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NONSURJECTIVE MAPS BETWEEN RECTANGULAR MATRIX SPACES PRESERVING DISJOINTNESS, TRIPLE PRODUCTS, OR NORMS

CHI-KWONG LI, MING-CHENG TSAI, YA-SHU WANG, and NGAI-CHING WONG

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ABSTRACT. Let $\mathbf{M}_{m,n}$ be the space of $m \times n$ real or complex rectangular matrices. Two matrices $A, B \in \mathbf{M}_{m,n}$ are disjoint if $A^*B = 0_n$ and $AB^* = 0_m$. In this paper, a characterization is given for linear maps $\Phi : \mathbf{M}_{m,n} \to \mathbf{M}_{r,s}$ sending disjoint matrix pairs to disjoint matrix pairs, i.e., $A, B \in \mathbf{M}_{m,n}$ are disjoint ensures that $\Phi(A), \Phi(B) \in \mathbf{M}_{r,s}$ are disjoint. More precisely, it is shown that Φ preserves disjointness if and only if Φ is of the form

$$\Phi(A) = U \begin{pmatrix} A \otimes Q_1 & 0 & 0 \\ 0 & A^t \otimes Q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V$$

for some unitary matrices $U \in \mathbf{M}_{r,r}$ and $V \in \mathbf{M}_{s,s}$, and positive diagonal matrices Q_1, Q_2, where Q_1 or Q_2 may be vacuous. The result is used to characterize nonsurjective linear maps that preserve the JB^*-triple product, or just the zero triple product, on rectangular matrices, defined by $\{A, B, C\} = \frac{1}{2}(AB^*C + CB^*A)$. The result is also applied to characterize linear maps between rectangular matrix spaces of different sizes preserving the Schatten p-norms or the Ky Fan k-norms.

KEYWORDS: *orthogonality preservers; matrix spaces; norm preservers; Ky Fan k-norms; Schatten p-norms; JB*-triples.*

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1. Introduction

The fruitful history of linear preserver problems starts with a rather surprising result of Frobenius. He showed in [15] that a linear map $\Phi : \mathbf{M}_n(\mathbb{C}) \to \mathbf{M}_n(\mathbb{C})$ of $n \times n$ complex matrices preserving determinant, i.e., $\det(A) = \det(\Phi(A))$, must be of the form $A \mapsto MAN$ or $A \mapsto MA^tN$ for some matrices $M, N \in \mathbf{M}_n(\mathbb{C})$ with $\det(MN) = 1$. Another seminal work is due to Kadison. In [19], Kadison showed that a unital surjective isometry between two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ must be a C^*-isomorphism; in particular, a linear map $\Phi : \mathbf{M}_n(\mathbb{C}) \to \mathbf{M}_n(\mathbb{C})$ leaving the operator norm invariant must be of the form $A \mapsto UAV$ or $A \mapsto UA^tV$ for some unitary matrices $U, V \in \mathbf{M}_n(\mathbb{C})$.

Researchers have developed many results and techniques in the study of linear preserver problems; see, e.g., [2, 22, 26]. Many of the results have been extended in different directions and applied to other topics such as geometrical structure of Banach spaces, and quantum mechanics; see, e.g., [13, 14, 30]. In spite of these advances, there are some intriguing basic linear preserver problems which remain open. In particular, characterizing linear preservers between different matrix or operator spaces without the surjectivity assumption is very challenging and sometimes intractable; see, for example, [3, 7, 23, 24, 32–34]. Even for finite dimensional spaces, the problem is highly non-trivial. For instance, there is no easy description of a linear norm preserver $\Phi : M_n \to M_r$ if $n \neq r$; see [8].

In this paper, we study nonsurjective linear maps between rectangular matrix spaces preserving disjointness, the Schatten p-norms, or the Ky-Fan k-norms. The result is used to characterize linear maps that preserve the JB^*-triple product, or just the zero triple product. Note that there are interesting results on disjointness preserving maps on different kinds of products over general operator spaces or algebras, see, e.g., [16, 17, 21, 27, 28]. However, the basic problem on disjointness preservers from a rectangular matrix space to another rectangular matrix space is unknown, and the existing results do not cover this case. It is our hope that our study will lead to some general techniques for the study of disjointness preservers in a more general context, say, for general JB^*-triples, to supplement those established in the few literature, e.g., [1].

To better describe the questions addressed in this paper, we introduce some notation. Let $\mathbf{M}_{m,n}$ be the set of $m \times n$ real or complex rectangular matrices, and let $\mathbf{M}_n = \mathbf{M}_{n,n}$. A pair of matrices $A, B \in \mathbf{M}_{m,n}$ are *disjoint*, denoted by

$$A \perp B, \quad \text{if} \quad A^*B = 0_n \text{ and } AB^* = 0_m.$$

Here the adjoint A^* of a rectangular matrix A is its conjugate transpose $\overline{A^t}$. If A is a real matrix, then A^* reduces to A^t, the transpose of A. Clearly, A and B are disjoint if and only if they have orthogonal ranges and initial spaces. A rectangular matrix A is called a *partial isometry* if $AA^*A = A$. In this case, A^*A is the *range projection* and AA^* is the *initial projection* of A. Two partial isometries are disjoint if and only if they have orthogonal range and initial projections.

We will characterize linear maps $\Phi : \mathbf{M}_{m,n} \to \mathbf{M}_{r,s}$ that preserve disjointness, i.e., $\Phi(A) \perp \Phi(B)$ whenever $A \perp B$, and apply the result to some related topics. In particular, we show in Section 2 that such a map has the form

$$\Phi(A) = U \begin{pmatrix} A \otimes Q_1 & 0 & 0 \\ 0 & A^t \otimes Q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V$$

for some unitary (orthogonal in the real case) matrices $U \in \mathbf{M}_r, V \in \mathbf{M}_s$ and diagonal (square) matrices Q_1, Q_2 with positive diagonal entries, where Q_1 or Q_2 may be vacuous.

In Section 3, we regard the space of rectangular matrices as JB*-triples carrying the Jordan triple product $\{A, B, C\} = \frac{1}{2}(AB^*C + CB^*A)$, and use our result in Section 2 to study JB*-triple homomorphisms on rectangular matrices, i.e., linear maps $\Phi : \mathbf{M}_{m,n} \to \mathbf{M}_{r,s}$ satisfying

$$\Phi(AB^*C + CB^*A) = \Phi(A)\Phi(B)^*\Phi(C) + \Phi(C)\Phi(B)^*\Phi(A), \quad \forall A, B, C \in \mathbf{M}_{m,n},$$

and also linear maps preserving matrix triples with zero Jordan triple product.

We also apply our result in Section 2 to study linear maps $\Phi : \mathbf{M}_{m,n} \to \mathbf{M}_{r,s}$ preserving the Schatten p-norms and the Ky Fan k-norms in Section 4. Open problems and future research possibilities are mentioned in Section 5.

Throughout the paper, we will always assume that m, n, r, s are positive integers, and use the following notation.

- $\mathbf{M}_{m,n} = \mathbf{M}_{m,n}(\mathbb{F})$: the vector space of $m \times n$ matrices over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
- $\mathbf{M}_n = \mathbf{M}_n(\mathbb{F})$: the set of $n \times n$ matrices over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
- $\mathbf{U}_n = \mathbf{U}_n(\mathbb{F}) = \{A \in \mathbf{M}_n : A^*A = I_n\}$: the set of real orthogonal or complex unitary matrices depending on $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
- $\mathbf{H}_n = \mathbf{H}_n(\mathbb{F}) = \{A \in \mathbf{M}_n : A = A^*\}$: the set of real symmetric or complex Hermitian matrices depending on $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

2. Nonsurjective preservers of disjointness

In this section, we will prove the following.

THEOREM 2.1. *A linear map $\Phi : \mathbf{M}_{m,n} \to \mathbf{M}_{r,s}$ preserves disjointness, i.e.,*

$$AB^* = 0_m \text{ and } A^*B = 0_n$$
$$\implies \Phi(A)\Phi(B)^* = 0_r \text{ and } \Phi(A)^*\Phi(B) = 0_s, \quad \forall A, B \in \mathbf{M}_{m,n},$$

if and only if there exist $U \in \mathbf{U}_r, V \in \mathbf{U}_s$ and diagonal matrices Q_1, Q_2 with positive diagonal entries such that

$$\Phi(A) = U \begin{pmatrix} A \otimes Q_1 & 0 & 0 \\ 0 & A^t \otimes Q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V \qquad \text{for all } A \in \mathbf{M}_{m,n}. \tag{2.1}$$

Here Q_1 or Q_2, may be vacuous.

Several remarks are in order concerning Theorem 2.1. (i) Observing the symmetry and avoiding the triviality, we can assume that $2 \le m \le n$.

(ii) $AB^* = 0_m$ and $A^*B = 0_n$ mean that A and B have orthogonal ranges and orthogonal initial spaces. This amounts to saying that we can obtain their singular value decompositions, $UAV = \sum_{j=1}^{k} a_j E_{jj}$ and $UBV = \sum_{j=k+1}^{p} b_j E_{jj}$, for some positive scalars $a_1, .., a_k$, $b_{k+1}, ..., b_p$, and unitary matrices $U \in \mathbf{U}_m$ and $V \in \mathbf{U}_n$.

(iii) In view of the singular value decompositions, (2.1) in Theorem 2.1 holds if the condition

$$\Phi(E) \perp \Phi(F) \quad \text{whenever} \quad E \perp F$$

is verified just for rank one disjoint partial isometries E, F in $\mathbf{M}_{m,n}$.

(iv) In Theorem 2.1, unless $r \geq m$ and $s \geq n$, or $s \geq m$ and $r \geq n$, Φ will be the zero map. If $(m, n) = (r, s)$ (resp. (s, r)) and $m \neq n$, then Φ will be the zero map or of the form $A \mapsto UAV$ (resp. $A \mapsto UA^tV$) with $U \in \mathbf{U}_r, V \in \mathbf{U}_s$.

(v) By relaxing the terminology, the rectangular matrix $A \otimes Q_1$ is permutationally similar to $q_1 A \oplus \cdots \oplus q_r A$ if $Q_1 = \operatorname{diag}(q_1, \ldots, q_r)$. Similarly $A^t \otimes Q_2$ is permutationally similar to a direct sum of positive multiples of A^t. So, the theorem asserts that up to a fixed unitary equivalence $\Phi(A)$ is a direct sum of positive multiples of A and A^t.

(vi) In addition to real and complex rectangular matrices, the conclusions in Theorem 2.1 is also valid with the same proof for a real linear map $\Phi : \mathbf{H}_n \to \mathbf{M}_{r,s}$ preserving disjointness. We can further assume that the co-domain is $\mathbf{H}_r$, i.e., $\Phi : \mathbf{H}_n \to \mathbf{H}_r$. In this case, the disjointness assumption on Φ reduces to that $AB = 0$ implies $\Phi(A)\Phi(B) = 0$. Adjusting the proof of Theorem 2.1, we can achieve the equality $U = V^*$, at the expenses that the diagonal matrices Q_1, Q_2 may have negative entries.

(vii) If the domain is the set $\mathbf{M}_n(\mathbb{C})$ of $n \times n$ complex matrices or the set $\mathbf{H}_n(\mathbb{C})$ of $n \times n$ complex Hermitian matrices, our results can be deduced from the abstract theorems on C^*-algebras; e.g., see [4,20,21,28], and also [6,27]. However, the proofs there do not seem to work for rectangular matrix spaces, or real square matrix spaces.

(viii) Our proof is computational and long. It would be nice to have some short and conceptual proofs.

The rest of the section is devoted to the proof of Theorem 2.1. We describe our proof strategy. Let $\{E_{11}, E_{12}, \ldots, E_{mn}\}$ be the standard basis for $\mathbf{M}_{m,n}$. We will show that one can apply a series of replacements of Φ by mappings of the form $X \mapsto \tilde{U}\Phi(X)\tilde{V}$ for some $\tilde{U} \in \mathbf{U}_r, \tilde{V} \in \mathbf{U}_s$ so that the resulting map satisfies

$$E_{ij} \mapsto \begin{pmatrix} E_{ij} \otimes Q_1 & 0 & 0 \\ 0 & E_{ji} \otimes Q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{for all } 1 \leq i \leq m,\ 1 \leq j \leq n.$$

The result will then follow. We carry out the above scheme with an inductive argument, and divide the proofs into several lemmas.

