notation and preliminary

We fix some notations and recall some facts of non-commutative L^p-spaces. The material here are mainly taken from [25] and [32]. Let M be a von Neumann algebra with predual M_*, let $\mathcal{P}(M)$ be the set of projections in M and let $\mathcal{Z}(M)$ be the center of M. We fix a normal semifinite faithful weight φ on M, and consider the modular automorphism group α corresponding to φ. There exists a normal faithful semifinite trace τ on the von Neumann algebra crossed product $\check{M} := M\bar{\rtimes}_\alpha\mathbb{R}$ satisfying some compatibility condition with φ. Denote by $L^0(\check{M}, \tau)$ the completion of $\check{M}$ under the vector topology defined by a neighborhood basis at 0 of the form

$$U(\epsilon, \delta) := \{x \in \check{M} : \|xp\| \leq \epsilon \text{ and } \tau(1-p) \leq \delta, \text{ for a projection } p \in \check{M}\}.$$

Then the *-algebra structure of $\check{M}$ extends to a *-algebra structure of $L^0(\check{M}, \tau)$.

If M is faithfully represented on a Hilbert space $\mathfrak{H}$, then elements in $L^0(\check{M}, \tau)$ can be regarded as closed operators on $L^2(\mathbb{R}; \mathfrak{H})$, the Hilbert space of square integrable $\mathfrak{H}$-valued functions on $\mathbb{R}$. More precisely, let T be a densely defined closed operator on $L^2(\mathbb{R}; \mathfrak{H})$ affiliated with $\check{M}$, and $|T|$ be its absolute value with spectral measure $E_{|T|}$. Then T corresponds uniquely to an element in $L^0(\check{M}, \tau)$ if and only if $\tau\big(1 - E_{|T|}([0, \lambda])\big) < \infty$ when λ is large. Conversely, every element in $L^0(\check{M}, \tau)$ arises from a closed operator in this way. Under this identification, the *-operation on $L^0(\check{M}, \tau)$ coincides with the adjoint. The addition and the multiplication on $L^0(\check{M}, \tau)$ are the closures of the corresponding operations for closed operators. Denote by $L^0_+(\check{M}, \tau)$ the set of all positive self-adjoint operators in $L^0(\check{M}, \tau)$.

The dual action $\hat{\alpha} : \mathbb{R} \to \mathrm{Aut}(\check{M})$ extends to an action on $L^0(\check{M}, \tau)$. For any $p \in [1, \infty]$, we set

$$L^p(M) := \big\{T \in L^0(\check{M}, \tau) : \hat{\alpha}_s(T) = e^{-s/p}T, \text{ for all } s \in \mathbb{R}\big\}$$

(where, by convention, $e^{-s/\infty} = 1$). Then $L^\infty(M)$ coincides with the subalgebra M of $\check{M} \subseteq L^0(\check{M}, \tau)$. Moreover, if $T \in L^0(\check{M}, \tau)$ and $T = u|T|$ is the polar decomposition, then $T \in L^p(M)$ if and only if $|T| \in L^p(M)$. The product of an element in $L^\infty(M)$ with an element in $L^p(M)$ (in whatever order) is again in $L^p(M)$. Hence, $L^p(M)$ is canonically an M-bimodule. Let $L^p_{\mathrm{sa}}(M)$ denote the set of all self-adjoint operators in $L^p(M)$ and put $L^p_+(M) := L^p(M) \cap L^0_+(\check{M}, \tau)$.

When $q \in (0,\infty)$, the Mazur map

$$S \mapsto S^{\frac{1}{q}} \qquad \big(S \in L^0_+(\check{M},\tau)\big)$$

restricts to a bijection from $L^1_+(M)$ onto $L^q_+(M)$. Since we will use this connection between $L^1_+(M)$ and $L^q_+(M)$ frequently,

> *elements in $L^q_+(M)$ may sometimes be written in the form $S^{\frac{1}{q}}$ (for a unique element $S \in L^1_+(M)$).*

Throughout this article, we identify $\big(L^1(M), L^1_+(M)\big)$ with $(M_*, (M_*)_+)$ as ordered vector spaces. Hence, $\big(L^1(M), L^1_+(M)\big)$ becomes an ordered Banach space with the norm $\|\cdot\|_1$ induced from M_*. When $p \in (1,\infty)$, the function:

$$\|T\|_p := \big\||T|^p\big\|_1^{\frac{1}{p}}$$

is a norm on $L^p(M)$, and $\big(L^p(M), L^p_+(M)\big)$ becomes an ordered Banach space. It is well-known that this ordered Banach space is independent of the choices of φ and τ (up to isometric order isomorphisms).

For any $p,q \in (1,\infty)$ satisfying $1/p + 1/q = 1$, if $S \in L^p(M)$ and $T \in L^q(M)$, then $ST \in L^1(M)$. The function $T \mapsto \mathrm{Tr}(T) := T(1)$ on $L^1(M) = M_*$ is called the "Haagerup trace", and the assignment $S \mapsto \mathrm{Tr}(S\cdot)$ defines a bijection from $L^p(M)$ to $(L^q(M))^*$ that sends $L^p_{\mathrm{sa}}(M)$ and $L^p_+(M)$ onto the set of hermitian functionals and the set of positive functionals on $L^q(M)$, respectively.

For $R \in L^p_{\mathrm{sa}}(M)$, we denote by $\mathbf{s}_R$ and by $\mathbf{z}_R$ the *support* and the *central support* of R, respectively; namely, $\mathbf{s}_R$ is the smallest element in $\mathcal{P}(M)$ satisfying $\mathbf{s}_R R = R$ and $\mathbf{z}_R$ is the smallest element in $\mathcal{P}(M) \cap \mathcal{Z}(M)$ satisfying $\mathbf{z}_R R = R$. It is easy to see that if $T \in L^1_+(M)$, then $\mathbf{s}_{T^{\frac{1}{p}}} = \mathbf{s}_T$ and $\mathbf{z}_{T^{\frac{1}{p}}} = \mathbf{z}_T$.

The following lemma is a reformulation of [25, Proposition A.2] together with some well-known facts (see e.g. [25, Fact 1.3]).

Lemma 2.1. *Let $p \in (1,\infty)$.*

(a) Suppose that $R_1, R_2 \in L^p_{\mathrm{sa}}(M)$. If $\mathbf{s}_{R_1}\mathbf{s}_{R_2} = 0$, then $\|R_1 + R_2\|_p^p = \|R_1\|_p^p + \|R_2\|_p^p$. Conversely, if $p \neq 2$ and $\|R_1 + R_2\|_p^p = \|R_1 - R_2\|_p^p = \|R_1\|_p^p + \|R_2\|_p^p$, then $\mathbf{s}_{R_1}\mathbf{s}_{R_2} = 0$.

(b) For $T_1, T_2 \in L^1_+(M)$, the following statements are equivalent.

(1) $\mathbf{s}_{T_1} \cdot \mathbf{s}_{T_2} = 0$.
(2) $T_1^{\frac{1}{p}} T_2^{\frac{1}{p}} = 0$.
(3) $\|T_1^{\frac{1}{p}} + T_2^{\frac{1}{p}}\|_p^p = \|T_1^{\frac{1}{p}}\|_p^p + \|T_2^{\frac{1}{p}}\|_p^p$.
(4) $\|T_1 - T_2\|_1 = \|T_1\|_1 + \|T_2\|_1$.

(c) $S \mapsto S^{\frac{1}{p}}$ is a homeomorphism from $L^1_+(M)$ onto $L^p_+(M)$.

The next lemma should also be well-known, but since we cannot find a precise reference for it in the literature, we give its justification here.

