ISOMETRIC EMBEDDINGS OF BANACH BUNDLES

Ming-Hsiu Hsu and Ngai-Ching Wong

Abstract. We show in this paper that every bijective linear isometry between the continuous section spaces of two non-square Banach bundles gives rise to a Banach bundle isomorphism. This is to support our expectation that the geometric structure of the continuous section space of a Banach bundle determines completely its bundle structures. We also describe the structure of an into isometry from a continuous section space into an other. However, we demonstrate by an example that a non-surjective linear isometry can be far away from a subbundle embedding.

1. INTRODUCTION

Let \( \langle B_X, \pi_X \rangle \) be a Banach bundle over a locally compact Hausdorff space \( X \). For each \( x \) in \( X \), denote by \( B_x = \pi_X^{-1}(x) \) the fiber Banach space. A continuous section \( f \) of the Banach bundle \( \langle B_X, \pi_X \rangle \) is a continuous function from \( X \) into \( B_X \) such that \( \pi_X(f(x)) = x \), i.e., \( f(x) \in B_x \) for all \( x \) in \( X \). Denote by \( \Gamma_X \) the Banach space of all continuous sections of \( \langle B_X, \pi_X \rangle \) vanishing at infinity, i.e. those \( f \) with \( \lim_{x \to \infty} \|f(x)\| = 0 \).

Let \( \langle B_Y, \pi_Y \rangle \) be an other Banach bundle over a locally compact Hausdorff space \( Y \) with continuous section space \( \Gamma_Y \). Assume that \( \Gamma_X \) is isometrically isomorphic to \( \Gamma_Y \) as Banach spaces. We want to assert whether \( \langle B_X, \pi_X \rangle \) is isometrically isomorphic to \( \langle B_Y, \pi_Y \rangle \) as Banach bundles (see \$2 for definitions). In other words, we expect that the geometric structure of the continuous sections of a Banach bundle determines its bundle structure.

Example 1.1. (Trivial line bundles). Let \( B_X = X \times \mathbb{K} \) and \( B_Y = Y \times \mathbb{K} \), where the underlying field \( \mathbb{K} \) is either the real \( \mathbb{R} \) or the complex \( \mathbb{C} \). The continuous section...
spaces are $C_0(X)$ and $C_0(Y)$, the Banach spaces of continuous scalar functions vanishing at infinity, respectively. The classical Banach-Stone Theorem (see, e.g., [1]) asserts that every linear isometry $T$ from $C_0(X)$ onto $C_0(Y)$ is a weighted composition operator:

$$Tf(y) = h(y)f(\varphi(y)), \quad \forall f \in C_0(X), \quad y \in Y.$$  

Here, $\varphi$ is a homeomorphism from $Y$ onto $X$, and $h$ is a continuous scalar function on $Y$ with $|h(y)| = 1$, $\forall y \in Y$. This induces an isometric bundle isomorphism $\Phi : B_X \rightarrow B_Y$ from $B_X = X \times \mathbb{K}$ onto $B_Y = Y \times \mathbb{K}$ defined by

$$\Phi(x, \alpha) = (\varphi^{-1}(x), h(\varphi^{-1}(x))\alpha), \quad \forall (x, \alpha) \in X \times \mathbb{K}.$$  

Hence, the trivial line bundles $\langle X \times \mathbb{K}, \pi_X \rangle$ and $\langle Y \times \mathbb{K}, \pi_Y \rangle$ are isometrically isomorphic if and only if they have isometrically isomorphic continuous section spaces.

Recall that a Banach space $E$ is strictly convex if $\|x + y\| < 2$ whenever $x \neq y$ in $E$ with $\|x\| = \|y\| = 1$. A Banach space $E$ is said to be non-square if $E$ does not contain a copy of the two-dimensional space $\mathbb{K} \oplus \mathbb{K}$ equipped with the norm $\|(a, b)\| = \max\{|a|, |b|\}$. In other words, if $x$ and $y$ are unit vectors in $E$, at least one of $\|x + y\|$ and $\|x - y\|$ is less than 2. Note that a Banach space $E$ is non-square if $E$ or its dual $E^*$ is strictly convex.