Note that in this section only the linearity and the disjointness structure of the rectangular matrices are concerned. As will be shown below, the (real or complex) matrix space $\mathbf{M}_2 = \operatorname{span}\{E_{11}, E_{12}, E_{21}, E_{22}\}$ and the matrix space $\operatorname{span}\{E_{ij}, E_{ik}, E_{lj}, E_{lk}\}$ can be considered as the same object during our discussion.

LEMMA 2.2. *Let $i \neq l$ and $j \neq k$. The bijective linear map*

$$\Psi : \mathbf{M}_2 \to \operatorname{span}\{E_{ij}, E_{ik}, E_{lj}, E_{lk}\},$$

sending $E_{11}, E_{12}, E_{21}, E_{22}$ to $E_{ij}, E_{ik}, E_{lj}, E_{lk} \in \mathbf{M}_{m,n}$ respectively, preserves the disjointness in two directions, i.e.,

$$A \perp B \quad \Longleftrightarrow \quad \Psi(A) \perp \Psi(B) \qquad \textit{for all } A, B \in \mathbf{M}_2.$$

Proof. The assertion follows from the fact that $\Psi(A) = UAV$, where $U = E_{i1} + E_{l2} \in \mathbf{M}_{m,2}$ and $V = E_{1j} + E_{2k} \in \mathbf{M}_{2,n}$ are partial isometries such that $U^*U = VV^* = I_2$, the 2×2 identity matrix. ■

The technical lemma below will be used heavily in the subsequent proofs. Although the statement is stated and proved for the case when the domain is $\mathbf{M}_2$, it is indeed valid for all the rectangular matrix space $\operatorname{span}\{E_{ij}, E_{ik}, E_{lj}, E_{lk}\}$ due to Lemma 2.2. In the future application, the lemma ensures that if $\Phi(E_{ij})$ and $\Phi(E_{lk})$ have some nice structure for a disjointness preserving linear map $\Phi : \mathbf{M}_{m,n} \to \mathbf{M}_{r,s}$, then much can be said about $\Phi(E_{ik} + E_{lj})$ and $\Phi(E_{ik} - E_{lj})$. One can then compose Φ with some unitaries so that all $\Phi(E_{ij}), \Phi(E_{ik}), \Phi(E_{lj})$ and $\Phi(E_{lk})$ have simple structure.

LEMMA 2.3. *Let $\Phi : \mathbf{M}_2 \to \mathbf{M}_{r,s}$ be a nonzero linear map preserving disjointness such that*

$$\Phi(E_{11}) = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & 0_\ell & 0 \\ 0 & 0 & 0_{r-k-\ell,s-k-\ell} \end{pmatrix} \quad \textit{and} \quad \Phi(E_{22}) = \begin{pmatrix} 0_k & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & 0_{r-k-\ell,s-k-\ell} \end{pmatrix},$$

where $D_1 \in \mathbf{M}_k, D_2 \in \mathbf{M}_\ell$ are diagonal matrices with positive diagonal entries arranged in descending order, and $D_1 = \alpha_1 I_{u_1} \oplus \cdots \oplus \alpha_v I_{u_v}$ with $\alpha_1 > \cdots > \alpha_v > 0$ and $u_1 + \cdots + u_v = k$.

(a) We have $D_1 = D_2$. Moreover,

$$\Phi(E_{12} + E_{21}) = \begin{pmatrix} 0_k & B_{12} & 0 \\ B_{12}^* & 0_k & 0 \\ 0 & 0 & 0_{r-2k,s-2k} \end{pmatrix} \quad \textit{and}$$

$$\Phi(E_{12} - E_{21}) = \begin{pmatrix} 0_k & \widehat{C}_{12} & 0 \\ -\widehat{C}_{12}^* & 0_k & 0 \\ 0 & 0 & 0_{r-2k,s-2k} \end{pmatrix},$$

where $B_{12} = \alpha_1 W_1 \oplus \cdots \oplus \alpha_v W_v$ and $\widehat{C}_{12} = \alpha_1 W_1 V_1 \oplus \cdots \oplus \alpha_v W_v V_v$ with $W_j, V_j \in \mathbf{U}_{u_j}$.

(b) There are unitaries $R_1, R_2 \in \mathbf{U}_k$ and a permutation $P \in \mathbf{M}_k$ such that the map

$$X \mapsto (P^* R_2^* R_1^* \oplus P^* R_2^* \oplus I_{r-2k}) \Phi(X) (R_1 R_2 P \oplus R_2 P \oplus I_{s-2k})$$

satisfies

$$E_{11} \mapsto \begin{pmatrix} Q_1 \oplus Q_2 & 0_k & 0 \\ 0_k & 0_k & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{12} \mapsto \begin{pmatrix} 0_k & Q_1 \oplus 0_{k_2} & 0 \\ 0_{k_1} \oplus Q_2 & 0_k & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_{21} \mapsto \begin{pmatrix} 0_k & 0_{k_1} \oplus Q_2 & 0 \\ Q_1 \oplus 0_{k_2} & 0_k & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{22} \mapsto \begin{pmatrix} 0_k & 0_k & 0 \\ 0_k & Q_1 \oplus Q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $Q_1 \in \mathbf{M}_{k_1}$, $Q_2 \in \mathbf{M}_{k_2}$, $k_1 + k_2 = k$, are diagonal matrices with positive diagonal entries from $\{\alpha_1, \ldots, \alpha_v\}$ arranged in descending order.

Proof. (a) Suppose $\Phi : \mathbf{M}_2 \to \mathbf{M}_{r,s}$ satisfies the assumption. Let

$$\Phi(E_{12} + E_{21}) = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix},$$

where $B_{11} \in \mathbf{M}_k$, $B_{22} \in \mathbf{M}_\ell$. For every nonzero $\gamma \in \mathbb{R}$, the pair of the matrices

$$Z_1 = \begin{pmatrix} \gamma & 1 \\ 1 & \frac{1}{\gamma} \end{pmatrix} \quad \text{and} \quad Z_2 = \begin{pmatrix} \frac{1}{\gamma} & -1 \\ -1 & \gamma \end{pmatrix}$$

are disjoint, and so are the pair $T_1 = \Phi(Z_1)$ and $T_2 = \Phi(Z_2)$. Considering the $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, $(3,3)$ blocks of the matrix $T_1^* T_2$, we get the following:

$$\begin{aligned} 0_k &= D_1^2 + \frac{1}{\gamma} B_{11}^* D_1 - \gamma D_1 B_{11} - B_{11}^* B_{11} - B_{21}^* B_{21} - B_{31}^* B_{31}, \\ 0_{k,\ell} &= \gamma(B_{21}^* D_2 - D_1 B_{12}) - B_{11}^* B_{12} - B_{21}^* B_{22} - B_{31}^* B_{32}, \\ 0_{\ell,k} &= \frac{1}{\gamma}(B_{12}^* D_1 - D_2 B_{21}) - B_{12}^* B_{11} - B_{22}^* B_{21} - B_{32}^* B_{31}, \\ 0_\ell &= D_2^2 - \frac{1}{\gamma} D_2 B_{22} + \gamma B_{22}^* D_2 - B_{22}^* B_{22} - B_{12}^* B_{12} - B_{32}^* B_{32}, \\ 0_{s-k-\ell} &= -B_{13}^* B_{13} - B_{23}^* B_{23} - B_{33}^* B_{33}. \end{aligned}$$

Considering the $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, $(3,3)$ blocks of the matrix $T_1 T_2^*$, we get the following:

$$\begin{aligned} 0_k &= D_1^2 + \frac{1}{\gamma} B_{11} D_1 - \gamma D_1 B_{11}^* - B_{11} B_{11}^* - B_{12} B_{12}^* - B_{13} B_{13}^*, \\ 0_{k,\ell} &= \gamma(B_{12} D_2 - D_1 B_{21}^*) - B_{11} B_{21}^* - B_{12} B_{22}^* - B_{13} B_{23}^*, \\ 0_{\ell,k} &= \frac{1}{\gamma}(B_{21} D_1 - D_2 B_{12}^*) - B_{21} B_{11}^* - B_{22} B_{12}^* - B_{23} B_{13}^*, \\ 0_\ell &= D_2^2 - \frac{1}{\gamma} D_2 B_{22}^* + \gamma B_{22} D_2 - B_{21} B_{21}^* - B_{22} B_{22}^* - B_{23} B_{23}^*, \\ 0_{r-k-\ell} &= -B_{31} B_{31}^* - B_{32} B_{32}^* - B_{33} B_{33}^*. \end{aligned}$$

In view of the $(3,3)$ blocks of $T_1^* T_2$ and $T_1 T_2^*$ being zero blocks, we see that $B_{13}, B_{23}, B_{33}, B_{31}, B_{32}$ are zero blocks. Since $0 \neq \gamma$ is arbitrary and D_1, D_2 are invertible, we see that

$$B_{11} = 0_k, \quad B_{22} = 0_\ell,$$

$$B_{12}B_{12}^* = B_{21}^*B_{21} = D_1^2 \in \mathbf{M}_k, \quad B_{12}^*B_{12} = B_{21}B_{21}^* = D_2^2 \in \mathbf{M}_\ell, \tag{2.2}$$

$$D_1B_{12} = B_{21}^*D_2, \quad \text{and} \quad B_{12}D_2 = D_1B_{21}^*. \tag{2.3}$$

Note that $B_{12}B_{12}^*$ and $B_{12}^*B_{12}$ have the same nonzero eigenvalues (counting multiplicities). Because D_1, D_2 have positive diagonal entries arranged in descending order, it follows from (2.2) that $k = \ell$ and $D_1 = D_2$.

We can now assume that $D_1 = D_2 = \alpha_1 I_{u_1} \oplus \cdots \oplus \alpha_v I_{u_v}$ with $\alpha_1 > \cdots > \alpha_v > 0$ and $u_1 + \cdots + u_v = k$. Furthermore, from (2.2) the matrices B_{12}, B_{12}^*, B_{21} and B_{21}^* have orthogonal columns with Euclidean norms equal to the diagonal entries of D_1. By (2.3), we see that

$$B_{12} = B_{21}^* = \alpha_1 W_1 \oplus \cdots \oplus \alpha_v W_v$$

for some $W_1 \in \mathbf{U}_{u_1}, \ldots, W_v \in \mathbf{U}_{u_v}$.

Let $R_1 = W_1 \oplus \cdots \oplus W_v$. Replace Φ by $X \mapsto (R_1^* \oplus I_{r-k})\Phi(X)(R_1 \oplus I_{s-k})$. We may assume that $B_{12} = B_{21}^* = D_1$. Let

$$\Phi(E_{12} - E_{21}) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix},$$

where $C_{11} \in \mathbf{M}_k, C_{22} \in \mathbf{M}_\ell$.