Lemma 2.2. *Let $q \in (0,\infty)$. If $R, T \in L^1(M)_+$ with $\mathbf{s}_R\mathbf{s}_T = 0$, then $(R+T)^q = R^q + T^q$.*

Proof. Let $\mathfrak{K}_R := \mathbf{s}_R(L^2(\mathbb{R};\mathfrak{H}))$ and $\mathfrak{K}_T := \mathbf{s}_T(L^2(\mathbb{R};\mathfrak{H}))$. Let $\mathfrak{K}_0$ be the orthogonal complement of $\mathfrak{K}_R + \mathfrak{K}_T$. As $R = \mathbf{s}_R R \mathbf{s}_R$, the restriction, R_1, of R on $\mathfrak{K}_R$ is a densely defined positive self-adjoint operator. The same is true for the restriction, T_1, of T on $\mathfrak{K}_T$. One may then identify R, T and $R + T$

with $R_1 \oplus 0_{\mathfrak{K}_T} \oplus 0_{\mathfrak{K}_0}$, $0_{\mathfrak{K}_R} \oplus T_1 \oplus 0_{\mathfrak{K}_0}$ and $R_1 \oplus T_1 \oplus 0_{\mathfrak{K}_0}$, respectively. Thus, $R^q + T^q$ can be identified with the closed operator $R_1^q \oplus T_1^q \oplus 0_{\mathfrak{K}_0}$, which clearly coincides with $(R+T)^q$. □

3. A preparation: extension to an order preserving linear isometry

We will show in this section that when $p \in (1, \infty)$, the metric preserving bijection Φ extends to a linear isometric order isomorphism from $L^p_{\mathrm{sa}}(M)$ onto $L^p_{\mathrm{sa}}(N)$.

The first ingredient that we needed is the following lemma concerning automatic affineness, that generalises a result of Baker in [1]. However, we do not find our generalization explicitly stated or used in any literature. Observe that our proof is completely different from the arguments in [1], which seemingly does not apply to our case.

Lemma 3.1. *Let X and Y be real Banach spaces with Y being strictly convex. Suppose that E is a (not necessarily convex) nonempty subset of X and $f : E \to Y$ is a metric preserving map. For any $x, y \in E$, one has*

$$f(sx + (1-s)y) = sf(x) + (1-s)f(y) \quad \textit{whenever } s \in (0,1) \textit{ satisfying } sx + (1-s)y \in E. \tag{3.1}$$

Proof. It suffices to consider the case when $y \neq x$. Observe that

$$\begin{aligned} \big\|\big(f(x) - f(y)\big) - \big(f(sx + (1-s)y) - f(y)\big)\big\| &= \|x - (sx + (1-s)y)\| \ = \ (1-s) \cdot \|x - y\| \\ &= \|f(x) - f(y)\| - \|f(sx + (1-s)y) - f(y)\| \end{aligned} \tag{3.2}$$

Hence, the strict convexity of Y produces $\delta \in \mathbb{R}_+$ such that

$$\big(f(x) - f(y)\big) - \big(f(sx + (1-s)y) - f(y)\big) = \delta\big(f(sx + (1-s)y) - f(y)\big).$$

It follows again from (3.2) that

$$(1-s) \cdot \|x - y\| = \big\|\big(f(x) - f(y)\big) - \big(f(sx + (1-s)y) - f(y)\big)\big\| \ = \ \delta s \cdot \|x - y\|,$$

and so $\delta = (1-s)/s$. Hence, $f(sx + (1-s)y) = sf(x) + (1-s)f(y)$ as required. □

Our second result is easy. In fact, if we set $\bar{f}(z) := mf(z/m)$ when $z \in K_0^m$ for some $m \in \mathbb{N}$, then $\bar{f}$ is well-defined and will satisfy the requirement in the statement.

Lemma 3.2. *Let X and Y be two Banach spaces, and let $K \subseteq X$ and $L \subseteq Y$ be (not necessarily proper nor closed) cones. If $f : K_0^1 \to L_0^1$ is an affine map (not necessarily surjective) with $f(0) = 0$, then f extends uniquely to an affine map $\bar{f}$ from K to L. If, in addition, f preserves metric, then so is $\bar{f}$.*

Proposition 3.3. *Let X and Y be strictly convex Banach spaces. Suppose that $K \subseteq X$ and $L \subseteq Y$ are (not necessarily proper nor closed) cones such that the subspace generated by K and the one by L both have dimensions (as real vector spaces) greater than one. Let $\epsilon \in (0,1]$. If $f : K^1_{1-\epsilon} \to L^1_{1-\epsilon}$ is a metric preserving surjection, then f can be extended to a metric preserving affine surjection from K onto L sending 0 to 0.*

Proof. For simplicity we set $\upsilon := 1 - \epsilon$. With Lemma 3.1, we only verify that f extends to a metric preserving map sending 0 to 0. Let us first show that

$$f(K_1^1) = L_1^1 \quad \text{and} \quad f(K_\upsilon^\upsilon) = L_\upsilon^\upsilon. \tag{3.3}$$

Consider an arbitrary element $x \in K_1^1$. If $\|f(x)\| \in (\upsilon, 1)$, then $f(x)$ is the mid-point of two distinct elements in K_υ^1, and by Lemma 3.1 (applied to f^{-1}), the element $x \in K_1^1$ is also the mid-point of two distinct elements in K_υ^1, which is impossible (as X is strictly convex). Consequently, $f(K_1^1) \subseteq L_\upsilon^\upsilon \cup L_1^1$. Moreover, since K_1^1 is path-connected and f is continuous, one sees that

$$\text{either} \quad f(K_1^1) \subseteq L_\upsilon^\upsilon \quad \text{or} \quad f(K_1^1) \subseteq L_1^1.$$

If $\upsilon = 0$, then L_υ^υ contains only one point, and hence $f(K_1^1) \nsubseteq L_\upsilon^\upsilon$ (because the subspace generated by K_1^1 has dimension strictly bigger than one). Suppose that $\upsilon > 0$, and consider two distinct elements $x, y \in K_1^1$ which are so close to each other that the line segment joining x and y lies inside K_υ^1. Then Lemma 3.1 tells us that the line segment joining $f(x)$ and $f(y)$ lies inside L_υ^1, which forbids both $f(x)$ and $f(y)$ belonging to L_υ^υ (because of the strict convexity of Y). This means that $f(K_1^1) \subseteq L_1^1$. By considering f^{-1}, we obtain the asserted equality $f(K_1^1) = L_1^1$.

In order to establish $f(K_\upsilon^\upsilon) = L_\upsilon^\upsilon$, it suffices to show that $f(K_\upsilon^\upsilon) \subseteq L_\upsilon^\upsilon$ (again, because f^{-1} preserves metric). Suppose on the contrary that there exists $x \in K_\upsilon^\upsilon$ with $\|f(x)\| \in (\upsilon, 1)$ (observe that $\|f(x)\| \neq 1$ since $f(K_1^1) = L_1^1$). Then

$$\left\| \frac{f(x)}{\|f(x)\|} - f(x) \right\| \ = \ \left(\frac{1}{\|f(x)\|} - 1 \right) \|f(x)\| \ = 1 - \|f(x)\| < \ 1 - \upsilon.$$

However, for any $y \in K_1^1$, one has $\|y - x\| \geq 1 - \upsilon$, and this contradicts $f(K_1^1) = L_1^1$ (because $\frac{f(x)}{\|f(x)\|} \in L_1^1$). Consequently, Relation (3.3) is verified.

Next, we define $\bar{f} : K \to L$ by setting $\bar{f}(0) = 0$ as well as

$$\bar{f}(x) := \|x\| f(x/\|x\|) \qquad (x \in K \setminus \{0\}). \tag{3.4}$$

We claim that $\bar{f}$ is a metric preserving map extending f. Indeed, if $\upsilon = 0$, then $f(0) = 0$ (because $K_0^0 = \{0\}$ and $L_0^0 = \{0\}$), and by Lemma 3.1, we know that f is an affine map on K_0^1, and the assertion on $\bar{f}$ follows from Lemma 3.2.

Suppose that $\upsilon > 0$. Pick an arbitrary element $x \in K_1^1$. It follows from Relation (3.3) that

$$\|f(x)\| = 1 = (1 - \upsilon) + \upsilon = \|x - \upsilon x\| + \|f(\upsilon x)\| = \|f(x) - f(\upsilon x)\| + \|f(\upsilon x)\|,$$

and this, together with the strict convexity of Y, gives $f(x) - f(\upsilon x) = \delta f(\upsilon x)$ for some $\delta \in \mathbb{R}_+$. Consequently, Relation (3.3) tells us that $\delta = (1 - \upsilon)/\upsilon$, which means that $f(\upsilon x) = \upsilon f(x)$. Hence, Lemma 3.1 ensures that

$$f(\gamma x) = \gamma f(x) \qquad (\gamma \in [\upsilon, 1]; x \in K_1^1). \tag{3.5}$$

Thus, $\bar{f}$ extends f.

For each $k \in \mathbb{Z}$, we set

$$K_k := K_{\upsilon^{-k+1}}^{\upsilon^{-k}}, \quad L_k := L_{\upsilon^{-k+1}}^{\upsilon^{-k}} \quad \text{and} \quad f_k := \bar{f}|_{K_k}.$$

It follows from (3.4) and (3.5) that

$$f_k(x) = f(\upsilon^k x)/\upsilon^k \qquad (x \in K_k).$$

Thus, the metric preserving property of f implies that f_k preserves metric.