**Example 1.2.** (Trivial bundles). Let $E$ and $F$ be Banach spaces. We consider the trivial bundles $B_X = X \times E$ and $B_Y = Y \times F$. The continuous section spaces are $C_0(X, E)$ and $C_0(Y, F)$, the Banach spaces of continuous vector-valued functions vanishing at infinity, respectively. If $E$ and $F$ are strictly convex, by a result of Jerison [9] we know that every linear isometry $T$ from $C_0(X, E)$ onto $C_0(Y, F)$ is of the form:

$$Tf(y) = h_y(f(\varphi(y))), \quad \forall f \in C_0(X, E), \quad y \in Y.$$  

Here, $\varphi$ is a homeomorphism from $Y$ onto $X$, and $h_y$ is a linear isometry from $E$ onto $F$ for all $y$ in $Y$. Moreover, the map $y \mapsto h_y$ is SOT continuous on $Y$. In the case both the Banach dual spaces $E^*$ and $F^*$ are strictly convex, Lau gets the same representation (1.2) in [10]. It is further extended that the same conclusion holds whenever $E$ and $F$ are non-square in [8] or the centralizers of $E$ and $F$ are one dimensional in [1]. The representation (1.2) induces an isometric bundle isomorphism $\Phi : B_X \rightarrow B_Y$ from $B_X = X \times E$ onto $B_Y = Y \times F$ defined by

$$\Phi(x, e) = (\varphi^{-1}(x), h_{\varphi^{-1}(x)}(e)), \quad \forall (x, e) \in X \times E.$$  

Hence, the trivial bundles $\langle X \times E, \pi_X \rangle$ and $\langle Y \times F, \pi_Y \rangle$ are isometrically isomorphic if and only if they have isometrically isomorphic continuous section spaces. We note
that if $E$ or $F$ is not non-square, the above assertion (1.2) can be false as shown in Example 3.4.

In this paper, we discuss the general Banach bundle case. Motivated by Example 1.2, we call a Banach bundle $\langle B_X, \pi_X \rangle$ non-square (resp. strictly convex) if every fiber Banach space $B_x = \pi_X^{-1}(x)$ is non-square (resp. strictly convex). The proof of the following theorem will be given in Section 3.

**Theorem 1.3.** Two non-square Banach bundles $\langle B_X, \pi_X \rangle$ and $\langle B_Y, \pi_Y \rangle$ are isometrically isomorphic as Banach bundles if and only if their continuous section spaces $\Gamma_X$ and $\Gamma_Y$ are isometrically isomorphic as Banach spaces.

We also consider the case when the continuous section space $\Gamma_X$ is embedded into $\Gamma_Y$ as a Banach subspace. We want to see whether $\langle B_X, \pi_X \rangle$ embedded into $\langle B_Y, \pi_Y \rangle$ as a subbundle. Assume $F$ is strictly convex. It is shown in [2, 5, 7] that every linear isometry from $C_0(X, E)$ into $C_0(Y, F)$ induces a continuous function $\varphi$ from a nonempty subset $Y_1$ of $Y$ onto $X$ and a field $y \mapsto h_y$ of norm one linear operators from $E$ into $F$ on $Y_1$, such that

\begin{align}
Tf(y) &= h_y(f(\varphi(y))), \quad \forall f \in C_0(X, E), \; y \in Y_1,
\end{align}

and

\begin{align}
\|Tf|_{Y_1}\| &= \sup\{\|Tf(y)\| : y \in Y_1\} = \|f\|, \quad \forall f \in C_0(X, E).
\end{align}

When $F$ is not strictly convex, the conclusion does not hold (see [7]).

In Theorem 3.1, we extend the above representation (1.3) and (1.4) to the general strictly convex Banach bundle case. Supposing all $h_y$ are isometries, we can consider $\langle B_X, \pi_X \rangle$ to be embedded into $\langle B_Y, \pi_Y \rangle$ as a subbundle. However, in Example 3.2 we have a linear into isometry between trivial bundles with all fiber maps $h_y$ not being isometric.