Now, the pair of matrices

$$Z_3 = \begin{pmatrix} \gamma & -1 \\ 1 & -\frac{1}{\gamma} \end{pmatrix} \quad \text{and} \quad Z_4 = \begin{pmatrix} \frac{1}{\gamma} & 1 \\ -1 & -\gamma \end{pmatrix}$$

are disjoint, and so are the pair of matrices $T_3 = \Phi(Z_3)$ and $T_4 = \Phi(Z_4)$. Consider the $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, $(3,3)$ blocks of the matrix $T_3^*T_4$. By the fact that $k = \ell$ and $D_1 = D_2$, we get the following:

$$\begin{aligned}
0_k &= D_1^2 - \frac{1}{\gamma}C_{11}^*D_1 + \gamma D_1C_{11} - C_{11}^*C_{11} - C_{21}^*C_{21} - C_{31}^*C_{31}, \\
0_k &= \gamma(D_1C_{12} + C_{21}^*D_2) - C_{11}^*C_{12} - C_{21}^*C_{22} - C_{31}^*C_{32}, \\
0_k &= -\frac{1}{\gamma}(C_{12}^*D_1 + D_2C_{21}) - C_{12}^*C_{11} - C_{22}^*C_{21} - C_{32}^*C_{31}, \\
0_k &= D_2^2 - \frac{1}{\gamma}D_2C_{22} + \gamma C_{22}^*D_2 - C_{22}^*C_{22} - C_{12}^*C_{12} - C_{32}^*C_{32}, \\
0_{s-2k} &= -C_{13}^*C_{13} - C_{23}^*C_{23} - C_{33}^*C_{33}.
\end{aligned}$$

Consider the $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, $(3,3)$ blocks of the matrix $T_3T_4^*$. We get the following:

$$\begin{aligned}
0_k &= D_1^2 - \frac{1}{\gamma}C_{11}D_1 + \gamma D_1C_{11}^* - C_{11}C_{11}^* - C_{12}C_{12}^* - C_{13}C_{13}^*, \\
0_k &= \gamma(D_1C_{21}^* + C_{12}D_2) - C_{11}C_{21}^* - C_{12}C_{22}^* - C_{13}C_{23}^*,
\end{aligned}$$

$$\begin{aligned}
0_k &= -\frac{1}{\gamma}(C_{21}D_1 + D_2C_{12}^*) - C_{21}C_{11}^* - C_{22}C_{12}^* - C_{23}C_{13}^*,\\
0_k &= D_2^2 - \frac{1}{\gamma}D_2C_{22}^* + \gamma C_{22}D_2 - C_{21}C_{21}^* - C_{22}C_{22}^* - C_{23}C_{23}^*,\\
0_{r-2k} &= -C_{31}C_{31}^* - C_{32}C_{32}^* - C_{33}C_{33}^*.
\end{aligned}$$

By a similar argument for the pair (T_1, T_2), we conclude that C_{11}, C_{22}, C_{13}, C_{23}, C_{33}, C_{31} and C_{32} are zero blocks. Furthermore,

$$\begin{aligned}
&C_{21}^*C_{21} = C_{12}C_{12}^* = C_{21}C_{21}^* = C_{12}^*C_{12} = D_1^2,\\
&D_1C_{12} = -C_{21}^*D_1, \quad C_{12}D_1 = -D_1C_{21}^*.
\end{aligned}$$

Now, $C_{21}, C_{12}^*, C_{21}^*, C_{12}$ have orthogonal columns with Euclidean norms equal to the diagonal entries of D_1, and together with the fact that $D_1C_{12} = -C_{21}^*D_1$, and $C_{12}D_1 = -D_1C_{21}^*$, we see that

$$C_{12} = -C_{21}^* = \alpha_1 V_1 \oplus \cdots \oplus \alpha_v V_v \in \mathbf{M}_{u_1} \oplus \cdots \oplus \mathbf{M}_{u_v},$$

where $V = D_1^{-1}C_{12} = V_1 \oplus \cdots \oplus V_v$ is unitary. Thus in its original form, we see that

$$\widehat{C}_{12} = -\widehat{C}_{21}^* = \alpha_1 W_1 V_1 \oplus \cdots \oplus \alpha_v W_v V_v.$$

(b) Continue the arguments in (a), and in particular assume that $B_{12} = B_{21}^* = D_1$ and $C_{12} = -C_{21}^* = \alpha_1 V_1 \oplus \cdots \oplus \alpha_v V_v = D_1 V$. There is a unitary matrix $R_2 = U_1 \oplus \cdots \oplus U_v \in \mathbf{U}_k$ with $U_1 \in \mathbf{M}_{u_1}, \ldots, U_v \in \mathbf{M}_{u_v}$ such that $R_2^* V R_2 = \operatorname{diag}(g_1, \ldots, g_k) = G \in \mathbf{U}_k$. Now, we may replace Φ by the map $X \mapsto (R_2^* \oplus R_2^* \oplus I_{r-2k})\Phi(X)(R_2 \oplus R_2 \oplus I_{s-2k})$ and assume that $C_{12} = -C_{21}^* = D_1 G$. In particular,

$$\Phi(E_{12}+E_{21}) = \begin{pmatrix} 0_k & D_1 & 0\\ D_1 & 0_k & 0\\ 0 & 0 & 0_{r-2k,s-2k}\end{pmatrix} \quad \text{and}$$

$$\Phi(E_{12}-E_{21}) = \begin{pmatrix} 0_k & D_1G & 0\\ -D_1G^* & 0_k & 0\\ 0 & 0 & 0_{r-2k,s-2k}\end{pmatrix}.$$

We claim that G is permutationally similar to $I_{k_1} \oplus -I_{k_2}$ with $k_1 + k_2 = k$. To see this, consider the pair

$$\Phi(E_{12}) = \begin{pmatrix} 0_k & \frac{D_1(I_k+G)}{2} & 0\\ \frac{D_1(I_k-G^*)}{2} & 0_k & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \Phi(E_{21}) = \begin{pmatrix} 0_k & \frac{D_1(I_k-G)}{2} & 0\\ \frac{D_1(I_k+G^*)}{2} & 0_k & 0\\ 0 & 0 & 0\end{pmatrix}.$$

One readily checks that the pair are disjoint if and only if $(I_k + G)(I_k - G^*) = 0_k$, equivalently, G is a real diagonal unitary matrix. Thus, there is a permutation matrix $P \in \mathbf{M}_k$ such that $P^tGP = I_{k_1} \oplus -I_{k_2}$ with $k_1 + k_2 = k$. With a further permutation, we can assume $P^tD_1GP = Q_1 \oplus -Q_2 \in \mathbf{M}_k$ so that Q_1, Q_2 are diagonal matrices with descending positive diagonal entries.

We may replace Φ by a map

$$X \mapsto (P^t \oplus P^t \oplus I_{r-2k})\Phi(X)(P \oplus P \oplus I_{s-2k})$$

so that

$$\Phi(E_{11}) = \begin{pmatrix} Q_1 \oplus Q_2 & 0_k & 0 \\ 0_k & 0_k & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \Phi(E_{12}+E_{21}) = \begin{pmatrix} 0_k & Q_1 \oplus Q_2 & 0 \\ Q_1 \oplus Q_2 & 0_k & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Phi(E_{22}) = \begin{pmatrix} 0_k & 0_k & 0 \\ 0_k & Q_1 \oplus Q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \Phi(E_{12}-E_{21}) = \begin{pmatrix} 0_k & Q_1 \oplus -Q_2 & 0 \\ -Q_1 \oplus Q_2 & 0_k & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Adding and subtracting the matrices $\Phi(E_{12}+E_{21})$ and $\Phi(E_{12}-E_{21})$, we get the desired forms of $\Phi(E_{12})$ and $\Phi(E_{21})$. The result follows. ∎

LEMMA 2.4. *Theorem 2.1 holds if $m = n \geq 2$.*

Proof. We prove the result by induction on $m = n \geq 2$. Suppose $m = n = 2$. We may choose $V_1 \in \mathbf{U}_r, V_2 \in \mathbf{U}_s$ such that

$$Y_1 = V_1\Phi(E_{11})V_2 = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & 0_\ell & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Y_2 = V_1\Phi(E_{22})V_2 = \begin{pmatrix} 0_k & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $D_1 \in \mathbf{M}_k, D_2 \in \mathbf{M}_\ell$ are diagonal matrices with positive diagonal entries arranged in descending order. We may replace Φ by the map $X \mapsto V_1\Phi(X)V_2$ so that the resulting map will preserve disjointness and send E_{jj} to Y_j for $j = 1,2$. By Lemma 2.3, we can modify V_1 and V_2 so that the resulting map satisfies

$$\Phi(E_{11}) = \begin{pmatrix} Q_1 \oplus Q_2 & 0_k & 0 \\ 0_k & 0_k & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Phi(E_{12}) = \begin{pmatrix} 0_k & Q_1 \oplus 0_{k_2} & 0 \\ 0_{k_1} \oplus Q_2 & 0_k & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Phi(E_{21}) = \begin{pmatrix} 0_k & 0_{k_1} \oplus Q_2 & 0 \\ Q_1 \oplus 0_{k_2} & 0_k & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Phi(E_{22}) = \begin{pmatrix} 0_k & 0_k & 0 \\ 0_k & Q_1 \oplus Q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for some diagonal matrices Q_1, Q_2 with descending positive diagonal entries.

Now, we can find a permutation matrix $\hat{P} \in \mathbf{M}_{2k}$ satisfying

$$[X_1|X_2|X_3|X_4]\hat{P} = [X_1|X_3|X_2|X_4]$$

whenever $X_1, X_3 \in \mathbf{M}_{2k,k_1}, X_2, X_4 \in \mathbf{M}_{2k,k_2}$. Then the map

$$X \mapsto (\hat{P} \oplus I_{r-2k})^t\Phi(X)(\hat{P} \oplus I_{s-2k})$$

will satisfy

$$E_{ij} \mapsto \begin{pmatrix} E_{ij} \otimes Q_1 & 0_{2k_1,2k_2} & 0_{2k_1,s-2k} \\ 0_{2k_2,2k_1} & E_{ji} \otimes Q_2 & 0_{2k_2,s-2k} \\ 0_{r-2k,2k_1} & 0_{r-2k,2k_2} & 0_{r-2k,s-2k} \end{pmatrix} \qquad \text{for } 1 \leq i,j \leq 2.$$

This establishes the assertion for the case when $m = n = 2$.

Now, suppose the result holds for square matrices of size smaller than n with $n > 2$. Then the restriction of Φ on matrices $A \in \mathbf{M}_n$ with the last row and last column equal to zero verifies the conclusion. So, there exist $U \in \mathbf{U}_r$ and $V \in \mathbf{U}_s$ such that

$$U\Phi(E_{ij})V = \begin{pmatrix} \hat{E}_{ij} \otimes Q_1 & 0_{(n-1)k_1,(n-1)k_2} & 0 \\ 0_{(n-1)k_2,(n-1)k_1} & \hat{E}_{ji} \otimes Q_2 & 0 \\ 0 & 0 & 0_{r-(n-1)k,s-(n-1)k} \end{pmatrix}$$

for $1 \leq i,j < n$. Here, $\{E_{ij} : 1 \leq i,j \leq n\}$ is the standard basis for $\mathbf{M}_n$, and $\{\hat{E}_{ij} : 1 \leq i,j \leq n-1\}$ is the standard basis for $\mathbf{M}_{n-1}$, $Q_1 \in \mathbf{M}_{k_1}, Q_2 \in \mathbf{M}_{k_2}$ are diagonal matrices with positive diagonal entries, and $k = k_1 + k_2$.

Note that E_{nn} and E_{ij} are disjoint for all $1 \leq i,j < n$. So, we may assume that

$$\Phi(E_{nn}) = \begin{pmatrix} 0_{(n-1)k} & 0 \\ 0 & Y \end{pmatrix}$$

for some matrix $Y \in \mathbf{M}_{r-(n-1)k,s-(n-1)k}$. There exist

$$U_1 \in \mathbf{U}_{r-(n-1)k} \quad \text{and} \quad V_1 \in \mathbf{U}_{s-(n-1)k}$$

such that

$$U_1 Y V_1 = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where D is a diagonal matrix with positive diagonal entries arranged in descending order. We may replace Φ by the map

$$X \mapsto (I_{(n-1)k} \oplus U_1)\Phi(X)(I_{(n-1)k} \oplus V_1)$$

and assume that $U_1 = I_{r-(n-1)k}$ and $V_1 = I_{s-(n-1)k}$.