Fix arbitrary distinct elements $x, y \in K \setminus \{0\}$ with $\|x\| \leq \|y\|$. Notice that the assignment

$$\nu : s \mapsto \|sx + (1 - s)y\|$$

is a continuous map from $[0, 1]$ to $\mathbb{R}_+$. There exist $k_1 \leq k_2$ in $\mathbb{Z}$ such that

$$\upsilon^{-k_1+1} < \|x\| \leq \upsilon^{-k_1} \quad \text{and} \quad \upsilon^{-k_2+1} \leq \|y\| < \upsilon^{-k_2}.$$

If $k_1 = k_2$, then $x, y \in K_{k_1}$ and we have $\|\bar{f}(x) - \bar{f}(y)\| = \|x - y\|$. Assume that $k_1 < k_2$. One can find $s_1, \ldots, s_{k_2-k_1} \in (0, 1)$ such that $s_1 < s_2 < \cdots < s_{k_2-k_1}$ and that $\nu(s_i) = \upsilon^{-k_1-i+1}$. Denote

$$z_0 := x, \quad z_{k_2-k_1+1} := y \quad \text{and} \quad z_i := s_i x + (1 - s_i) y \quad (i = 1, \ldots, k_2 - k_1).$$

It follows that $z_i, z_{i+1} \in K_{k_1+i}$ $(i = 0, 1, \ldots, k_2 - k_1)$, and we have

$$\|\bar{f}(z_i) - \bar{f}(z_{i+1})\| = \|f_{k_1+i}(z_i) - f_{k_1+i}(z_{i+1})\| = \|z_i - z_{i+1}\|.$$

Moreover, since

$$\|(sx + (1-s)y) - (s'x + (1-s')y)\| = (s' - s)\|x - y\| \quad \text{whenever} \quad s \leq s',$$

we see that

$$\|z_0 - z_1\| + \cdots + \|z_{k_2-k_1} - z_{k_2-k_1+1}\| = \|x - y\|.$$

Thus,

$$\|\bar{f}(x) - \bar{f}(y)\| \leq \|\bar{f}(z_0) - \bar{f}(z_1)\| + \cdots + \|\bar{f}(z_{k_2-k_1}) - \bar{f}(z_{k_2-k_1+1})\| = \|x - y\|.$$

Furthermore, it follows from the definition of $\bar{f}$ that $\|\bar{f}(sx)\| = \|sx\|$. From these, we conclude that $\bar{f}$ is contractive. By considering $\bar{f}^{-1}$, we conclude that $\bar{f}$ is a metric preserving bijection extending f. $\square$

The above shows that Φ can be extended to a metric preserving bijection from $L^p_+(M)$ onto $L^p_+(N)$. In order to further extend this map to $L^p_{\text{sa}}(M)$, let us recall the following well-known information about projections. Denote by

$$\mathcal{P}_\sigma(M) := \{\mathbf{s}_T : T \in L^1_+(M) = (M_*)_+\}.$$

Elements in $\mathcal{P}_\sigma(M)$ are called *σ-finite*. By Zorn's lemma, for any projection $e \in \mathcal{P}(M)$, one has

$$e \;=\; \sup\{f \in \mathcal{P}_\sigma(M) : f \leq e\}, \tag{3.6}$$

and e can be written as an orthogonal sum of σ-finite projections.

Definition 3.4 (Dye [6])**.** A bijection $\Upsilon : \mathcal{P}(M) \to \mathcal{P}(N)$ is called an *orthoisomorphism* if for every p and q in $\mathcal{P}(M)$, one has

$$pq = 0 \quad \text{is equivalent to} \quad \Upsilon(p)\Upsilon(q) = 0. \tag{3.7}$$

Proposition 3.5. *Let $p \in (1, \infty)$, and let M and N be von Neumann algebras of dimensions at least 2. Suppose that $\epsilon \in (0, 1]$, and $\Phi : L^p_+(M)^1_{1-\epsilon} \to L^p_+(N)^1_{1-\epsilon}$ is a metric preserving surjection. Then Φ extends to an isometric order isomorphism from $L^p_{\text{sa}}(M)$ onto $L^p_{\text{sa}}(N)$. Moreover, there exists an orthoisomorphism $\Upsilon : \mathcal{P}(M) \to \mathcal{P}(N)$ such that $\Upsilon(\mathbf{s}_T) = \mathbf{s}_{\Phi(T)}$ for all $T \in L^p_+(M)^1_{1-\epsilon}$.*

Proof. For any $T \in L^p(M)_{\text{sa}}$, we know that $|T| \in L^p(M)_+$. Denote by T_+ and T_-, respectively, the positive part and the negative part of the self-adjoint operator T. It is well-known that $T_\pm = \frac{|T| \pm T}{2}$ as elements in $L_0(\check{M}, \tau)$. Moreover, one has $T_\pm \in L^p(M)_+$ with $\mathbf{s}_{T_+}\mathbf{s}_{T_-} = 0$,

$$\|T\|_p^p = \|T_+\|_p^p + \|T_-\|_p^p \quad \text{and} \quad \|T_+ + T_-\|_p^p = \|T_+\|_p^p + \|T_-\|_p^p.$$

Conversely, if $T \in L^p(M)_{\text{sa}}$ and $R, S \in L^p(M)_+$ satisfying $\mathbf{s}_R\mathbf{s}_S = 0$ and $T = R - S$, then we have $R + S = |T|$ (as $(R + S)^2 = (R - S)^2 = T^2$ and one can apply [3, Theorem 12]), as well as

$$R = T_+ \qquad \text{and} \qquad S = T_-. \tag{3.8}$$

It is well-known that $L^p_{\text{sa}}(M)$ is strictly convex (see e.g., Section 5 of [24]). By Proposition 3.3, the map Φ extends to a metric preserving affine surjection, again denoted by Φ, from $L^p_+(M)$ to $L^p_+(N)$ with $\Phi(0) = 0$.

As Φ is affine, one has

$$\|\Phi(R) + \Phi(S)\|_p = \|\Phi(R + S)\|_p = \|R + S\|_p \qquad (R, S \in L^p_+(M)). \tag{3.9}$$

Let us define $\tilde{\Phi} : L^p_{\text{sa}}(M) \to L^p_{\text{sa}}(N)$ by

$$\tilde{\Phi}(T) := \Phi(T_+) - \Phi(T_-) \qquad (T \in L^p_{\text{sa}}(M)).$$

Clear, $\tilde{\Phi}$ is a linear extension of Φ. On the other hand, Relation (3.9) implies

$$\|\Phi(T_+) + \Phi(T_-)\|_p^p = \|T_+ + T_-\|_p^p = \|T_+\|_p^p + \|T_-\|_p^p = \|\Phi(T_+)\|_p^p + \|\Phi(T_-)\|_p^p.$$

By Lemma 2.1(b), we have $\mathbf{s}_{\Phi(T_+)}\mathbf{s}_{\Phi(T_-)} = 0$. Thus, the uniqueness of $\tilde{\Phi}(T)_\pm$ (see (3.8)) ensures that $\tilde{\Phi}(T)_\pm = \Phi(T_\pm)$ for any $T \in L^p_{\rm sa}(M)$. Moreover, $\tilde{\Phi}$ is surjective because Φ is surjective. Furthermore, for any $R, S \in L^p_{\rm sa}(M)$, one has

$$\begin{aligned}\|\tilde{\Phi}(R) - \tilde{\Phi}(S)\| = \|\Phi(R_+) - \Phi(R_-) - \Phi(S_+) + \Phi(S_-)\| \ &= \ \|\Phi(R_+ + S_-) - \Phi(R_- + S_+)\| \\ = \|(R_+ + S_-) - (R_- + S_+)\| \ &= \ \|R - S\|.\end{aligned}$$

Finally, using Lemma 2.1(b) and Relation (3.9), one sees that $\Upsilon_\sigma : \mathbf{s}_T \mapsto \mathbf{s}_{\Phi(T)}$ is a well-defined bijection from $\mathcal{P}_\sigma(M)$ onto $\mathcal{P}_\sigma(N)$ such that Relation (3.7) holds. By Relation (3.6), the map Υ_σ extends to a bijection Υ from $\mathcal{P}(M)$ onto $\mathcal{P}(N)$ that satisfies Relation (3.7). $\square$

As said in the Introduction, it is not at all obvious that the complexification of an isometry from $L^p_{\rm sa}(M)$ onto $L^p_{\rm sa}(N)$ is an isometry from $L^p(M)$ onto $L^p(N)$. If it is true, then with Proposition 3.5 we can apply directly the main result of [27] to the obtain Theorem 1.3 for the case when $p \neq 2$ (even without assuming M to be approximately semifinite).