2. Preliminaries

Let $K = \mathbb{R}$ or $\mathbb{C}$ be the underlying field. Let $X$ be a locally compact Hausdorff space. A Banach bundle (see, e.g. [4]) over $X$ is a pair $\langle B_X, \pi_X \rangle$, where $B_X$ is a topological space and $\pi_X$ is a continuous open surjective map from $B_X$ onto $X$, such that, for all $x$ in $X$, each fiber $B_x = \pi_X^{-1}(x)$ carries a Banach space structure in the subspace topology and satisfying the following conditions:

1. Scalar multiplication, addition and the norm on $B_X$ are all continuous whenever they are defined.
2. If $x \in X$ and $\{b_i\}$ is any net in $B_X$ such that $\|b_i\| \to 0$ and $\pi(b_i) \to x$ in $X$, then $b_i \to 0_x$ (the zero element of $B_x$) in $B_X$. 

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The condition (2) above ensures that the zero section is in $\Gamma_X$.

**Definition 2.1.** ([4], p. 128). A Banach bundle $\langle B_X, \pi_X \rangle$ is said to be \textit{isometrically isomorphic} to a Banach bundle $\langle B_Y, \pi_Y \rangle$ if there are homeomorphisms $\Phi : B_X \to B_Y$ and $\psi : X \to Y$ such that

(a) $\pi_Y \circ \Phi = \psi \circ \pi_X$, i.e., $\Phi(B_x) = B_{\psi(x)}$, $\forall x \in X$;

(b) $\Phi|_{B_x}$ is a linear map from $B_x$ onto $B_{\psi(x)}$, $\forall x \in X$;

(c) $\Phi$ preserves norm, i.e., $\|\Phi(b)\| = \|b\|$, $\forall b \in B_X$.

Clearly, all the fiber linear maps $\Phi|_{B_x}$ are surjective isometries. In fact, an isometrical bundle isomorphism $\langle \Phi, \psi \rangle : \langle B_X, \pi_X \rangle \to \langle B_Y, \pi_Y \rangle$ induces a linear isometry $T$ from $\Gamma_X$ onto $\Gamma_Y$ defined by setting $\varphi = \psi^{-1} : Y \to X$, $h_y = \Phi|_{B_{\varphi(y)}} : B_{\varphi(y)} \to B_y$, and

\[(2.1) \quad Tf(y) = \Phi(f(\varphi(y))) = h_y(f(\varphi(y))), \quad \forall f \in \Gamma_X, \; y \in Y.\]

In other words, isometrically isomorphic Banach bundles have isometrically isomorphic continuous section spaces. We want to establish the converse of this observation.

In general, let $\varphi : Y \to X$ be a continuous map, and let $y \mapsto h_y$ be a field of fiber linear maps $h_y : B_{\varphi(y)} \to B_y$, $\forall y \in Y$. We can define a linear map $T$ sending vector sections $f$ in $\langle B_X, \pi_X \rangle$ to vector sections $Tf$ in $\langle B_Y, \pi_Y \rangle$ by setting $Tf(y) = h_y(f(\varphi(y)))$, $\forall y \in Y$. The field $y \mapsto h_y$ is said to be \textit{continuous} if $y_\lambda \to y$ implies $h_{y_\lambda}(f(\varphi(y_\lambda))) \to h_y(f(\varphi(y)))$, and \textit{uniformly bounded} if $\sup_{y \in Y} \|h_y\| < +\infty$. When $B_X = X \times E$ and $B_Y = Y \times F$, the continuity of a field $y \mapsto h_y$ of fiber linear maps reduces to the usual SOT continuity. In general, assuming $\varphi$ is proper, i.e., $\lim_{y \to \infty} \varphi(y) = \infty$, if the field $y \mapsto h_y$ is uniformly bounded and continuous on $Y$, then $T(\Gamma_X) \subseteq \Gamma_Y$. Conversely, we will see in Theorem 3.1 that every linear into isometry $T : \Gamma_X \to \Gamma_Y$ defines a continuous field $y \mapsto h_y$ of fiber linear maps with all $\|h_y\| = 1$, provided that $\langle B_Y, \pi_Y \rangle$ is strictly convex.

In terms of Banach bundles, Example 1.1 says that trivial line bundles are completely determined by the geometric structure of its continuous sections. It is also the case for trivial Banach bundles $X \times E$ and $Y \times F$ whenever $E$ and $F$ are non-square, as demonstrated in Example 1.2. In attacking the general Banach bundle case, we need the following result of Fell [4].