Consider the restriction of the map on the span$\{E_{11}, E_{1n}, E_{n1}, E_{nn}\}$. Applying the proof of Lemma 2.3 to the restriction map, we see that there is a permutation matrix P such that $D = P^t(Q_1 \oplus Q_2)P$. Now, replace Φ by the map

$$X \mapsto ((I_{n-1} \otimes P^t) \oplus I_{r-(n-1)k})\Phi(X)((I_{n-1} \otimes P) \oplus I_{s-(n-1)k}).$$

After a further permutation, we can replace $\hat{E}_{ij}$ with E_{ij} for $1 \leq i,j < n$, and the resulting map Φ satisfies

$$E_{jj} \mapsto \begin{pmatrix} E_{jj} \otimes D & 0 \\ 0 & 0 \end{pmatrix}, \quad j = 1, \ldots, n, \tag{2.4}$$

$$E_{ij} + E_{ji} \mapsto \begin{pmatrix} (E_{ij} + E_{ji}) \otimes D & 0 \\ 0 & 0_{r-(n-1)k,s-(n-1)k} \end{pmatrix}, \quad 1 \leq i \leq j < n,$$

$$E_{ij} - E_{ji} \mapsto \begin{pmatrix} (E_{ij} - E_{ji}) \otimes \hat{D} & 0 \\ 0 & 0_{r-(n-1)k,s-(n-1)k} \end{pmatrix}, \quad 1 \leq i < j < n,$$

where $\hat{D} = P^t(Q_1 \oplus -Q_2)P$.

For $j = 1, 2, \ldots, n-1$, apply Lemma 2.3(a) to the restriction map on the rectangular matrix space span$\{E_{jj}, E_{jn}, E_{nj}, E_{nn}\}$. We see that

$$\Phi(E_{jn} + E_{nj}) = \begin{pmatrix} E_{jn} \otimes B_{jn} + E_{nj} \otimes B_{jn}^* & 0 \\ 0 & 0 \end{pmatrix},$$

$$\Phi(E_{jn} - E_{nj}) = \begin{pmatrix} E_{jn} \otimes C_{jn} - E_{nj} \otimes C_{jn}^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Here, $B_{jn}, C_{jn} \in \mathbf{M}_k$, and $D^{-1}B_{jn}, D^{-1}C_{jn} \in \mathbf{U}_k$ commute with D.

Because every matrix in the range of the map Φ has its last $r - nk$ rows and last $s - nk$ columns equal to zero, we will assume that $r = nk$ and $s = nk$ for simplicity (by removing the last $r - nk$ rows and $s - nk$ columns from every matrix in the range space). Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{C}^n$. For $j = 2, \ldots, n-1$, consider the disjoint pair

$$X_1 = (e_1 + e_j + e_n)(e_1 + e_j + e_n)^t \quad \text{and} \quad X_2 = (2e_1 - e_j - e_n)(2e_1 - e_j - e_n)^t.$$

Then $\Phi(X_1)$ and $\Phi(X_2)$ are disjoint. If we partition $\Phi(X_1), \Phi(X_2)$ as $n \times n$ block matrices $Z = (Z_{ij})_{1 \le i,j \le n}$ such that each block is in $\mathbf{M}_k$, then all the blocks are zero except for the (p, q) blocks with $p, q \in \{1, j, n\}$. Deleting all the zero blocks, we get the following two 3×3 block matrices.

$$Z_1 = \begin{pmatrix} D & D & B_{1n} \\ D & D & B_{jn} \\ B_{1n}^* & B_{jn}^* & D \end{pmatrix} \quad \text{and} \quad Z_2 = \begin{pmatrix} 4D & -2D & -2B_{1n} \\ -2D & D & B_{jn} \\ -2B_{1n}^* & B_{jn}^* & D \end{pmatrix}.$$

Both the $(1,1)$ and $(1,2)$ blocks of $Z_1 Z_2^*$ equal 0_k, i.e.,

$$0_k = 2D^2 - 2B_{1n}B_{1n}^* = -D^2 + B_{1n}B_{jn}^*.$$

We see that $B_{1n}B_{1n}^* = D^2 = B_{1n}B_{jn}^*$. Since B_{1n} is the product of D and a unitary matrix, it is invertible. So, $B_{1n} = B_{jn}$ for $j = 2, \ldots, n-1$.

Similarly, we can consider the disjoint pair

$$X_3 = (e_1 + e_j + e_n)(-e_1 - e_j + e_n)^t \quad \text{and} \quad X_4 = (e_1 + e_j - 2e_n)(e_1 + e_j + 2e_n)^t.$$

Then removing the zero blocks of $\Phi(X_3)$ and $\Phi(X_4)$, we get

$$Z_3 = \begin{pmatrix} -D & -D & C_{1n} \\ -D & -D & C_{jn} \\ -C_{1n}^* & -C_{jn}^* & D \end{pmatrix} \quad \text{and} \quad Z_4 = \begin{pmatrix} D & D & 2C_{1n} \\ D & D & 2C_{jn} \\ -2C_{1n}^* & -2C_{jn}^* & -4D \end{pmatrix}.$$

Both the $(1,1)$ and $(1,2)$ blocks of $Z_3 Z_4^*$ equal 0_k, i.e.,

$$0_k = -2D^2 + 2C_{1n}C_{1n}^* = -2D^2 + 2C_{1n}C_{jn}^*.$$

We see that $C_{1n}C_{1n}^* = D^2 = C_{1n}C_{jn}^*$. Since C_{1n} is the product of D and a unitary (real orthogonal) matrix, it is invertible. Thus, $C_{1n} = C_{jn}$ for $j = 2, \ldots, n-1$.

Let W be the unitary matrix $D^{-1}B_{1n} \in \mathbf{M}_n$. Replace Φ by the map $X \mapsto (I_{(n-1)k} \oplus W)\Phi(X)(I_{(n-1)k} \oplus W^*)$. Then with $\hat{C} = C_{jn}W^*$ for $j = 1, \ldots, n-1$, we have

$$\Phi(E_{ij} + E_{ji}) = (E_{ij} + E_{ji}) \otimes D, \quad 1 \le i \le j \le n,$$

$$\Phi(E_{ij} - E_{ji}) = (E_{ij} - E_{ji}) \otimes \hat{D}, \quad 1 \le i < j \le n-1,$$

$$\Phi(E_{jn} - E_{nj}) = E_{jn} \otimes \hat{C} - E_{nj} \otimes \hat{C}^*, \quad j = 1, \ldots, n-1.$$

Recall that P is a permutation matrix such that $D = P^t(Q_1 \oplus Q_2)P$. Now replace Φ by $X \mapsto (I_n \otimes P)\Phi(X)(I_n \otimes P^t)$. Then

$$\Phi(E_{ij} + E_{ji}) = (E_{ij} + E_{ji}) \otimes (Q_1 \oplus Q_2), \quad 1 \le i \le j \le n,$$

$$\Phi(E_{ij} - E_{ji}) = (E_{ij} - E_{ji}) \otimes (Q_1 \oplus -Q_2), \quad 1 \le i < j \le n-1,$$

$$\Phi(E_{jn} - E_{nj}) = E_{jn} \otimes G - E_{nj} \otimes G^*, \quad j = 1, \ldots, n-1,$$

where $G = P\hat{C}P^t$.

It remains to show that $G = Q_1 \oplus -Q_2$ so that $E_{jn} \otimes G - E_{nj} \otimes G^* = (E_{jn} - E_{nj}) \otimes (Q_1 \oplus -Q_2)$. To this end, consider the disjoint pair $X_5 = E_{22} + E_{nn} - E_{2n} - E_{n2}$ and $X_6 = E_{12} + E_{1n} - E_{21} - E_{n1}$. Then $Z_5 = \Phi(X_5)$ and $Z_6 = \Phi(X_6)$ are disjoint. If we partition $\Phi(X_5), \Phi(X_6)$ as $n \times n$ block matrices $Z = (Z_{ij})_{1 \le i,j \le n}$ such that each block is in $\mathbf{M}_k$, then all the blocks are zero except for the (p,q) blocks with $p, q \in \{1, 2, n\}$. Let $Q = Q_1 \oplus Q_2$ and $C_{12} = Q_1 \oplus -Q_2$. Deleting all the zero blocks, we get the following two matrices.

$$Z_5 = \begin{pmatrix} 0_k & 0_k & 0_k \\ 0_k & Q & -Q \\ 0_k & -Q & Q \end{pmatrix} \quad \text{and} \quad Z_6 = \begin{pmatrix} 0_k & C_{12} & G \\ -C_{12}^* & 0_k & 0_k \\ -G^* & 0_k & 0_k \end{pmatrix}.$$

Now, the $(1,2)$ block of $Z_6Z_5^*$ is zero, i.e., $C_{12}Q = GQ$. It follows that $G = C_{12} = Q_1 \oplus -Q_2$. Thus, the desired result follows. ■

To prove the theorem when the domain is $\mathbf{M}_{m,n}$ with $m < n$, we can apply the result for the restriction of Φ to the subspace spanned by $\{E_{ij} : 1 \le i, j \le m\}$ and assume the restriction map has nice structure. Then we have to show that $\Phi(E_{il})$ also has a nice form for $l > m$. To do that we need another technical lemma showing that if $\Phi(E_{ij})$ and $\Phi(E_{kj})$ have nice forms, then $\Phi(E_{il})$ and $\Phi(E_{kl})$ also have nice forms. We state and prove the results for a special case in the following, in view of Lemma 2.2.

LEMMA 2.5. *Let $Q_1 \in \mathbf{M}_{k_1}, Q_2 \in \mathbf{M}_{k_2}$ with $k_1 + k_2 = k$ be diagonal matrices with positive diagonal entries arranged in descending order. Let $\Phi : \mathbf{M}_2 \to \mathbf{M}_{r,s}$ be a nonzero linear map preserving disjointness.*

(a) Assume

$$\Phi(E_{11}) = \begin{pmatrix} Q_1 & 0 & 0 & 0 \\ 0 & 0_{k_1,k_2} & 0 & 0 \\ 0 & Q_2 & 0_{k_2} & 0 \\ 0 & 0 & 0 & 0_{r_1,s_1} \end{pmatrix}, \quad \Phi(E_{21}) = \begin{pmatrix} 0_{k_1} & 0 & 0 & 0 \\ Q_1 & 0_{k_1,k_2} & 0 & 0 \\ 0 & 0_{k_2} & Q_2 & 0 \\ 0 & 0 & 0 & 0_{r_1,s_1} \end{pmatrix},$$

where $(r_1, s_1) = (r - 2k_1 - k_2, s - k_1 - 2k_2)$*. Then there exist* $R = R_1 \oplus R_2 \in \mathbf{U}_{k_1} \oplus \mathbf{U}_{k_2}$*,* $U \in \mathbf{U}_{r-k}$*,* $V \in \mathbf{U}_{s-k}$ *such that*

$$R_1^* Q_1 R_1 = Q_1, \quad R_2^* Q_2 R_2 = Q_2,$$
$$U \begin{pmatrix} Q_1 \\ 0_{r-k-k_1,k_1} \end{pmatrix} R_1 = \begin{pmatrix} Q_1 \\ 0_{r-k-k_1,k_1} \end{pmatrix},$$

and

$$R_2^*(Q_2 \mid 0_{k_2,s-k-k_2})V = (Q_2 \mid 0_{k_2,s-k-k_2});$$

moreover, if $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ *with* $U_{11} \in \mathbf{M}_{k_1}$*, then the modified map* Ψ *defined by*

$$X \mapsto \begin{pmatrix} R_1^* & 0 & 0 & 0 \\ 0 & U_{11} & 0 & U_{12} \\ 0 & 0 & R_2^* & 0 \\ 0 & U_{21} & 0 & U_{22} \end{pmatrix} \Phi(X) \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & V \end{pmatrix}$$

satisfies

$$\Psi(E_{11}) = \Phi(E_{11}), \quad \Psi(E_{21}) = \Phi(E_{21}),$$

$$\Psi(E_{12}) = \begin{pmatrix} 0_{k_1} & 0 & 0 & Q_1 & 0 \\ 0 & 0_{k_1,k_2} & 0 & 0 & 0 \\ 0 & 0 & 0_{k_2} & 0 & 0 \\ 0 & Q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0_{r-2k,s-2k} \end{pmatrix},$$

$$\Psi(E_{22}) = \begin{pmatrix} 0_{k_1} & 0 & 0 & 0 & 0 \\ 0 & 0_{k_1,k_2} & 0 & Q_1 & 0 \\ 0 & 0 & 0_{k_2} & 0 & 0 \\ 0 & 0 & Q_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0_{r-2k,s-2k} \end{pmatrix}.$$

Consequently, before the modification we have

$$\Phi(E_{12}) = \begin{pmatrix} 0_{k_1} & 0 & 0 & \hat{Y}_1 \\ 0 & 0_{k_1,k_2} & 0 & 0 \\ 0 & 0 & 0_{k_2} & 0 \\ 0 & \hat{Y}_2 & 0 & 0_{r_1,s_1} \end{pmatrix}, \ \Phi(E_{22}) = \begin{pmatrix} 0_{k_1} & 0 & 0 & 0 \\ 0 & 0_{k_1,k_2} & 0 & \hat{Z}_1 \\ 0 & 0 & 0_{k_2} & 0 \\ 0 & 0 & \hat{Z}_2 & 0_{r_1,s_1} \end{pmatrix},$$

where $\hat{Y}_1, \hat{Z}_1$ have singular values equal to the diagonal entries of Q_1, and $\hat{Y}_2, \hat{Z}_2$ have singular values equal to the diagonal entries of Q_2.