4. The first main result

Let us now consider Theorem 1.2 for the case of $p = 1$. In order to obtain a proof for this case, we need the following proposition from [19, Proposition 2.2], which is a variant of the main result in [6].

Proposition 4.1. *(Dye) Suppose that there is an orthoisomorphism Δ from $\mathcal{P}(M)$ onto $\mathcal{P}(N)$. Then M and N are Jordan *-isomorphic.*

The following result establishes the case of $p = 1$ in Theorem 1.2. Notice that the situation when $\epsilon = 0$ was already verified in [19, Corollary 3.11].

Theorem 4.2. *Let $\epsilon \in (0, 1]$. If there is a metric preserving bijection $\Phi : L^1_+(M)^1_{1-\epsilon} \to L^1_+(N)^1_{1-\epsilon}$, then M and N are Jordan *-isomorphic.*

Proof. If M is one dimensional, then $L^1_+(M)^1_{1-\epsilon}$ is an interval when $\epsilon > 0$ and is a singleton set when $\epsilon = 0$. This implies that $L^1_+(N)^1_{1-\epsilon}$ is homeomorphic to an interval or a singleton set respectively. Thus N is also one dimensional, and hence isomorphic to M. We assume that both M and N are of dimension greater than one in the following.

Set $\mathcal{S}_M := \{R \in L^1_+(M)^1_1 : \mathbf{s}_R \neq 1\}$. For any $R \in L^1_+(M)^1_{1-\epsilon}$, it is easy to see, via Lemma 2.1(b), that $R \in \mathcal{S}_M$ if and only if there exists $T \in L^1_+(M)^1_{1-\epsilon}$ such that $\|R - T\|_1 = 2$. In this case, $T \in \mathcal{S}_M$ and $\mathbf{s}_R \cdot \mathbf{s}_T = 0$. Hence, by considering Φ and Φ^{-1}, one has $\Phi(\mathcal{S}_M) = \mathcal{S}_N$.

Let us formally define a map

$$\Delta_0 : \mathcal{P}_\sigma(M) \setminus \{1\} \to \mathcal{P}_\sigma(N) \setminus \{1\}$$

by $\Delta_0(e) := \mathbf{s}_{\Phi(R)}$, where $R \in \mathcal{S}_M$ satisfying $\mathbf{s}_R = e$. To show that Δ_0 is well-defined, let us consider another element $R' \in \mathcal{S}_M$ with $\mathbf{s}_{R'} = e$. Pick any projection $f \in \mathcal{P}_\sigma(N)$ with $\mathbf{s}_{\Phi(R)} \cdot f = 0$. Suppose that $T \in \mathcal{S}_M$ satisfying $\mathbf{s}_{\Phi(T)} = f$. Lemma 2.1(b) implies

$$\|R - T\|_1 = \|\Phi(R) - \Phi(T)\|_1 = 2,$$

and $e \cdot \mathbf{s}_T = 0$. Hence we have $\|\Phi(R') - \Phi(T)\|_1 = \|R' - T\|_1 = 2$, which gives $\mathbf{s}_{\Phi(R')} \cdot f = 0$. From this and (3.6), we conclude that $\mathbf{s}_{\Phi(R')} = \mathbf{s}_{\Phi(R)}$, and Δ_0 is well-defined. Suppose that $e_1, e_2 \in \mathcal{P}_\sigma(M) \setminus \{1\}$

such that $e_1 \cdot e_2 = 0$. If $R_1, R_2 \in \mathcal{S}_M$ satisfying $\mathbf{s}_{R_i} = e_i$ for $i = 1, 2$, then $\|\Phi(R_1) - \Phi(R_2)\|_1 = 2$, which gives $\Delta_0(e_1) \cdot \Delta_0(e_2) = 0$. By considering Φ^{-1}, we know that if $\Delta_0(e_1) \cdot \Delta_0(e_2) = 0$, then $e_1 \cdot e_2 = 0$

Now, we extend Δ_0 to $\Delta : \mathcal{P}(M) \to \mathcal{P}(N)$ by setting $\Delta(e)$ to be the supremum in $\mathcal{P}(N)$ of the set $\{\Delta_0(e') : e' \in \mathcal{P}_\sigma(N); e' \leq e\}$. In particular, $\Delta(1) = 1$. Using (3.6), it is not hard to show that Δ satisfies Relation (3.7) and the conclusion follows from Proposition 4.1. □

Next, we consider the case when $p = \infty$. For this case, we need the following result of Mankiewicz from [22, Theorem 2], which can also be found in [2, Theorem 14.1].

Proposition 4.3. *(Mazur-Ulam-Mankiewicz) Let U be a non-empty open connected subset of a normed space X, and let W be an open subset of a normed space Y. Then every isometry from U onto W can be extended uniquely to an affine isometry from X onto Y.*

Under the identification of $(L^\infty(M), L^\infty(M)_+)$ and (M, M_+) as ordered Banach spaces, the following result gives the case of $p = \infty$ in Theorem 1.2.

Theorem 4.4. *Let A and B be unital C^*-algebras. Assume $\epsilon \in (0, 1]$. If there is a metric preserving bijection $\Phi : (A_+)^1_{1-\epsilon} \to (B_+)^1_{1-\epsilon}$, then A and B are Jordan *-isomorphic. Indeed, Φ extends to a Jordan *-isomorphism from A onto B after translation and multiplication by a central symmetry.*

Proof. For $y \in B_+$ and $r > 0$, we set

$$D_B(y, r) := \{z \in B_{\rm sa} : \|z - y\| < r\}$$

as well as

$$V(y, r) := D_B(y, r) \cap (B_+)^1_{1-\epsilon}.$$

Clearly, $\{V(x, r) : r > 0\}$ is a neighbourhood basis of an element x in $(B_+)^1_{1-\epsilon}$. Moreover, notice that

$$(B_+)^1_0 = \{z \in B_{\rm sa} : \|z - 1/2\| \leq 1/2\}$$

(this can be verified by considering the C^*-subalgebra generated by z, when z runs through all elements in $(B_+)^1_0$). In other words, $(B_+)^1_0$ is the closure of $D_B(1/2, 1/2)$. Let us also put

$$O := D_B(1/2, 1/2) \setminus (B_+)^{1-\epsilon}_0, \quad B_1 := \{y \in B_{\rm sa} : \|y - 1/2\| = 1/2; \|y\| > 1 - \epsilon\} \quad \text{and} \quad B_2 := (B_+)^{1-\epsilon}_{1-\epsilon}.$$

Clearly, O is open in $B_{\rm sa}$ and $(B_+)^1_{1-\epsilon} = O \cup B_1 \cup B_2$. It is not hard to see that O is dense in $(B_+)^1_{1-\epsilon}$.

Next, we want to find an element c in $(A_+)^1_{1-\epsilon}$ (as a subset of $A_{\rm sa}$) and a scalar $t > 0$ such that $D_A(c, t) \subseteq (A_+)^1_{1-\epsilon}$ and $\Phi\big(D_A(c, t)\big)$ is an open subset of $B_{\rm sa}$. Let us first consider an arbitrary element a in the open set $U := D_A(1/2, 1/2) \setminus (A_+)^{1-\epsilon}_0$ of $A_{\rm sa}$. If $\Phi(a) \in O$, then we may take $c = a$ and it is clear that such a scalar $t > 0$ can be found. Suppose that $\Phi(a) \notin O$. The density of O in $(B_+)^1_{1-\epsilon}$ tells us that $O \cap V(\Phi(a), s) \neq \emptyset$ for all $s > 0$. We choose $s > 0$ so that $D_A(a, s) \subseteq U$, and pick an arbitrary element $d \in O \cap V(\Phi(a), s)$. Then $c := \Phi^{-1}(d) \in D_A(a, s)$. One may then find small enough $t > 0$ with $D_B(d, t) \subseteq O$ and $D_B(c, t) \subseteq U$.