**Proposition 2.2.** ([4], p. 129). Let $\{s_i\}$ ($i \in I$) be a net of elements of $B_X$ and $s$ an element of $B_X$ such that $\pi_X(s_i) \to \pi_X(s)$. Suppose further that for each $\epsilon > 0$ we can find a net $\{u_i\}$ of elements of $B_X$ (indexed by the same $I$) and an element $u$ of $B_X$ such that: (1) $u_i \to u$ in $B_X$, (2) $\pi_X(u_i) = \pi_X(s_i)$ for each $i$, (3) $\|s - u\| < \epsilon$, and (4) $\|s_i - u_i\| < \epsilon$ for all large enough $i$. Then $s_i \to s$ in $B_X$. 

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3. The Results

First, we discuss the into isometry case. We shall write $E^*$ and $S_E$ for the Banach dual space and the unit sphere of a Banach space $E$, respectively.

**Theorem 3.1.** Suppose $\langle B_X, \pi_X \rangle$ and $\langle B_Y, \pi_Y \rangle$ are Banach bundles such that $\langle B_Y, \pi_Y \rangle$ is strictly convex. Let $T : \Gamma_X \to \Gamma_Y$ be a linear into isometry. Then there exist a continuous map $\varphi$ from a nonempty subset $Y_1$ of $Y$ onto $X$, and a field of norm one linear operators $h_y : B_{\varphi(y)} \to B_y$, for all $y$ in $Y_1$, such that

$$Tf(y) = h_y(f(\varphi(y))), \quad \forall \ f \in \Gamma_X, \ y \in Y_1,$$

and

$$\|Tf\|_{Y_1} = \|Tf\|, \quad \forall \ f \in \Gamma_X.$$

**Proof.** We employ the notations developed in [7, 8]. For $x$ in $X$, $y$ in $Y$, $\mu$ in $S_{B_x^*}$ and $\nu$ in $S_{B_y^*}$, we set

$$S_{x, \mu} = \{f \in \Gamma_X : \mu(f(x)) = \|f\| = 1\},$$

$$R_{g, \nu} = \{g \in \Gamma_Y : \nu(g(y)) = \|g\| = 1\},$$

$$Q_{x, \mu} = \{y \in Y : T(S_{x, \mu}) \subseteq R_{g, \nu} \text{ for some } \nu \text{ in } S_{B_y^*}\},$$

and

$$Q_x = \bigcup_{\mu \in S_{B_x^*}} Q_{x, \mu}.$$

As in [8], it is not difficult to see that

(a) For all $x$ in $X$, the set $S_{x, \mu} \neq \emptyset$ for some $\mu$ in $S_{B_x^*}$;

(b) If $S_{x, \mu} \neq \emptyset$, then so is $Q_{x, \mu}$.

By the strict convexity of $\langle B_Y, \pi_Y \rangle$, we have

(c) $Q_{x_1} \cap Q_{x_2} = \emptyset$ for all $x_1 \neq x_2$. Set

$$Y_1 = \bigcup_{x \in X} Q_x = \bigcup_{x \in X} \bigcup_{\mu \in S_{B_x^*}} Q_{x, \mu}.$$

From (c), we can define a map $\varphi$ from $Y_1$ onto $X$ by

$$\varphi(y) = x \quad \text{if } y \in Q_x.$$

Using the strict convexity of $\langle B_Y, \pi_Y \rangle$ again, we also have

(d) $f(\varphi(y)) = 0$ implies $Tf(y) = 0$, i.e. $\ker \delta_{\varphi(y)} \subseteq \ker(\delta_y \circ T)$. 
Then there exists a linear operator $h_y : B_{\varphi(y)} \to B_y$ such that
\[ \delta_y \circ T = h_y \circ \delta_{\varphi(y)}, \quad \forall \ y \in Y_1. \]

In other words,
\[ Tf(y) = h_y(f(\varphi(y))), \quad \forall \ f \in \Gamma_X, \ y \in Y_1. \]

For all $b$ in $B_{\varphi(y)}$, choose an element $f$ in $\Gamma_X$ such that $f(\varphi(y)) = b$ and $\|f\| = \|b\|$. It follows
\[ \|h_y(b)\| = \|h_y(f(\varphi(y)))\| = \|Tf(y)\| \leq \|Tf\| = \|f\| = \|b\|. \]

Since $y \in Y_1$, there exist $x$ in $X$, $\mu$ in $S_{B_y^*}$ and $\nu$ in $S_{B_y^*}$ such that
\[ \nu(Tf(y)) = \mu(f(x)) = 1, \quad \forall \ f \in S_{x,\mu}, \]
and hence
\[ \|h_y(f(x))\| = \|Tf(y)\| = 1. \]

This shows that $\|h_y\| = 1$.