(b) Suppose

$$\Phi(E_{ij}) = \begin{pmatrix} E_{ij}\otimes Q_1 & 0 & 0 \\ 0 & E_{ji}\otimes Q_2 & 0 \\ 0 & 0 & 0_{r_2,s_2} \end{pmatrix} \quad \text{for } (i,j) \in \{(1,1),(2,1),(2,2)\}, \tag{2.5}$$

and $(r_2, s_2) = (r-2k, s-2k)$. Then $\Phi(E_{12})$ also satisfies (2.5).

Proof. (a) By Lemma 2.3, we know that the disjoint matrices $\Phi(E_{22})$ and $\Phi(E_{11})$ have the same rank. So, $r, s \geq 2k$. Let $P_1 \in \mathbf{M}_{2k}$ be a permutation matrix such that $[X_1|X_2|X_3|X_4]P_1 = [X_1|X_3|X_2|X_4]$ whenever $X_1, X_2 \in \mathbf{M}_{2k,k_1}$ and $X_3, X_4 \in \mathbf{M}_{2k,k_2}$. Then the map $\hat{\Phi}$ defined by $\hat{\Phi}(X) = (P_1^t \oplus I_{r-2k})\Phi(X)$ will still preserve disjointness such that $\hat{\Phi}(E_{11})$ and $\hat{\Phi}(E_{21})$ equal

$$\hat{\Phi}(E_{11}) = \begin{pmatrix} Q_1 & 0 & 0 & 0 \\ 0 & Q_2 & 0 & 0 \\ 0 & 0 & 0_{k_1,k_2} & 0 \\ 0 & 0 & 0 & 0_{r_1,s_1} \end{pmatrix} \text{ and } \hat{\Phi}(E_{21}) = \begin{pmatrix} 0_{k_1} & 0 & 0 & 0 \\ 0 & 0_{k_2} & Q_2 & 0 \\ Q_1 & 0 & 0_{k_1,k_2} & 0 \\ 0 & 0 & 0 & 0_{r_1,s_1} \end{pmatrix}.$$

Suppose $P_2 \in \mathbf{M}_k$ is a permutation matrix such that $D_1 = P_2^t(Q_1 \oplus Q_2)P_2$ has diagonal entries arranged in descending order. We can then find $U_1 \in \mathbf{U}_{r-k}$ and $V_1 \in \mathbf{U}_{s-k}$ such that

$$(P_2^t \oplus U_1)\hat{\Phi}(E_{22})(P_2 \oplus V_1) = \begin{pmatrix} 0_k & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where D_2 is a diagonal matrix with positive diagonal entries arranged in descending order.

Applying Lemma 2.3, we can find $S_2 \in \mathbf{U}_k, U_2 \in \mathbf{U}_{r-k}, V_2 \in \mathbf{U}_{s-k}$ such that the map Ψ_1 defined by

$$X \mapsto (S_2^* \oplus U_2)(P_2^t \oplus U_1)\hat{\Phi}(X)(P_2 \oplus V_1)(S_2 \oplus V_2)$$

satisfies

$$E_{ij} \mapsto (E_{ij} \otimes (\hat{Q}_1 \oplus 0_{\ell_2}) + E_{ji} \otimes (0_{\ell_1} \oplus \hat{Q}_2)), \quad 1 \leq i,j \leq 2,$$

where $\hat{Q}_1 \in \mathbf{M}_{\ell_1}$ and $\hat{Q}_2 \in \mathbf{M}_{\ell_2}$ are diagonal matrices with positive diagonal entries arranged in descending order. Let Ψ be defined by $\Psi(X) = \Psi_1(X)(I_k \oplus P_3 \oplus I_{s-2k})$, where $P_3 \in \mathbf{M}_k$ is a permutation matrix such that $[X_1|X_2]P_3 = [X_2|X_1]$ whenever $X_1 \in \mathbf{M}_{k,k_1}$ and $X_2 \in \mathbf{M}_{k,k_2}$. Then the map Ψ satisfies

$$E_{ij} \mapsto (E_{ij} \otimes (\hat{Q}_1 \oplus 0_{\ell_2}) + E_{ji} \otimes (0_{\ell_1} \oplus \hat{Q}_2))(I_k \oplus P_3 \oplus I_{s-2k}), \quad 1 \leq i,j \leq 2.$$

Let $R = P_2S_2 \in \mathbf{M}_k$, $V = V_1V_2(P_3 \oplus I_{s-2k}) \in \mathbf{M}_{s-k}$, and $U = U_2U_1 \in \mathbf{M}_{r-k}$. Then

$$\Psi(X) = (R^* \oplus U)\hat{\Phi}(X)(R \oplus V) \quad \text{for all } X \in \mathbf{M}_2.$$

If we partition $\Psi(X)$ into a 2×2 block matrix such that the $(1,1)$ block lies in $\mathbf{M}_k$, then the diagonal entries of $\hat{Q}_1$ are the singular values of the $(2,1)$ block of $\hat{\Phi}(E_{21})$ (using the same partition). So, $\hat{Q}_1 = Q_1$ and $\hat{Q}_2 = Q_2$. Hence, $\hat{\Phi}(E_{21}) = \Psi(E_{21})$. It follows that

(2.6)

$$R^*\begin{pmatrix} 0_{k_1,k_2} & 0_{k_1,s_1} \\ Q_2 & 0_{k_2,s_1}\end{pmatrix} V = \begin{pmatrix} 0_{k_1,k_2} & 0_{k_1,s_1} \\ Q_2 & 0_{k_2,s_1}\end{pmatrix},\ U\begin{pmatrix} Q_1 & 0_{k_1,k_2} \\ 0_{r_1,k_1} & 0_{r_1,k_2}\end{pmatrix} R = \begin{pmatrix} Q_1 & 0_{k_1,k_2} \\ 0_{r_1,k_1} & 0_{r_1,k_2}\end{pmatrix}.$$

As a result,

$$R^*\begin{pmatrix} 0_{k_1} & 0 \\ 0 & Q_2^2\end{pmatrix} R = \begin{pmatrix} 0_{k_1} & 0 \\ 0 & Q_2^2\end{pmatrix} \quad \text{and} \quad R^*\begin{pmatrix} Q_1^2 & 0 \\ 0 & 0_{k_2}\end{pmatrix} R = \begin{pmatrix} Q_1^2 & 0 \\ 0 & 0_{k_2}\end{pmatrix}.$$

Thus, $R = R_1 \oplus R_2$ with $R_1 \in \mathbf{M}_{k_1}, R_2 \in \mathbf{M}_{k_2}$. Since Q_1 and Q_2 are diagonal matrices with positive diagonal entries, we see that $R_1^* Q_1 R_1 = Q_1$ and $R_2^* Q_2 R_2 = Q_2$. Moreover, by (2.6) we have

$$U\begin{pmatrix} Q_1 \\ 0_{r-k-k_1,k_1}\end{pmatrix} R_1 = \begin{pmatrix} Q_1 \\ 0_{r-k-k_1,k_1}\end{pmatrix} \text{ and } R_2^*(Q_2 \mid 0_{k_2,s-k-k_2})V = (Q_2 \mid 0_{k_2,s-k-k_2}).$$

One can then check that the modified map $\hat{\Psi}(X) = (P_1^t \oplus I_{r-2k})\Psi(X)$ has the desired property.

Now, we turn to $\Phi(E_{12})$ and $\Phi(E_{21})$. If $U = (U_{ij})_{1\le i,j\le 3} \in \mathbf{M}_{r-k}$ with $U_{11} \in \mathbf{M}_{k_1}, U_{22} \in \mathbf{M}_{k_2}$, and $V = (V_{ij})_{1\le i,j\le 3} \in \mathbf{M}_{s-k}$ with $V_{11} \in \mathbf{M}_{k_2}, V_{22} \in \mathbf{M}_{k_1}$, then

$$\hat{\Phi}(E_{12}) = (R \oplus U^*)\Psi(E_{12})(R^* \oplus V^*) = \begin{pmatrix} 0_k & F_{12} \\ F_{21} & 0_{r-k,s-k}\end{pmatrix},$$

where

$$F_{12} = R\begin{pmatrix} 0 & Q_1 & 0 \\ 0_{k_2} & 0_{k_2,k_1} & 0_{k_2,s-2k}\end{pmatrix} V^* = \begin{pmatrix} R_1Q_1V_{12}^* & R_1Q_1V_{22}^* & R_1Q_1V_{32}^* \\ 0_{k_2} & 0_{k_2,k_1} & 0_{k_2,s-2k}\end{pmatrix},$$

$$F_{21} = U^*\begin{pmatrix} 0_{k_1} & 0 \\ 0 & Q_2 \\ 0 & 0\end{pmatrix} R^* = \begin{pmatrix} 0_{k_1} & U_{21}^*Q_2R_2^* \\ 0_{k_2,k_1} & U_{22}^*Q_2R_2^* \\ 0_{r-2k,k_1} & U_{23}^*Q_2R_2^*\end{pmatrix}.$$

Note that $\hat{\Phi}(E_{12})$ and $\hat{\Phi}(E_{21})$ are disjoint. So, $U_{21}^*Q_2R_2^*, R_1Q_1V_{12}^* \in \mathbf{M}_{k_1,k_2}$ are zero blocks. Since R_1Q_1 and $Q_2R_2^*$ are invertible, we see that

$$U_{21}^* = 0_{k_1,k_2} \quad \text{and} \quad V_{12}^* = 0_{k_1,k_2}. \tag{2.7}$$

As a result, $\Phi(E_{12}) = (P_1 \oplus I_{r-2k})\hat{\Phi}(E_{12})$ has the asserted form with

$$\hat{Y}_1 = (R_1Q_1V_{22}^* \mid R_1Q_1V_{32}^*) \quad \text{and} \quad \hat{Y}_2 = \begin{pmatrix} U_{22}^*Q_2R_2^* \\ U_{23}^*Q_2R_2^*\end{pmatrix}.$$