Finally, Proposition 4.3 tells us that $\Phi|_{D_A(c,t)}$ extends to a bijective isometry from $A_{\rm sa}$ onto $B_{\rm sa}$, and a well-known result of Kadison (see [15, Theorem 2]) gives the desired conclusion. □

Observe that $\big\{b \in B_+ : 1 - \epsilon < \|b\| < 1\big\}$ is not an open subset of $B_{\rm sa}$ (actually, this set coincides with $O \cup B_1 \setminus (B_+)^1_1$). Thus, the above argument remains almost the same if we assume that Φ is a metric preserving bijection from $\big\{b \in B_+ : 1 - \epsilon < \|b\| < 1\big\}$ onto $\big\{a \in A_+ : 1 - \epsilon < \|a\| < 1\big\}$ instead.

We now turn to the case when $p \in (1, \infty)$. Since it is very rare that $L^p_+(M)$ contains an open subset of $L^p_{\mathrm{sa}}(M)$, Proposition 4.3 cannot be employed in this case. Instead, we need Proposition 3.3 and the following result from [19, Theorem 3.2(a)], which is another variant of Dye's theorem in [6].

Proposition 4.5. *Suppose that there is a bijection $\Lambda : L^1_+(M)^1_1 \to L^1_+(N)^1_1$ satisfying: for every $R, T \in L^1_+(M)^1_1$, one has*

$$\mathbf{s}_R \cdot \mathbf{s}_T = 0 \quad \text{if and only if} \quad \mathbf{s}_{\Lambda(R)} \cdot \mathbf{s}_{\Lambda(T)} = 0.$$

Then M and N are Jordan *-isomorphic.

Theorem 4.6. *Let $p \in (1, \infty)$ and $\epsilon \in (0, 1]$. If there is a metric preserving bijection $\Phi : L^p_+(M)^1_{1-\epsilon} \to L^p_+(N)^1_{1-\epsilon}$, then M and N are Jordan *-isomorphic.*

Proof. If $M \cong \mathbb{C}$, then $L^p_+(M)^1_{1-\epsilon}$ is a closed and bounded interval. As Φ is a metric preserving bijection, the topological space $L^p_+(N)^1_{1-\epsilon}$ is also of Hausdorff dimension one, which implies that $N \cong \mathbb{C}$. The corresponding conclusion holds when $N \cong \mathbb{C}$. Therefore, we will only consider the cases when $M \ncong \mathbb{C}$ and $N \ncong \mathbb{C}$ in the following.

Proposition 3.5 ensures that Φ extends to a metric preserving affine bijection $\bar{\Phi}$ from $L^p_+(M)$ onto $L^p_+(N)$. Let us define a bijection $\Lambda : L^1_+(M)^1_1 \to L^1_+(N)^1_1$ by

$$\Lambda(S) := \big(\bar{\Phi}(S^{\frac{1}{p}})\big)^p \qquad (S \in L^1_+(M)^1_1),$$

where $S \mapsto S^{\frac{1}{p}}$ is the Mazur map.

Pick arbitrary elements $R, T \in L^1_+(M)^1_1$ with $\mathbf{s}_R \cdot \mathbf{s}_T = 0$. Lemma 2.1(b) gives $\big\|R^{\frac{1}{p}} + T^{\frac{1}{p}}\big\|_p^p = 2$, and we have

$$\big\|\Lambda(R)^{\frac{1}{p}} + \Lambda(T)^{\frac{1}{p}}\big\|_p^p = \big\|\bar{\Phi}(R^{\frac{1}{p}} + T^{\frac{1}{p}})\big\|_p^p = 2.$$

Therefore, Lemma 2.1(b) again produces $\mathbf{s}_{\Lambda(R)} \cdot \mathbf{s}_{\Lambda(T)} = 0$. By considering Φ^{-1}, we know that Λ satisfies the hypothesis of Proposition 4.5, and the required conclusion follows. $\square$

5. The second main result

In order to obtain Theorem 1.3, we need to deal with two cases separately. They are the case of algebras of type I_2 and the case of algebras having EP_1.

5.1. The case of type I_2 algebras.

In the following, $M_2(\mathbb{C})$ is the von Neumann algebra of 2×2 complex matrices. For $p \in (1, \infty)$, we denote by $\mathcal{S}^p_2$ the four dimensional real vector space $M_2(\mathbb{C})_{\mathrm{sa}}$ equipped with the Schatten p-norm. If (X, μ) is a semifinite measure space and $M := L^\infty(\mu, M_2(\mathbb{C}))$, then $L^p_{\mathrm{sa}}(M) = L^p(\mu; \mathcal{S}^p_2)$ and

$$L^p_+(M) = L^p_+(\mu; \mathcal{S}^p_2) := \{f \in L^p(\mu; \mathcal{S}^p_2) : f(x) \in M_2(\mathbb{C})_+ \ \mu\text{-a.e.}\}.$$

In this case, the center $\mathcal{Z}(M)$ can be identified with $L^\infty(\mu)$, and the central support $\mathbf{z}_g$ coincides with the indicator function $\mathbf{1}_{\{x \in X : g(x) \neq 0\}}$ of the cozero of g, for each $g \in L^p_+(M)$.

Lemma 5.1. *Let $q \in (1, \infty) \backslash \{2\}$. Then $\mathcal{S}^q_2$ cannot be written as an ℓ^q-direct sum of two proper subspaces.*

Proof. Suppose $\mathcal{X}$ and $\mathcal{Y}$ are two proper subspaces of $\mathcal{S}_2^q$ such that $\mathcal{S}_2^q = \mathcal{X} \oplus_{\ell^q} \mathcal{Y}$. Fixed an arbitrary $R \in \mathcal{X} \setminus \{0\}$. For every $T \in \mathcal{Y}$, we have $\|R+T\|_q^q = \|R\|_q^q + \|T\|_q^q = \|R-T\|_q^q$. By Lemma 2.1(a), one has $\mathbf{s}_R \cdot \mathbf{s}_T = 0$. Hence, if $\mathbf{s}_R = 1$, then $\mathcal{Y} = \{0\}$, which is a contradiction. This shows that $\mathbf{s}_R$ is a rank one projection, and for each $T \in \mathcal{Y} \setminus \{0\}$, the projection $\mathbf{s}_T = 1 - \mathbf{s}_R$ is also of rank one. Consequently, $\mathcal{Y} = (1-\mathbf{s}_R)\mathcal{S}_2^q(1-\mathbf{s}_R)$, and thus is of real dimension one. In the same way, $\mathcal{X}$ is of real dimension one. However, this contradicts to the fact that $\mathcal{S}_2^q$ has real dimension 4. $\square$

The following lemma should be well-known, but we give a simple argument here for completeness.

Lemma 5.2. *Let $q \in (1,\infty)$ and $\Lambda : \mathcal{S}_2^q \to \mathcal{S}_2^q$ be a surjective linear isometry with $\Lambda(M_2(\mathbb{C})_+) = M_2(\mathbb{C})_+$. Then Λ is an isometry on $M_2(\mathbb{C})_{\mathrm{sa}}$, when it is equipped with the operator norm.*

Proof. Let $e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Since e and $1-e$ are orthogonal projections, one can use Lemma 2.1(b) and the isometric assumption of Λ to show that $\mathbf{s}_{\Lambda(e)}\mathbf{s}_{\Lambda(1-e)} = 0$. This tells us that $\Lambda(e)$ and $\Lambda(1-e)$ are rank one positive matrices, and they can be simultaneously diagonalized. Therefore, one can find a unitary $U \in M_2(\mathbb{C})$ such that $U\Lambda(e)U^* = e$ (observe that $\|\Lambda(e)\|_p = 1$) and $U\big(\Lambda(1) - \Lambda(e)\big)U^* = 1-e$. Hence, $\Lambda(1) = 1$. Now, [15, Corollary 5] gives the conclusion. $\square$

In order to verify Theorem 1.3 for $M = M_2(\mathbb{C})$ when $p \neq 2$, we also need the following result (see [10] and [29]), which can be found in [9, Theorem 8.3.9].