For all $f$ in $\Gamma_X$ with norm one, $f \in S_{x,\mu}$ for some $x$ and $\mu$. As a result, $Tf \in R_{y,\nu}$ for some $y$ in $Y_1$ and $\nu$ in $S_{B_y^*}$. Thus,
\[ \nu(Tf(y)) = \mu(f(x)) = \|f\| = 1. \]

Therefore, $\|Tf\|_{Y_1} = 1 = \|f\| = \|Tf\|$.

It remains to show that the map $\varphi$ is continuous. Let $y_\lambda$ be a net converging to $y$ in $Y_1$. If $\varphi(y_\lambda)$ does not converge to $\varphi(y)$, then by passing to a subnet if necessary, we can assume it converges to an $x \neq \varphi(y)$ in $X_\infty$. Let $U_1$ and $U_2$ be disjoint neighborhoods of $x$ and $\varphi(y)$ in $X_\infty$, respectively. Let $f$ be an element of $S_{\varphi(y),\mu}$ supporting in $U_2$. Then $f(\varphi(y_\lambda)) = 0$ for large $\lambda$. By (d), $Tf(y_\lambda) = 0$ for large $\lambda$. The definition of $\varphi$ implies that there exists a $\nu$ in $S_{B_y^*}$ such that $\nu(Tf(y)) = \mu(f(\varphi(y))) = \|f\| = 1$. Hence, $\|Tf(y)\| = 1$, contradicting to the fact $Tf(y_\lambda) = 0$ for large $\lambda$.

**Example 3.2.** For each $\theta$ in $[0, 2\pi]$, let $P_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the orthogonal projection onto the one-dimensional subspace of $\mathbb{R}^2$ spanned by the unit vector $(\cos \theta, \sin \theta)$. Every element $f$ in $C(\{0\}, \mathbb{R}^2)$ is given by the vector $f(0) = (r \cos t, r \sin t)$ for some $r \geq 0$ and $t \in [0, 2\pi]$. Define a linear isometry $T : C(\{0\}, \mathbb{R}^2) \to C([0, 2\pi], \mathbb{R}^2)$ by
\[ T(f)(\theta) = P_\theta(f(0)) = P_\theta(r \cos t, r \sin t) = (r \cos(t - \theta) \cos \theta, r \cos(t - \theta) \sin \theta). \]
In the notations of Theorem 3.1, \( h_\theta = P_\theta, \ Y_\theta = [0, 2\pi], \) and
\[
T f(\theta) = h_\theta(f(0)), \quad \forall \ f \in C(\{0\}, \mathbb{R}^2), \ \theta \in [0, 2\pi].
\]
Note that \( h_\theta = P_\theta \) is not an isometry for every \( \theta \) in \( Y_\theta = [0, 2\pi] \).

Here comes the proof of our main result.

**Proof of Theorem 1.3.** Let \( \langle B_X, \pi_X \rangle \) and \( \langle B_Y, \pi_Y \rangle \) be two non-square Banach bundles and \( T \) a linear isometry from \( \Gamma_X \) onto \( \Gamma_Y \). Denote by
\[
K_X = \bigcup_{x \in X} (\{x\} \times U_{B_x^*}) \quad \text{and} \quad K_Y = \bigcup_{y \in Y} (\{y\} \times U_{B_y^*}),
\]
the disjoint unions of the compact sets \( \{x\} \times U_{B_x^*} \) and \( \{y\} \times U_{B_y^*} \), respectively.