Also, $\hat{\Phi}(E_{22}) = \begin{pmatrix} 0_k & 0 \\ 0 & G \end{pmatrix}$ with

$$\begin{aligned}
G &= U^* \begin{pmatrix} 0 & Q_1 & 0 \\ Q_2 & 0 & 0 \\ 0 & 0 & 0_{r-2k,s-2k} \end{pmatrix} V^* \\
&= U^* \begin{pmatrix} 0 & Q_1 & 0 \\ 0_{k_2} & 0 & 0 \\ 0 & 0 & 0_{r-2k,s-2k} \end{pmatrix} V^* + U^* \begin{pmatrix} 0 & 0_{k_1} & 0 \\ Q_2 & 0 & 0 \\ 0 & 0 & 0_{r-2k,s-2k} \end{pmatrix} V^* \\
&= U^* \begin{pmatrix} 0 & Q_1 \\ 0_{k_2} & 0 \\ 0 & 0 \end{pmatrix} R'^* R' \begin{pmatrix} V_{11}^* & V_{21}^* & V_{31}^* \\ V_{12}^* & V_{22}^* & V_{32}^* \end{pmatrix} \\
&\quad + \begin{pmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \\ U_{13}^* & U_{23}^* \end{pmatrix} R^* R \begin{pmatrix} 0 & 0_{k_1} & 0 \\ Q_2 & 0 & 0 \end{pmatrix} V^* \\
&= \begin{pmatrix} 0 & Q_1 \\ 0_{k_2} & 0 \\ 0 & 0 \end{pmatrix} R' \begin{pmatrix} V_{11}^* & V_{21}^* & V_{31}^* \\ V_{12}^* & V_{22}^* & V_{32}^* \end{pmatrix} + \begin{pmatrix} U_{11}^* & U_{21}^* \\ U_{12}^* & U_{22}^* \\ U_{13}^* & U_{23}^* \end{pmatrix} R^* \begin{pmatrix} 0_{k_1,k_2} & 0 & 0 \\ Q_2 & 0 & 0 \end{pmatrix}
\end{aligned}$$

by (2.6), where $R' = R_2 \oplus R_1$. Thus, by (2.7), we have

$$\begin{aligned}
G &= \begin{pmatrix} Q_1 R_1 V_{12}^* + U_{21}^* R_2^* Q_2 & Q_1 R_1 V_{22}^* & Q_1 R_1 V_{32}^* \\ U_{22}^* R_2^* Q_2 & 0 & 0 \\ U_{23}^* R_2^* Q_2 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0_{k_1,k_2} & Q_1 R_1 V_{22}^* & Q_1 R_1 V_{32}^* \\ U_{22}^* R_2^* Q_2 & 0 & 0 \\ U_{23}^* R_2^* Q_2 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

As a result, $\Phi(E_{22}) = (P_1 \oplus I_{r-2k})\hat{\Phi}(E_{22})$ has the asserted form with

$$\hat{Z}_1 = (Q_1 R_1 V_{22}^* \mid Q_1 R_1 V_{32}^*) \quad \text{and} \quad \hat{Z}_2 = \begin{pmatrix} U_{22}^* R_2^* Q_2 \\ U_{23}^* R_2^* Q_2 \end{pmatrix}.$$

(b) Applying a block permutation, we may assume that $\Phi(E_{11})$, $\Phi(E_{21})$, $\Phi(E_{22})$ equal

$$\begin{pmatrix} Q_1 \oplus Q_2 & 0_k & 0 \\ 0_k & 0_k & 0 \\ 0 & 0 & 0_{\hat{r},\hat{s}} \end{pmatrix}, \begin{pmatrix} 0_k & 0_{k_1} \oplus Q_2 & 0 \\ Q_1 \oplus 0_{k_2} & 0_k & 0 \\ 0 & 0 & 0_{\hat{r},\hat{s}} \end{pmatrix}, \begin{pmatrix} 0_k & 0_k & 0 \\ 0_k & Q_1 \oplus Q_2 & 0 \\ 0 & 0 & 0_{\hat{r},\hat{s}} \end{pmatrix},$$

respectively. We need to show that

$$\Phi(E_{12}) = \begin{pmatrix} 0_k & Q_1 \oplus 0_{k_2} & 0 \\ 0_{k_1} \oplus Q_2 & 0_k & 0 \\ 0 & 0 & 0_{\hat{r},\hat{s}} \end{pmatrix}.$$

Suppose $\hat{P} \in \mathbf{M}_k$ is a permutation matrix such that $\hat{D} = \hat{P}^t(Q_1 \oplus Q_2)\hat{P}$ is a diagonal matrix with entries in descending order. Applying Lemma 2.3 to the map

$$X \mapsto (\hat{P}^t \oplus \hat{P}^t \oplus I_{r-2k})\Phi(X)(\hat{P} \oplus \hat{P} \oplus I_{s-2k}),$$

we conclude that there exist a permutation $P \in \mathbf{M}_k$ and $W_1, W_2 \in \mathbf{U}_k$ commuting with $\hat{D}$ such that for $W = \hat{P}W_1W_2P \oplus \hat{P}W_2P \in \mathbf{M}_{2k}$, the map Ψ defined by $X \mapsto (W^* \oplus I_{r-2k})\Phi(X)(W \oplus I_{s-2k})$ has the form

$$E_{ij} \mapsto E_{ij} \otimes (\hat{Q}_1 \oplus 0_{\ell_2}) + E_{ji} \otimes (0_{\ell_1} \oplus \hat{Q}_2), \quad 1 \le i, j \le 2,$$

where $\hat{Q}_1 \in \mathbf{M}_{\ell_1}$ and $\hat{Q}_2 \in \mathbf{M}_{\ell_2}$ are diagonal matrices with positive diagonal entries arranged in descending order. Note that the diagonal entries of $\hat{Q}_1$ are the singular values of the $(1,2)$ block of $\Phi(E_{12})$. So, $\hat{Q}_1 = Q_1$ and $\hat{Q}_2 = Q_2$. Consequently,

$$\Phi(X) = (W \oplus I_{r-2k})\Psi(X)(W^* \oplus I_{s-2k})$$

has the asserted form. ∎

Proof of Theorem 2.1. Without loss of generality, we assume $2 \le m \le n$. We prove the result by induction on $n - m$. If $n - m = 0$, the result follows from Lemma 2.4. Suppose $n - m = \ell \ge 1$ and the result holds for the cases when $n - m < \ell$.

By the induction assumption on the restriction map of Φ on the span of $\mathcal{C}_n = \{E_{ij} : 1 \le i \le m, 1 \le j < n\}$, there are diagonal matrices $Q_1 \in \mathbf{M}_{k_1}, Q_2 \in \mathbf{M}_{k_2}$ with positive entries arranged in descending order, and $U_1 \in \mathbf{U}_r, V_1 \in \mathbf{U}_s$ such that the map $U_1\Phi(X)V_1$ satisfies

$$E_{ij} \mapsto \begin{pmatrix} \hat{E}_{ij} \otimes Q_1 & 0 & 0 \\ 0 & \hat{E}_{ji} \otimes Q_2 & 0 \\ 0 & 0 & 0_{\hat{r},\hat{s}} \end{pmatrix} \qquad \text{for all } E_{ij} \in \mathcal{C}_n, \tag{2.8}$$

where $\{E_{ij} : 1 \le i \le m, 1 \le j \le n\}$ is the standard basis for $\mathbf{M}_{m,n}$, $\{\hat{E}_{ij} : 1 \le i \le m, 1 \le j < n\}$ is the standard basis for $\mathbf{M}_{m,n-1}$, and $(\hat{r}, \hat{s}) = (r - mk_1 - (n-1)k_2, s - (n-1)k_1 - mk_2)$. For notational simplicity, we assume that $U_1 = I_r, V_1 = I_s$.

Consider the restriction of Φ on $\operatorname{span}\{E_{ij}, E_{in}, E_{mj}, E_{mn}\}$ for all $1 \le i < m, 1 \le j < n$. By Lemma 2.5 (a), we see that

$$\Phi(E_{mn}) = \begin{pmatrix} 0_{mk_1,(n-1)k_1} & 0 & Z_1 \\ 0 & 0_{(n-1)k_2,mk_2} & 0 \\ 0 & Z_2 & 0_{\hat{r},\hat{s}} \end{pmatrix}, \tag{2.9}$$

where only the last k_1 rows of Z_1 can be nonzero, and only the last k_2 columns of Z_2 can be nonzero.

Similarly,

$$\Phi(E_{1n}) = \begin{pmatrix} 0_{mk_1,(n-1)k_1} & 0 & Y_1 \\ 0 & 0_{(n-1)k_2,mk_1} & 0 \\ 0 & Y_2 & 0_{\hat{r},\hat{s}} \end{pmatrix} \tag{2.10}$$

where only the first k_1 rows of Y_1 can be nonzero, and only the first k_2 columns of Y_2 can be nonzero.

Now, consider the restriction of Φ on $\mathrm{span}\{E_{11}, E_{1n}, E_{m1}, E_{mn}\}$. By Lemma 2.5 (a), there exist $R = R_1 \oplus R_2 \in \mathbf{U}_{k_1} \oplus \mathbf{U}_{k_2}$, $U \in \mathbf{U}_{\hat{r}}$ and $V \in \mathbf{U}_{\hat{s}}$ such that

$$R_1^* Q_1 R_1 = Q_1, \quad R_2^* Q_2 R_2 = Q_2,$$

$$U \begin{pmatrix} Q_1 \\ 0_{\hat{r}-k_1,k_1} \end{pmatrix} R_1 = \begin{pmatrix} Q_1 \\ 0_{\hat{r}-k_1,k_1} \end{pmatrix}, \text{ and } R_2^*(Q_2 \mid 0_{k_2,\hat{s}-k_2})V = (Q_2 \mid 0_{k_2,\hat{s}-k_2});$$

moreover, if $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ with $U_{11} \in \mathbf{M}_{k_1}$, then

$$\begin{pmatrix} R_1^* & 0 & 0 & 0 \\ 0 & U_{11} & 0 & U_{12} \\ 0 & 0 & R_2^* & 0 \\ 0 & U_{21} & 0 & U_{22} \end{pmatrix} \begin{pmatrix} 0_{2k_1,k_1} & 0_{2k_1,2k_2} & Z_1 \\ 0 & 0_{k_2,2k_2} & 0 \\ 0 & Z_2 & 0_{\hat{r},\hat{s}} \end{pmatrix} \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & V \end{pmatrix}$$

$$= \begin{pmatrix} 0_{k_1} & 0 & 0 & 0 & 0 \\ 0 & 0_{k_1,k_2} & 0 & Q_1 & 0 \\ 0 & 0 & 0_{k_2} & 0 & 0 \\ 0 & 0 & Q_2 & 0_{k_2,k_1} & 0 \\ 0 & 0 & 0 & 0 & 0_{r_1,s_1} \end{pmatrix},$$

where $(r_1, s_1) = (\hat{r} - 2k_1, \hat{s} - 2k_2)$. Consequently, the modified map Ψ defined by

$$X \mapsto \begin{pmatrix} I_{m-1} \otimes R_1^* & 0 & 0 & 0 \\ 0 & U_{11} & 0 & U_{12} \\ 0 & 0 & I_{n-1} \otimes R_2^* & 0 \\ 0 & U_{21} & 0 & U_{22} \end{pmatrix} \Phi(X) \begin{pmatrix} I_{n-1} \otimes R_1 & 0 & 0 \\ 0 & I_{m-1} \otimes R_2 & 0 \\ 0 & 0 & V \end{pmatrix}$$

satisfies $\Psi(E_{ij}) = \Phi(E_{ij})$ for all $1 \le i \le m, 1 \le j < n-1$, and $\Psi(E_{mn})$ has the form (2.9) with

$$Z_1 = \begin{pmatrix} 0 & 0 \\ Q_1 & 0 \end{pmatrix} \text{ and } Z_2 = \begin{pmatrix} 0 & Q_2 \\ 0 & 0 \end{pmatrix}.$$

Let $\tilde{P} \in \mathbf{M}_s$ be the permutation matrix satisfying

$$[X_1|X_2|X_3|X_4]\tilde{P} = [X_1|X_3|X_2|X_4]$$

whenever $X_1 \in \mathbf{M}_{r,(n-1)k_1}$, $X_2 \in \mathbf{M}_{r,mk_2}$, $X_3 \in \mathbf{M}_{r,k_1}$, $X_4 \in \mathbf{M}_{r,\hat{s}-k_1}$. Then the map $\hat{\Psi}$ defined by $X \mapsto \Psi(X)\tilde{P}$ satisfies

$$\hat{\Psi}(E_{ij}) = \begin{pmatrix} E_{ij} \otimes Q_1 & 0 & 0 \\ 0 & E_{ji} \otimes Q_2 & 0 \\ 0 & 0 & 0_{\hat{r}-k_2,\hat{s}-k_1} \end{pmatrix} \tag{2.11}$$

for $(i,j) \in \{(u,v) : 1 \leq u \leq m, 1 \leq v < n\} \cup \{(m,n)\}$. For $j = 2, \ldots, n-1$, consider the restriction of Ψ on $\operatorname{span}\{E_{jj}, E_{jn}, E_{mj}, E_{mn}\}$. Thus, $\hat{\Psi}(E_{jj})$, $\hat{\Psi}(E_{mj})$ and $\hat{\Psi}(E_{mn})$ have the form (2.11), and so must $\hat{\Psi}(E_{jn})$ by Lemma 2.5 (b). As a result, $\hat{\Psi}(E_{ij})$ has the form in (2.11) for all $1 \leq i \leq m$, $1 \leq j \leq n$. ■

3. Nonsurjective (zero) Triple Product Preservers and JB*-homomorphisms on rectangular matrices

Notice that the set $\mathbf{M}_n(\mathbb{C})$ of complex square matrices is a C*-algebra. Let $T : \mathcal{A} \to \mathcal{B}$ be a bounded linear map between C*-algebras. In [31, Theorem 3.2], it was shown that T is a triple homomorphism with respect to the Jordan triple product,

$$\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a) \quad \text{for all } a, b, c \in \mathcal{A},$$

if and only if T preserves disjointness and $T^{**}(1)$ is a partial isometry in B^{**}. In the case that T is surjective, the condition on $T^{**}(1)$ can be dropped as shown in [20, Theorem 2.2], see also [27]. In [4], on the other hand, it is obtained a characterization of linear maps from C^*-algebras into JB*-triples that preserve disjointness with some conditions.