Proposition 5.3. *(Sourour-Greim) Suppose that $q \in [1,\infty) \setminus \{2\}$. Let (X_1,μ_1) and (X_2,μ_2) be finite measure spaces, and let E_1 and E_2 be two separable real Banach spaces such that neither of them can be decomposed into an ℓ^q-direct sum of two non-zero subspaces. Assume $\Psi : L^q(\mu_1, E_1) \to L^q(\mu_2, E_2)$ is a surjective linear isometry. Then there is a set isomorphism Ξ from measurable subsets of X_1 onto measurable subsets of X_2 as well as a strongly measurable map $V : X_2 \to \mathcal{B}(E_1; E_2)$ such that $V(y)$ is a surjective isometry μ_2-a.e. and that for any measurable subset Δ of X_1 and $a \in E_1$, one has*

$$\Psi(a\mathbf{1}_\Delta)(y) = \Big(\frac{d\mu_1 \circ \Xi^{-1}}{d\mu_2}\Big)^{1/q}(y)V(y)\big(a\mathbf{1}_{\Xi(\Delta)}(y)\big) \quad \text{for } \mu_2\text{-a.e. } y. \tag{5.1}$$

We say that a map $\Phi : L^p_+(M) \to L^p_+(N)$ *preserves central supports* if $\mathbf{z}_T = \mathbf{z}_{\Phi(T)}$ for any $T \in L^p_+(M)$.

Lemma 5.4. *Let (X,μ) be a finite measure space and $p \in (1,\infty)\setminus\{2\}$. If $\Phi : L^p(\mu;\mathcal{S}_2^p) \to L^p(\mu;\mathcal{S}_2^p)$ is a surjective linear isometry preserving central supports and satisfying $\Phi(L^p_+(\mu;\mathcal{S}_2^p)) = L^p_+(\mu;\mathcal{S}_2^p)$, then there is a Jordan *-isomorphism $\Theta : L^\infty(\mu; M_2(\mathbb{C})) \to L^\infty(\mu; M_2(\mathbb{C}))$ with $\Phi(f^{\frac{1}{p}}) = \Theta_*(f)^{\frac{1}{p}}$ $(f \in L^1_+(\mu;\mathcal{S}_2^p))$.*

Proof. Notice that since $\mathcal{S}_2^p$ is finite dimensional, the dual Banach space of $L^p(\mu;\mathcal{S}_2^p)$ is $L^q(\mu;\mathcal{S}_2^q)$ (where $1/p + 1/q = 1$) and the canonical bijective isometry between them will send the set of positive linear functionals on $L^p(\mu;\mathcal{S}_2^p)$ onto $L^q_+(\mu;\mathcal{S}_2^q)$. Therefore, the dual map Ψ of Φ is an order isomorphic isometry from $L^q(\mu;\mathcal{S}_2^q)$ to itself. It is easy to see that Ψ also preserves central supports.

By Lemma 5.1, we see that the hypothesis of Proposition 5.3 is satisfied. Since Ψ preserves central supports, we know from Relation (5.1) that the map Ξ in Proposition 5.3 will satisfy

$$\mu\big((\Delta \setminus \Xi(\Delta)) \cup (\Xi(\Delta) \setminus \Delta)\big) = 0 \qquad \text{for every measurable set } \Delta.$$

Thus, we may assume that Ξ is the identity map and obtain

$$\Psi(g)(x) = V(x)(g(x)) \quad \text{for } \mu\text{-almost every } x \in X \text{ and all } g \in L^q(\mu;\mathcal{S}_2^q)),$$

where V is the strongly measurable map in Proposition 5.3.

For any positive matrix $a \in M_2(\mathbb{C})_+$ with rational entries, by considering the constant function $g_a \in L^q(\mu; \mathcal{S}_2^q)$ taking the value a, the positivity of Ψ tells us that $V(x)(a) \geq 0$ for μ-a.e. x. As the set of positive matrices in $M_2(\mathbb{C})$ with rational entries is countable and dense in $M_2(\mathbb{C})_+$, we conclude from the continuity of the map $V(x)$ (on $\mathcal{S}_2^q$) that $V(x)(M_2(\mathbb{C})_+) \subseteq M_2(\mathbb{C})_+$ for almost all x. Thus, one may assume that $V(x) \geq 0$ for all $x \in X$. From Lemma 5.2, it is known that $V(x)$ is an isometric order isomorphism from $M_2(\mathbb{C})_{\rm sa}$ onto $M_2(\mathbb{C})_{\rm sa}$ (both equipped with the operator norms). Moreover, because $\mathcal{B}(M_2(\mathbb{C})_{\rm sa}) \cong \mathcal{B}(\mathcal{S}_2^q)$ as locally convex spaces, we know that V is a measurable map from X to $\mathcal{B}(M_2(\mathbb{C})_{\rm sa})$. Consequently, $\Theta(h)(x) := V(x)(h(x))$ ($h \in L^\infty(\mu; M_2(\mathbb{C}))$) is the Jordan *-isomorphism that satisfies the requirement. $\square$

The following lemma is a simple case of [6, Corollary 1].

Lemma 5.5. *Let (X, μ) be a semifinite measure space. If Υ is an orthoisomorphism from the projection lattice $\mathcal{P}(L^\infty(\mu))$ onto itself, then Υ extends to a *-isomorphism from $L^\infty(\mu)$ onto itself.*

Proposition 5.6. *Let M be a type I_2 von Neumann algebra and $\Phi : L^p_+(M)^1_{1-\epsilon} \to L^p_+(N)^1_{1-\epsilon}$ be a metric preserving surjection, where $\epsilon \in (0, 1]$. There exists a Jordan *-isomorphism $\Theta : N \to M$ such that $\Phi(S^{\frac{1}{p}}) = \Theta_*(S)^{\frac{1}{p}}$ ($S \in L^1_+(M)^1_{1-\epsilon^p}$).*

Proof. As the case of $p = 2$ follows directly from [4, Théorème 3.3] and Proposition 3.5, we will only consider the case of $p \neq 2$. It follows from Theorem 1.2 (which was established in Section 4 above) that there is a Jordan *-isomorphism from N to M. By composing Φ with this isomorphism, we may assume that $N = M$. By Proposition 3.5, the map Φ extends to an isometric order isomorphism from $L^p_{\rm sa}(M)$ onto itself.

Let $M = L^\infty(\mu) \bar{\otimes} M_2(\mathbb{C})$ for a semifinite measure space (X, μ). It follows from [6, Lemma 1] that the map Υ as given by Proposition 3.5 restricts to an orthoisomorphism from $\mathcal{P}(Z(M)) = \mathcal{P}(L^\infty(\mu))$ onto itself. Let $\Psi : L^\infty(\mu) \to L^\infty(\mu)$ be the *-isomorphism extending this restriction (as given in Lemma 5.5). Replacing Φ with its composition with $\Psi^{-1} \otimes \mathrm{id} : M \to M$, we may assume that Φ preserves central supports.

Consider a family $\{X_i\}_{i\in\mathfrak{I}}$ of pairwise disjoint measurable subsets of finite measures with $X = \bigcup_{i\in\mathfrak{I}} X_i$. If $\mu_i := \mu|_{X_i}$, one see from the central support preserving assumption of Φ that it restricts to an isometric order isomorphism from $L^p_{\rm sa}(\mu_i; M_2(\mathbb{C}))$ onto itself. Therefore, Lemma 5.4 produces a Jordan *-automorphism Θ_i on $L^\infty(\mu_i; M_2(\mathbb{C}))$ that implements $\Phi|_{L^p_{\rm sa}(\mu_i; M_2(\mathbb{C}))}$. Now, it is not hard to verify that the map from $M \cong \bigoplus_{i\in\mathfrak{I}}^{\ell^\infty} L^\infty_{\rm sa}(\mu_i; M_2(\mathbb{C}))$ to itself induced by $\{\Theta_i\}_{i\in\mathfrak{I}}$ is the Jordan *-isomorphism satisfying the asserted property. $\square$

5.2. **The case of algebras having EP_1.** In this section, we verify Theorem 1.3 for non-type I_2 algebras that satisfy an extra assumption, the so-called EP_1. Let us first give the reason why we need this assumption through the illustration of the commutative case.

Let (X, μ) and (Y, ν) be two semi-finite measure spaces. Let $p \in (1, \infty)$ and $\epsilon \in (0, 1)$. Suppose that $\Phi : L^p_+(\mu)^1_{1-\epsilon} \to L^p_+(\nu)^1_{1-\epsilon}$ is a metric preserving bijection. By Proposition 3.5, we can extend Φ to a metric preserving affine bijection Ψ from $L^p_+(\mu)$ onto $L^p_+(\nu)$. The map $\bar{\Psi} : f \mapsto \Psi(f^{1/p})^p$ is then a bijective map from $L^1_+(\mu)$ onto $L^1_+(\nu)$. However, we do not know a priori that this continuous bijection $\bar{\Psi}$ is isometric or affine. Nevertheless, it can be shown that convex combinations of elements with orthogonal supports are sent to the corresponding convex combinations under $\bar{\Psi}$. If it happens that every such "orthogonally affine" map is actually affine, then $\bar{\Psi}$ will restrict to an affine bijection from the normal state space $L^\infty(\mu)$ onto that of $L^\infty(\nu)$, and we can use a well-known result to obtain the

*-isomorphism from $L^\infty(\nu)$ onto $L^\infty(\mu)$ that induces Φ. Fortunately, some von Neumann algebras do satisfy this property (e.g. semi-finite ones), and they are studied under the name EP_1. In fact, the EP_1 property was first introduced by K. Watanabe (see [34]) and was extended to EP_p (for any $p \in [1,\infty)$) by D. Sherman (see [28]). Let us restate this property clearly in the following.