Note that both the Hausdorff spaces \( K_X \) and \( K_Y \) are locally compact. Define a linear isometry \( \Psi : \Gamma_Y \to C_0(K_Y) \) by
\[
\Psi(g)(y, \nu) = \nu(g(y)), \quad \forall \ g \in \Gamma_Y, (y, \nu) \in K_Y.
\]

Then \( \tilde{T} = \Psi \circ T \) is a linear isometry from \( \Gamma_X \) into \( C_0(K_Y) \). By Theorem 3.1, there exist a continuous map \( \tilde{\varphi} \) from a nonempty subset \( A_Y \) of \( K_Y \) onto \( X \) and bounded linear functionals \( \tilde{h}_{(y, \nu)} \in B_{\tilde{\varphi}(y, \nu)}^* \) such that
\[
(3.1) \quad \tilde{T} f(y, \nu) = \nu(T f(y)) = \tilde{h}_{(y, \nu)}(f(\tilde{\varphi}(y, \nu))), \quad \forall \ f \in \Gamma_X, (y, \nu) \in A_Y.
\]

Applying the same argument to \( T^{-1} \), there exist a continuous map \( \tilde{\psi} \) from a subset \( A_X \) of \( K_X \) onto \( Y \) and bounded linear functionals \( \tilde{k}_{(x, \mu)} \in B_{\tilde{\psi}(x, \mu)}^* \) such that
\[
\mu(T^{-1}g(x)) = \tilde{k}_{(x, \mu)}(g(\tilde{\psi}(x, \mu))), \quad \forall \ g \in \Gamma_Y, (x, \mu) \in A_X.
\]

Let
\[
C_y = \{ \nu \in S_{B_y^*} : (y, \nu) \in A_Y \},
\]
\[
X_I = \{ x \in X : \text{there exists a } \mu \text{ in } S_{B_x^*} \text{ such that } (x, \mu) \in A_X \},
\]
and
\[
Y_I = \{ y \in Y : \text{there exists a } \nu \text{ in } S_{B_y^*} \text{ such that } (y, \nu) \in A_Y \}.
\]

We make the following easy observations:

(I) \( X_I = X \) and \( Y_I = Y \);

(II) \( C_y \) is total in \( B_{y^*} \), for all \( y \) in \( Y \).
By modifying the arguments in [8], it is not difficult to show that if \((B_Y, \pi_Y)\) is non-square, \(\tilde{\varphi}(y, \nu_1) = \tilde{\varphi}(y, \nu_2)\) for all \(\nu_1, \nu_2 \in C_y\) and for all \(y \in Y\). Consequently, we can define a continuous map \(\varphi : Y \to X\) by
\[
\varphi(y) = \tilde{\varphi}(y, \nu), \text{ for some } \nu \in C_y.
\]
In view of (3.1) and (II), we have \(f(\varphi(y)) = 0\) implies \(Tf(y) = 0\). Then there exists a bounded linear operator \(h_y : B_{\varphi(y)} \to B_y\) such that
\[
(3.2) \quad Tf(y) = h_y(f(\varphi(y))), \; \forall f \in \Gamma_X, \; y \in Y.
\]
By symmetry, \(T^{-1}\) also carries a form
\[
T^{-1}g(x) = k_x(g(\psi(x))), \; \forall g \in \Gamma_Y, \; x \in X,
\]
for some continuous map \(\psi\) from \(X\) onto \(Y\), and bounded linear operators \(k_x\) from \(B_{\psi(x)}\) into \(B_x\), for all \(x \in X\). Consequently,
\[
f(x) = (T^{-1}(Tf))(x) = k_x(Tf(\psi(x))) = k_xh_{\psi(x)}(f(\varphi(\psi(x)))).
\]
This implies that \(\varphi\) is a homeomorphism with inverse \(\psi\), and \(h_y\) are bijective linear isometries with inverses \(k_{\varphi(y)}\) for all \(y \in Y\).

Let \(\Phi = (h_y^{-1})_{y \in Y}\), i.e. \(\Phi|_{B_y} = h_y^{-1}\). Then it defines a map from \(B_Y\) onto \(B_X\) as follows: for all \(b \in B_Y\) and \(\pi_Y(b) = y_0\). Choose a continuous section \(g\) in \(\Gamma_Y\) such that \(g(y_0) = b\). Then
\[
\Phi(b) = h_y^{-1}(g(y_0)) = T^{-1}(g)(\varphi(y_0)).
\]
We show that \(\Phi\) is a homeomorphism from \(B_Y\) to \(B_X\). By symmetry, it suffices to prove that \(\Phi\) is continuous. We shall make use of Proposition 2.2 in below.