In the following, we consider the Jordan triple product

$$\{A, B, C\} = \frac{1}{2}(AB^*C + CB^*A)$$

of real or complex matrices $A, B, C \in \mathbf{M}_{m,n}$. A (real or complex) linear map $\Psi : \mathbf{M}_{m,n} \to \mathbf{M}_{r,s}$ between rectangular matrices is called a JB*-*triple homomorphism* if

(3.1)
$$\Psi(AB^*C + CB^*A) = \Psi(A)\Psi(B)^*\Psi(C) + \Psi(C)\Psi(B)^*\Psi(A), \quad \forall A, B, C \in \mathbf{M}_{m,n}.$$

We have the polarization identity

$$2\{A, B, C\} = \{A + C, B, A + C\} - \{A, B, A\} - \{C, B, C\}, \quad \forall A, B, C \in \mathbf{M}_{m,n}.$$

In the complex case, letting the *cube* $A^{(3)} = AA^*A$, we have

$$4\{A, B, A\} = (B + A)^{(3)} + (B - A)^{(3)} - (B + iA)^{(3)} - (B - iA)^{(3)}, \ \forall A, B \in \mathbf{M}_{m,n}.$$

Therefore, a linear map Φ between rectangular matrices is a JB*-triple homomorphism exactly when $\Phi(AB^*A) = \Phi(A)\Phi(B)^*\Phi(A)$, and in the complex case exactly when $\Phi(AA^*A) = \Phi(A)\Phi(A)^*\Phi(A)$, for all $A, B \in \mathbf{M}_{m,n}$.

We say that the matrix triple (A, B, C) in $\mathbf{M}_{m,n}$ has *zero triple product* if

$$\{A, B, C\} = 0_{m,n}.$$

A linear map $\Phi : \mathbf{M}_{m,n} \to \mathbf{M}_{r,s}$ preserves zero triple products if

$$\{A, B, C\} = 0_{m,n} \implies \{\Phi(A), \Phi(B), \Phi(C)\} = 0_{r,s} \quad \text{for all } A, B, C \in \mathbf{M}_{m,n}.$$

For more information of JB*-triples, see, e.g., [9].

We have the following result concerning the zero triple product preservers and JB*-triple homomorphisms on rectangular matrices.

THEOREM 3.1. *Let $\Phi : \mathbf{M}_{m,n} \to \mathbf{M}_{r,s}$ be a linear map.*

(a) Φ preserves zero triple products if and only if there are $U \in \mathbf{U}_r, V \in \mathbf{U}_s$, and diagonal matrices Q_1, Q_2 with positive diagonal entries such that

$$\Phi(A) = U \begin{pmatrix} A \otimes Q_1 & 0 & 0 \\ 0 & A^t \otimes Q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V. \tag{3.2}$$

Here Q_1 or Q_2, may be vacuous.

(b) Φ is a JB-triple homomorphism if and only if there exist $U \in \mathbf{U}_r, V \in \mathbf{U}_s$, and nonnegative integers q_1, q_2 such that*

$$\Phi(A) = U \begin{pmatrix} A \otimes I_{q_1} & 0 & 0 \\ 0 & A^t \otimes I_{q_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} V, \tag{3.3}$$

where the size of the zero block at the bottom right corner is $(r - (q_1 m + q_2 n)) \times (s - (q_1 n + q_2 m))$.

To prove the above theorem, we need the following lemma, which is valid for both real and complex matrices. See [4, Lemma 1] for the complex case. Recall that $A^* = A^t$ in the real case.

LEMMA 3.2. *Let $A, B \in \mathbf{M}_{m,n}$. The following conditions are equivalent to each other.*

(a) $A^*B = 0_n$ *and* $AB^* = 0_m$.

(b) $AA^*B + BA^*A = 0_{m,n}$.

Proof. It suffices to prove (b) $\Longrightarrow$ (a). Observe that from (b) we have

$$0 \leq (B^*A)(B^*A)^* = B^*AA^*B = -(B^*B)(A^*A).$$

Taking adjoints of the Hermitian matrices, we have

$$(B^*B)(A^*A) = (A^*A)(B^*B).$$

Therefore, the positive semi-definite $n \times n$ matrices A^*A and B^*B commute. By spectral theory, the product $(B^*B)(A^*A) = -(B^*A)(B^*A)^*$ is also positive semi-definite, and thus $B^*A = 0$. Similarly, we have $AB^* = 0$. ∎

Proof of Theorem 3.1. (a) Suppose that Φ preserves zero triple products. By Lemma 3.2, if $A, B \in \mathbf{M}_{m,n}$ are disjoint, then $\Phi(A), \Phi(B) \in \mathbf{M}_{r,s}$ are disjoint. So, Φ has the asserted form by Theorem 2.1. The converse is clear.

(b) Suppose that Φ is a JB*-triple homomorphism. Then it will preserve zero triple products, and thus by (a), be of the form (3.2). Since $E_{11}^{(3)} = E_{11}$, we have

$\Phi(E_{11})^{(3)} = \Phi(E_{11})$. One gets the conclusions $Q_1 = I_{q_1}$ and $Q_2 = I_{q_2}$ as in (3.3). The converse is clear. ∎

Recall that a rectangular matrix A is called a *partial isometry* if $AA^*A = A$. Equivalently, A has singular values from the set $\{1, 0\}$. We state our result using the complex notation. Of course, in the real case, we have $X^* = X^t$, and a unitary matrix is a real orthogonal matrix. It turns out that JB*-triple homomorphisms are closely related to linear preservers of (disjoint) partial isometries. Some assertions in the following might be known to experts, at least in the complex case.

THEOREM 3.3. *Suppose $\Phi : \mathbf{M}_{m,n} \to \mathbf{M}_{r,s}$ is a linear map. The following conditions are equivalent.*

(a) *Φ maps partial isometries in $\mathbf{M}_{m,n}$ to partial isometries in $\mathbf{M}_{r,s}$.*
(b) *Φ sends disjoint (rank one) partial isometries to disjoint partial isometries.*
(c) *Φ preserves disjointness, and there is a nonzero partial isometry $P \in \mathbf{M}_{m,n}$ such that $\Phi(P)$ is a partial isometry.*
(d) *Φ preserves matrix triples with zero JB*-triple product, and there is a nonzero partial isometry $P \in \mathbf{M}_{m,n}$ such that $\Phi(P)$ is a partial isometry.*
(e) *Φ is a JB*-triple homomorphism and has the form* (3.3).

Proof. The implication (e) $\Longrightarrow$ (a) is clear.

(a) $\Longrightarrow$ (b): Let $A \in \mathbf{M}_{m,n}$ be a rank one partial isometry, and $\Phi(A) = U\begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}V$, where $U \in \mathbf{U}_r, V \in \mathbf{U}_s$. Suppose $B \in \mathbf{M}_{m,n}$ is a rank one partial isometry disjoint from A such that $\Phi(B) = U\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}V$ with $Y_{11} \in \mathbf{M}_k$. Because $\Phi(A) \pm \Phi(B)$ are partial isometries, we see that the Euclidean norm of each of the first k columns of $\Phi(A) + \Phi(B)$ and $\Phi(A) - \Phi(B)$ is not larger than one. Thus, Y_{11}, Y_{21} are zero matrices. Considering the norms of the first k rows of $\Phi(A) + \Phi(B)$, we see that Y_{12} is the zero matrix as well. Thus, $\Phi(A), \Phi(B)$ are disjoint partial isometries in $\mathbf{M}_{r,s}$. In general, due to the singular value decomposition, every rectangular matrix can be written as a linear sum of disjoint rank one partial isometries. Thus Φ sends disjoint partial isometries to disjoint partial isometries.

(b) $\Longrightarrow$ (c): Φ preserves disjointness of rank one partial isometries, and hence preserves disjointness due to the singular value decomposition. Evidently, it sends a nonzero partial isometry to a partial isometry.

(c) $\Longrightarrow$ (e): Because Φ preserves disjointness, Φ has the form described in Theorem 2.1. By the fact that Φ sends a nonzero partial isometry to a partial isometry, we see that Q_1, Q_2 are identity matrices. So, conditions (a), (b), (c) and (e) are equivalent.

By Lemma 3.2 we have (d) $\Longrightarrow$ (c). The implication (e) $\Longrightarrow$ (d) is also clear. ∎

Several remarks are in order. Theorem 3.1 and Theorem 3.3 are also valid for real linear maps $\Phi : \mathbf{H}_n \to \mathbf{M}_{r,s}$. Note that self-adjoint partial isometries are exactly differences $p - q$ of two orthogonal projections. Indeed, we can further assume that the co-domain is $\mathbf{H}_r$, i.e., $\Phi : \mathbf{H}_n \to \mathbf{H}_r$. Then we can arrange $U = V^*$ in (3.2) and (3.3), at the expenses that Q_1, Q_2 may have negative diagonal matrices in (3.2), and (3.3) may look like

$$\Phi(A) = U \begin{pmatrix} A \otimes I_{q_1^+} & 0 & 0 & 0 & 0 \\ 0 & -A \otimes I_{q_1^-} & 0 & 0 & 0 \\ 0 & 0 & A^t \otimes I_{q_2^+} & 0 & 0 \\ 0 & 0 & 0 & -A^t \otimes I_{q_2^-} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} U^*,$$

where $q_1^+, q_1^-, q_2^+, q_2^-$ are nonnegative integers and the zero block matrix in the bottom right corner has size $(r - ((q_1^+ + q_1^-)m + (q_2^+ + q_2^-)n)) \times (r - ((q_1^+ + q_1^-)n + (q_2^+ + q_2^-)m))$.

Theorem 3.1 (a) allows us to obtain the following general result on linear preserver of functions of JB^*-triple product on matrices.

COROLLARY 3.4. *Let ν_1, ν_2 be scalar functions on $M_{m,n}$ and $M_{r,s}$ such that*

$$\nu_j(A) = 0 \quad \text{if and only if} \quad A = 0$$

for all A in $M_{m,n}$ or $M_{r,s}$, respectively. Suppose a linear map $\Phi : M_{m,n} \to M_{r,s}$ satisfies

$$\nu_1(\{A, B, C\}) = \nu_2\{\Phi(A), \Phi(B), \Phi(C)\}) \qquad \text{for all } A, B, C \in M_{m,n}. \tag{3.4}$$

Then Φ has the form (3.2).