Definition 5.7. Let M be a von Neumann algebra.

(a) For a normed space X, a map $\tau : L^1_+(M)^1_1 \to X$ is said to be *orthogonally affine* if for every $s \in (0,1)$,
$$\tau(sR + (1-s)T) = s\tau(R) + (1-s)\tau(T) \quad \text{whenever } R, T \in L^1_+(M)^1_1 \text{ with } \mathbf{s}_R \cdot \mathbf{s}_T = 0.$$

(b) M is said to have EP_1 if every norm continuous orthogonally affine function $\kappa : L^1_+(M)^1_1 \to [0,1]$ is actually affine.

Remark 5.8. (a) Our definition of EP_1 is the same as the one introduced in [28]. In fact, suppose that $\kappa : L^1_+(M)^1_1 \to [0,1]$ is a norm continuous orthogonally affine function. We define $\rho : L^1_+(M) \to \mathbb{R}_+$ by
$$\rho(T) := \|T\|\kappa(T/\|T\|) \qquad (T \in L^1_+(M) \setminus \{0\}).$$
Since $\|sR + (1-s)T\| = s\|R\| + (1-s)\|T\|$ for any $R, T \in L^1_+(M)$, it is not hard to check that ρ will satisfy the four conditions in [28, Definition 4.1] for $C = 1$. Conversely, if a function $\rho : L^1_+(M) \to \mathbb{R}_+$ satisfies the four conditions in [28, Definition 4.1], and we define $\kappa : L^1_+(M)^1_1 \to [0,1]$ by
$$\kappa(T) := \rho(T)/C \qquad (T \in L^1_+(M)^1_1),$$
then κ is a norm continuous orthogonally affine map.

(b) It was shown in [28, Theorem 1.2] that all semifinite algebras without type I_2 summand, all hyperfinite algebras without type I_2 summand as well as all type III_0 factors with separable preduals have EP_1. In fact, all these algebras are approximately semifinite algebras, and it was shown in [28] that all approximately semifinite algebras with no type I_2 summand have EP_1 (the precise statement is stated in Proposition A.7). For the benefit of the reader, we will recall in the appendix some materials from [28] that lead to this fact.

Lemma 5.9. *Suppose that M has EP_1. Let $\Phi : L^1_+(M)^1_1 \to L^1_+(N)^1_1$ be a norm continuous orthogonally affine map (not assumed to be surjective). Then Φ is an affine map.*

Proof. Fix an arbitrary element $f \in L^1(N)^*_+$ with $\|f\| \leq 1$. Consider the map $\kappa : L^1_+(M)^1_1 \to [0,1]$ given by $\kappa(R) := f(\Phi(R))$. Clearly, κ is a norm-continuous orthogonally affine function. By the assumption, we know that κ is affine, and hence Φ is affine (since f is chosen arbitrarily). $\square$

As said in [28], the von Neumann algebra $M_2(\mathbb{C})$ does not have EP_1. In fact, Lemma 5.9 does not hold for $M = M_2(\mathbb{C})$, as shown in the following.

Example 5.10. Recall that in the so-called *Bloch sphere model* there is a metric preserving affine bijection from $L^1_+(M_2(\mathbb{C}))^1_1$ (considered as the state space of $M_2(\mathbb{C})$) onto the closed unit ball $\mathcal{B}$ of $\mathbb{R}^3$. More precisely, fix any $a \in M_2(\mathbb{C})_+$ with normalized trace being 1. There exist $u, v, w \in \mathbb{R}$ with $u^2+v^2+w^2 \leq 1$ such that
$$a = \frac{1}{2}\begin{pmatrix} 1-u & v+iw \\ v-iw & 1+u \end{pmatrix}.$$
Conversely, $\frac{1}{2}\begin{pmatrix} 1-u & v+iw \\ v-iw & 1+u \end{pmatrix}$ is positive when $u^2+v^2+w^2 \leq 1$. The assignment $R_a : b \mapsto \mathrm{Tr}(ba)$ is a state of $M_2(\mathbb{C})$ (i.e., it belongs to $L^1_+(M_2(\mathbb{C}))^1_1$ under the identification $L^1(M_2(\mathbb{C})) \cong M_2(\mathbb{C})_*$), and any state of $M_2(\mathbb{C})$ is of this form. Moreover, R_a is pure, i.e., $\mathbf{s}_{R_a}$ is a rank one projection, exactly when

$u^2+v^2+w^2=1$. We thus identify the state R_a with the point (u,v,w) in $\mathcal{B}$, and the set of pure states with the unit sphere $\mathcal{S}$. Furthermore, it is easy to see that for any other state $R_b\in L^1_+(M_2(\mathbb{C}))^1_1$, one has $\mathbf{s}_{R_a}\mathbf{s}_{R_b}=0$ if and only if $b=\frac{1}{2}\begin{pmatrix} 1+u & -v-iw \\ -v+iw & 1-u \end{pmatrix}$.

Now, consider a homeomorphism Γ from $\mathcal{S}$ onto itself that does not preserve the metric but satisfies

$$\Gamma(-(u,v,w))=-\Gamma((u,v,w))\qquad((u,v,w)\in\mathcal{S}).$$

Consider $\Phi:\mathcal{B}\to\mathcal{B}$ to be the map that sends $(2s-1)(u,v,w)$ to $(2s-1)\Gamma(u,v,w)$ for any $s\in[0,1]$ and $(u,v,w)\in\mathcal{S}$. It is easy to see that Φ is a continuous orthogonally affine map extending Γ. However, Φ cannot be affine, because continuous affine bijections between normal state spaces are metric preserving.

Proposition 5.11. *Let $p\in(1,\infty)$, and let M and N be von Neumann algebras such that M has EP_1 and $M\ncong\mathbb{C}$. Suppose that $\epsilon\in(0,1]$ and $\Phi:L^p_+(M)^1_{1-\epsilon}\to L^p_+(N)^1_{1-\epsilon}$ is a metric preserving surjection. There is a Jordan *-isomorphism $\Theta:N\to M$ satisfying*

$$\Phi(R^{\frac{1}{p}})=\Theta_*(R)^{\frac{1}{p}}\qquad(R^{\frac{1}{p}}\in L^p_+(M)^1_{1-\epsilon})\tag{5.2}$$

Proof. By Proposition 3.5, the map Φ extends to a metric preserving affine bijection $\bar{\Phi}:L^p_+(M)\to L^p_+(N)$. Since $\bar{\Phi}(0)=0$, we know that $\bar{\Phi}$ restricts to a bijection from $L^p_+(M)^1_1$ onto $L^p_+(N)^1_1$. Let $\Lambda:L^1_+(M)^1_1\to L^1_+(N)^1_1$ be the bijection defined by

$$\Lambda(S):=\bar{\Phi}(S^{\frac{1}{p}})^p\qquad(S\in L^1_+(M)^1_1).\tag{5.3}$$

Suppose that $s\in(0,1)$ and $R,T\in L^1_+(M)^1_1$ satisfying $\mathbf{s}_R\cdot\mathbf{s}_T=0$. It follows from Lemma 2.2 and the affineness of $\bar{\Phi}$ that

$$\begin{aligned}\Lambda(sR+(1-s)T)=\bar{\Phi}\big((sR+(1-s)T)^{\frac{1}{p}}\big)^p\ &=\ \bar{\Phi}\big(s^{\frac{1}{p}}R^{\frac{1}{p}}+(1-s)^{\frac{1}{p}}T^{\frac{1}{p}}\big)^p\\ =\big(s^{\frac{1}{p}}\bar{\Phi}(R^{\frac{1}{p}})+(1-s)^{\frac{1}{p}}\bar{\Phi}(T^{\frac{1}{p}})\big)^p\ &=\ s\Lambda(R)+(1-s)\Lambda(T).\end{aligned}$$

In other words, Λ is orthogonally affine. Moreover, we know from Lemma 2.1(c) that the bijection Λ is a homeomorphism. It now follows from Lemma 5.9 and the hypothesis that Λ is affine. Thus, [16, Theorem 4.5] gives a Jordan *-isomorphism $\Theta:N\to M$ such that for every $T\in L^1_+(M)^1_1$, one has $\Lambda(T)=\Theta_*(T)$, or equivalently, $\bar{\Phi}(T^{\frac{1}{p}})=\Theta_*(T)^{\frac{1}{p}}$. From this, one obtains Relation (5.2) (as $\bar{\Phi}$ is positively homogeneous). □

5.3. **The proof of the second main theorem.** Theorem 1.3 is a direct consequence of the following more general result. For a von Neumann algebra M if M_0 is the type I_2 part of M and $M=M_0\oplus M_1$, then M_1 is called the *non-type-*I_2 *part* of M. Note that by Proposition A.7 and Lemma A.6, if M is approximately semi-finite, then its non-type-I_2 part of M has EP_1.

Theorem 5.12. *Let $p\in(1,\infty)$ and $\epsilon\in(0,1]$. Suppose that M and N are von Neumann algebras with $M\ncong\mathbb{C}$ such that the non-type-*I_2 *part of M has EP_1. If $\Phi:L^p_+(M)^1_{1-\epsilon}\to L^p_+(N)^1_{1-\epsilon}$ is a metric preserving bijection, then there is a Jordan *-isomorphism $\Theta:N\to M$ satisfying $\Phi(R^{\frac{1}{p}})=\Theta_*(R)^{\frac{1}{p}}$, for any $R\in L^1_+(M)^{1^p}_{1-\epsilon^p}$.*

Proof. It follows from Proposition 3.5 that Φ extends to an isometric order isomorphism, again denoted by Φ, from $L^p_{\mathrm{sa}}(M)$ onto $L^p_{\mathrm{sa}}(N)$. Moreover, as in Proposition 3.5, the assignment $\mathbf{s}_T\mapsto\mathbf{s}_{\Phi(T)}$ induces an orthoisomorphism Υ from $\mathcal{P}(M)$ onto $\mathcal{P}(N)$.

Let e_0 be the central projection in M with e_0M being the type I_2 part of M. If $f_0:=\Upsilon(e_0)$, then f_0 is a central projection. Therefore, Φ can be written as a sum of an order preserving bijective isometry $\Phi_0: L^p_{\mathrm{sa}}(e_0M)\to L^p_{\mathrm{sa}}(f_0N)$ and order preserving bijective isometry $\Phi_1:L^p_{\mathrm{sa}}((1-e_0)M)\to L^p_{\mathrm{sa}}((1-f_0)N)$.

By Theorem 1.2, we know that e_0M and $(1-e_0)M$ are Jordan *-isomorphic to f_0N and $(1-f_0)N$, respectively. Thus, f_0N is the type I_2 part of N.

Now, Proposition 5.6 produces a Jordan *-isomorphism $\Theta_0 : f_0N \to e_0M$ such that $\Phi_0(S^{\frac{1}{p}}) = \Theta_0^*(S)^{\frac{1}{p}}$ for each $S \in L^1_+(e_0M)$, while Proposition 5.11 produces a Jordan *-isomorphism $\Theta_1 : (1-f_0)N \to (1-e_0)M$ such that $\Phi_1(T^{\frac{1}{p}}) = \Theta_1^*(T)^{\frac{1}{p}}$ for each $T \in L^1_+((1-e_0)M)$. Set $\Theta := \Theta_0 + \Theta_1$. As Φ is linear, one concludes that $\Phi(R^{\frac{1}{p}}) = \Theta^*(R)^{\frac{1}{p}}$ as required. $\square$

Appendix A. Approximately semifinte algebras and property EP_1

The notion of EP_1 is first introduced by Watanabe in [34] and further studied by Sherman in [28]. In [28, Theorem 1.2], some algebras with EP_1 were listed, and their proofs were given in the main body of [28] (in fact, the more general case of EP_p was considered there). In particular, it was shown that an approximately semifinite algebra with no type I_2 summand has EP_1. However, the proof for this fact scatters in [28] and is not easy to trace. For the benefit of the readers, we collect some facts as well as some arguments from both [28] and [34] that lead to the above statement. There is no new result nor new proof given in this appendix.

First of all, let us recall from [34, Theorem 4.8] the following result.

Lemma A.1. *Any von Neumann algebra with a normal faithful tracial state and with no type* I_2 *summand has* EP_1.

Secondly, we recall the following lemma from [28, Theorem 5.3(a)].

Lemma A.2. *Let M be a von Neumann algebra. Suppose that there is an increasing family $\{M_i\}_{i\in\mathfrak{I}}$ of von Neumann subalgebras (of M) having EP_1 such that $\bigcup_{i\in\mathfrak{I}} M_i$ is $\sigma(M, M_*)$-dense in M, and that for each $i \in \mathfrak{I}$, there is a normal conditional expectation $E_i : M \to M_i$ with $E_i(1)$ being the identity of M_i and $E_i \circ E_j = E_i$ whenever $i \leq j$. Then M has EP_1.*

Suppose now that M is a semifinite algebra without type I_2 summand. Let M_1 and M_2 be the type I and the type II parts of M respectively. Clearly, qM_2q does not have any type I_2 summand, for any $q \in \mathcal{P}(M_2)$. On the other hand, M_1 can be decomposed as $\bigoplus_{\lambda\in\Lambda} L^\infty(X_\lambda, \mathcal{L}(\mathfrak{H}_\lambda))$ with $\dim \mathfrak{H}_\lambda \neq 2$ for every $\lambda \in \Lambda$. Thus, there exists an increasing net $\{p_i\}_{i\in\mathfrak{I}}$ in the set

$$\{p \in \mathcal{P}(M) : pMp \text{ has a normal faithful tracial state and does not have any type } \mathrm{I}_2 \text{ summand}\}$$

that $\sigma(M, M_*)$-converges to 1. This, together with Lemmas A.1 and A.2, gives the following.

Proposition A.3. *If M is a semifinite von Neumann algebra with no type* I_2 *summand, then M has* EP_1.

Our next lemma follows readily from the definition of EP_1, because all elements in $L^1_+(M)^1_1$ have disjoint supports from elements in $L^1_+(N)^1_1$.

Lemma A.4. *If M and N are two von Neumann algebras with EP_1, then $M \oplus N$ has EP_1.*

Let us now recall the definition of approximately semifinite algebras from [28].

Definition A.5. A von Neumann algebra M is said to be *approximately semifinite* if there is an increasing family $\{M_i\}_{i\in\mathfrak{I}}$ of semifinite von Neumann subalgebras as well as a net $\{E_i\}_{i\in\mathfrak{I}}$ of normal conditional expectations satisfying the conditions as in Lemma A.2. In this case, $\{(M_i, E_i)\}_{i\in\mathfrak{I}}$ is called a *semifinite paving* for M.

Lemma A.6. *If N and L are von Neumann algebras with $L \oplus N$ being approximately semifinite, then N is approximately semifinite.*

Indeed, if $\{(M_i, E_i)\}_{i\in\mathfrak{I}}$ is a semifinite paving for $L \oplus N$, and $P : L \oplus N \to N$ is the canonical projection, then $\{(P(M_i), P \circ E_i|_N)\}_{i\in\mathfrak{I}}$ is a semifinite paving for N.

Proposition A.7. *If M is an approximately semifinite von Neumann algebra with no type I_2 summand, then M has EP_1.*

In fact, we consider L and N to be the finite part and the properly infinite part of M, respectively. It follows from Proposition A.3 that L has EP_1. Moreover, by Lemma A.6, the algebra N is approximately semifinite. If $\{(N_i, E_i)\}_{i\in\mathfrak{I}}$ is a semifinite paving for N, then $\{(N_i \otimes M_3(\mathbb{C}), E_i \otimes \mathrm{id})\}_{i\in\mathfrak{I}}$ is a semifinite paving for $N \otimes M_3(\mathbb{C}) \cong N$ (because N is properly infinite). Since the semifinite algebra $N_i \otimes M_3(\mathbb{C})$ can never have a type I_2 summand, we know from Proposition A.3 and Lemma A.2 that N has EP_1. Now, it follows from Lemma A.4 that M has EP_1.

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