This corollary can be used to determine the structure of linear preservers of functions on triple product of matrices easily. We mention a few examples in the following related to the study in [5, 10–12, 16, 17, 23] and their references.

Suppose a linear map $\Phi : M_{m,n} \to M_{r,s}$ satisfies (3.4), where ν_1, ν_2 are norms on matrices. Then Φ has the form (3.2). From this, one may easily deduce the conditions on U, V, Q_1, Q_2, etc. to ensure the converse of the statement. For example, if ν_1, ν_2 are the operator norms, then U, V can be any unitary matrices and the operator norm of $D_1 \oplus D_2$ has to be one.

Suppose $(m, r) = (n, s)$, $\mathbb{F} = \mathbb{C}$, and ν_1, ν_2 are the numerical radius. Then $\Phi : M_n \to M_r$ satisfies (3.4) if and only if Φ has the form (3.2) with $V = RU^*$ for a diagonal matrix R such that $((I_n \otimes Q_1) \oplus (I_n \otimes Q_2) \oplus 0)R$ has numerical radius 1. From this, one may further deduce that when $(m, r) = (n, s)$, $\mathbb{F} = \mathbb{C}$, and ν_1, ν_2 are the numerical range, $\Phi : M_n \to M_r$ satisfies (3.4) if and only if Φ has the form (3.3) with $V = U^*$. Similarly, we can treat the linear preservers $\Phi : M_n \to M_r$ leaving invariant the pseudo spectral radius, pseudo spectrum, and other types of scalar or non-scalar functions.

4. Nonsurjective norm preservers

Denote the singular values of $A \in \mathbf{M}_{m,n}$ by $s_1(A) \geq \cdots \geq s_h(A)$ for $h = \min\{m,n\}$. For $p > 0$, let

$$S_p(A) = \left(\sum_{j=1}^{h} s_j(A)^p\right)^{1/p}.$$

If $p \geq 1$, then $S_p(A)$ is known as the *Schatten p-norm*. In particular, $S_2(A) = (\sum_{j=1}^{h} s_j(A))^{1/2} = (\operatorname{tr}(A^*A))^{1/2}$, which is called the *Frobenius norm*, equips $\mathbf{M}_{m,n}$ as a Hilbert space. For $1 \leq p < +\infty$ but $p \neq 2$, a linear operator $\Psi : \mathbf{M}_{m,n} \to \mathbf{M}_{m,n}$ satisfies $S_p(\Psi(A)) = S_p(A)$ for all $A \in \mathbf{M}_{m,n}$ if and only if Ψ has the form $A \mapsto UAV$, or $A \mapsto UA^tV$ in case $m = n$, for some $U \in \mathbf{U}_m, V \in \mathbf{U}_n$ (see, e.g., [5,25]).

It is more difficult to characterize linear isometries from $\mathbf{M}_{m,n}$ to $\mathbf{M}_{r,s}$ for $(m,n) \neq (r,s)$. Only very few results are known; see, for example, [8,23]. With Theorem 2.1, we get the following result.

THEOREM 4.1. *Suppose $m,n \geq 2$, $p \in (0,2) \cup (2,+\infty)$, and $\Phi : \mathbf{M}_{m,n} \to \mathbf{M}_{r,s}$ is a linear map. The following conditions are equivalent.*

(a) *$S_p(\Phi(A)) = S_p(A)$ for all $A \in \mathbf{M}_{m,n}$.*
(b) *$S_p(\Phi(A)) = S_p(A)$ for all $A \in \mathbf{M}_{m,n}$ with rank at most 2.*
(c) *There are $U \in \mathbf{U}_r, V \in \mathbf{U}_s$, and diagonal matrices $Q_1 \in \mathbf{M}_{q_1}, Q_2 \in \mathbf{M}_{q_2}$ with positive diagonal entries such that $S_p(Q_1 \oplus Q_2) = 1$ and*

$$\Phi(A) = U \begin{pmatrix} A \otimes Q_1 & 0 & 0 \\ 0 & A^t \otimes Q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V \qquad \text{for all } A \in \mathbf{M}_{m,n}.$$

Here Q_1 or Q_2 may be vacuous.

Proof. The implications (c) $\implies$ (a) $\implies$ (b) are clear. For the implication (b) $\implies$ (c), it follows from a result of McCarthy [29, Theorem 2.7] that Φ preserves disjointness for rank one matrix pairs. By Theorem 2.1, we get the form of Φ. Applying the fact that $S_p(\Phi(E_{11})) = S_p(E_{11})$, we easily deduce that $S_p(Q_1 \oplus Q_2) = 1$. ■

For $1 \leq k \leq \min\{m,n\}$, the *Ky Fan k-norm* of A is defined by

$$F_k(A) = \sum_{j=1}^{k} s_j(A).$$

Linear isometries for the Ky Fan k-norm have been studied. Seeing Theorem 4.1, one may think that a similar extension for the Ky Fan k-norm can be obtained by similar arguments. It turns out that this can only be done for the complex case because there are real linear isometries for Ky Fan k-norms that do not preserve disjointness; see [18,25]. This reinforces the fact that proof techniques for complex matrices may not apply to real matrices, and it is quite remarkable that a uniform proof of Theorem 2.1 can be used for both real and complex matrices. In any event, we have the following theorem supplementing [23, Theorem 1.1], in which the linear map $\Phi : \mathbf{M}_{m,n}(\mathbb{C}) \to \mathbf{M}_{r,s}(\mathbb{C})$ is assumed to satisfy that

$$F_k(\Phi(A)) = F_{k'}(A), \quad \text{for all } A \in \mathbf{M}_{m,n}(\mathbb{C}).$$

THEOREM 4.2. *Suppose* $2 \le k' \le \min\{m,n\}$ *and* $1 \le k \le \min\{r,s\}$. *The following conditions are equivalent for a linear map* $\Phi : \mathbf{M}_{m,n}(\mathbb{C}) \to \mathbf{M}_{r,s}(\mathbb{C})$.

(a) $F_k(\Phi(A)) = F_{k'}(A)$ *for all* $A \in \mathbf{M}_{m,n}(\mathbb{C})$ *with rank at most* 2.

(b) There are unitary matrices $U \in \mathbf{M}_r(\mathbb{C})$, $V \in \mathbf{M}_s(\mathbb{C})$ *and positive-definite diagonal matrices* Q_1, Q_2 *(maybe vacuous) of size* q_1, q_2 *such that* $k \ge 2(q_1+q_2)$, $Q_1 \oplus Q_2$ *has trace* 1, *and*

$$\Phi(A) = U \begin{pmatrix} A \otimes Q_1 & 0 & 0 \\ 0 & A^t \otimes Q_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} V. \tag{4.1}$$

Proof. The implication (b) $\implies$ (a) is plain.

(a) $\implies$ (b). By [23, Lemma 2.2], Φ preserves disjoint rank one pairs. By Theorem 2.1, Φ carries the form (4.1). Consider $A_\epsilon = E_{11} + \epsilon E_{22}$ for $0 \le \epsilon < 1$. Using (4.1), we can assume

$$\Phi(A_\epsilon) = \lambda_1 A_\epsilon \oplus \lambda_2 A_\epsilon \oplus \cdots \oplus \lambda_q A_\epsilon \oplus 0$$

for some fixed scalars $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_q > 0$ with $q = q_1 + q_2$.

Suppose $k \le q$ first. Since $k' \ge 2$, we have

$$\begin{aligned} 1+\epsilon = F_{k'}(A_\epsilon) &= F_k(\lambda_1 A_\epsilon \oplus \lambda_2 A_\epsilon \oplus \cdots \oplus \lambda_q A_\epsilon \oplus 0) \\ &= \lambda_1 + \lambda_2 + \cdots + \lambda_k, \quad \text{when } 0 \le \epsilon\lambda_1 \le \lambda_k. \end{aligned}$$

This yields a contradiction, because $[0, \lambda_k/\lambda_1]$ contains infinitely many points ϵ.

Suppose $0 < r = k - q < q$. Then we have

$$\begin{aligned} 1+\epsilon &= F_{k'}(A_\epsilon) = F_k(\lambda_1 A_\epsilon \oplus \lambda_2 A_\epsilon \oplus \cdots \oplus \lambda_q A_\epsilon \oplus 0) \\ &= \begin{cases} \lambda_1 + \lambda_2 + \cdots + \lambda_q, & \text{when } \epsilon = 0, \\ \lambda_1 + \lambda_2 + \cdots + \lambda_q + \epsilon\lambda_1 + \cdots + \epsilon\lambda_r, & \text{when } \epsilon\lambda_{r+1} \le \lambda_q. \end{cases} \end{aligned}$$

This implies $\lambda_1 + \lambda_2 + \cdots + \lambda_q = 1$, and $1 + \epsilon = 1 + \epsilon\lambda_1 + \cdots + \epsilon\lambda_r$ for all $0 < \epsilon \le \lambda_q/\lambda_{r+1}$. This gives us the contradiction that $\lambda_{r+1} = \cdots = \lambda_q = 0$.

Hence, $k \ge 2q$. In this case, we have

$$\begin{aligned} 1+\epsilon = F_{k'}(A_\epsilon) &= F_k(\lambda_1 A_\epsilon \oplus \lambda_2 A_\epsilon \oplus \cdots \oplus \lambda_q A_\epsilon \oplus 0) \\ &= (1+\epsilon)(\lambda_1 + \lambda_2 + \cdots + \lambda_q), \quad \text{when } \epsilon \in [0,1). \end{aligned}$$

This gives $1 = \lambda_1 + \lambda_2 + \cdots + \lambda_q$, which equals the trace of $Q_1 \oplus Q_2$. ∎

5. Final remarks and future research

It would be interesting to extend our results in Sections 2 and 3 to the (real or complex) linear space $B(H,K)$ of bounded linear operators between infinite dimensional Banach spaces H and K, or to general JB*-triples. Our approach depends on the singular value decomposition of matrices, which is a finite dimensional feature. New techniques will be needed to extend our results.

To conclude the paper, we list several comments and questions concerning the results in Section 4.

(i) As pointed out in [8], the problem for the operator norm, i.e., Ky Fan 1-norm, is difficult.

(ii) Many real linear isometries for Ky Fan k-norms also preserve disjointness (although there are exceptions). It would be nice to investigate a version of Theorem 4.2 such that the conclusion also hold for real matrices.

(iii) For any linear isometry which preserves disjoint rank one pairs, we can apply Theorem 2.1. It is interesting to characterize such norms other than the Schatten p-norms and the Ky Fan k-norms. Suggested by the asserted form (4.1), we should put emphasis on unitarily invariant norms.

(iv) We have similar results for real symmetric and complex Hermitian matrices. Besides $S_p(A)$ and $F_k(A)$, can we do it for the k-numerical radius on Hermitian matrices $\mathbf{H}_n$ defined by

$$w_k(A) = \max\{\operatorname{tr}(AR) : R^* = R = R^2, \operatorname{tr} R = k\}?$$

(v) In fact, one can also ask for characterizations of k-numerical radius preservers $\Phi : \mathbf{M}_n \to \mathbf{M}_r$.

(vi) One may consider linear preservers or non-linear preservers for other types of norms or functions on rectangular matrices, Hermitian, symmetric, or skew-symmetric matrix spaces that are related to disjointness preserving maps.

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CHI-KWONG LI, DEPARTMENT OF MATHEMATICS, THE COLLEGE OF WILLIAM & MARY, WILLIAMSBURG, VA 13185, USA.
E-mail address: ckli@math.wm.edu

MING-CHENG TSAI, GENERAL EDUCATION CENTER, TAIPEI UNIVERSITY OF TECHNOLOGY 10608, TAIWAN.
E-mail address: mctsai2@mail.ntut.edu.tw

YA-SHU WANG, DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL CHUNG HSING UNIVERSITY, TAICHUNG 40227, TAIWAN.
E-mail address: yashu@nchu.edu.tw

NGAI-CHING WONG, DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL SUN YAT-SEN UNIVERSITY, KAOHSIUNG, 80424, TAIWAN.
E-mail address: wong@math.nsysu.edu.tw

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