Let \(b_i \to b\) in \(B_Y\). We show that \(\Phi(b_i) \to \Phi(b)\) in \(B_X\). Since \(\pi_Y\) and \(\varphi\) are continuous, we have \(\pi_Y(b_i) \to \pi_Y(b)\) and \(\varphi(\pi_Y(b_i)) \to \varphi(\pi_Y(b))\). Let \(s_i = \Phi(b_i)\) and \(s = \Phi(b)\). Choose a continuous section \(g\) in \(\Gamma_Y\) such that \(g(\pi_Y(b_i)) = b_i\). Then, for all \(\epsilon > 0\), we have \(\|g(\pi_Y(b_i)) - b_i\| < \epsilon\) for all large enough \(i\). The fact \(\pi_X \circ \Phi = \varphi \circ \pi_Y\) (this follows from (3.2)) implies that \(\pi_X(s_i) = \pi_X(\Phi(b_i)) = \varphi(\pi_Y(b_i))\) approaches \(\varphi(\pi_Y(b)) = \pi_X(\Phi(b)) = \pi_X(s)\). Let \(u_i = \Phi(g(\pi_Y(b_i)))\) and \(u = \Phi(g(\pi_Y(b)))\). Since \(\Phi|_{B_x}\) is an isometry, we have \(\|u_i - s_i\| = \|g(\pi_Y(b_i)) - b_i\| < \epsilon\), for all large enough \(i\). And
\[
u_i = \Phi(g(\pi_Y(b_i))) = h_{\pi_Y(b_i)}^{-1}(g(\pi_Y(b_i))) = f(\varphi(\pi_Y(b_i))),
\]
for some \(f\) in \(\Gamma_X\), which converges to
\[
f(\varphi(\pi_Y(b))) = h_{\pi_Y(b)}^{-1}(g(\pi_Y(b))) = \Phi(b) = u
\]
in \(B_X\). By Proposition 2.2, we have \(\Phi(b_i) = s_i \to s = \Phi(b)\) in \(B_X\). This shows that \(\Phi\) is continuous and complete the proof of Theorem 1.3.
**Corollary 3.3.** Assume \( \langle B_X, \pi_X \rangle \) and \( \langle B_Y, \pi_Y \rangle \) are two non-square Banach bundles over locally compact Hausdorff spaces with isometrically isometric continuous sections. If \( \langle B_X, \pi_X \rangle \) is locally trivial, then so is \( \langle B_Y, \pi_Y \rangle \).

The following example shows that the conclusion in Theorem 1.3 might not hold if \( \langle B_X, \pi_X \rangle \) or \( \langle B_Y, \pi_Y \rangle \) is not non-square.

**Example 3.4.** Let \( \pi_i \) be the \( i \)-th coordinate map of \( \mathbb{R} \oplus \mathbb{R} \), \( i = 1, 2 \). Each element \( f \) in \( C(\{0\}, \mathbb{R} \oplus \mathbb{R}) \) is given by the vector \( f(0) \) in \( \mathbb{R} \oplus \mathbb{R} \). Define a linear map \( T : C(\{0\}, \mathbb{R} \oplus \mathbb{R}) \rightarrow C(\{1, 2\}, \mathbb{R}) \) by

\[
Tf(i) = \pi_i(f(0)), \quad \forall f \in C(\{0\}, \mathbb{R} \oplus \mathbb{R}), \ i = 1, 2.
\]

It is easy to see that \( T \) is an isometrical isomorphism, but \( \mathbb{R} \oplus \mathbb{R} \) and \( \mathbb{R} \) are not isomorphic as Banach spaces. In particular, \( \mathbb{R} \oplus \mathbb{R} \) is not isometrically isomorphic to \( \{1, 2\} \times \mathbb{R} \) as Banach bundles, although they have isometrically isomorphic continuous section spaces.

**REFERENCES**


Ming-Hsiu Hsu and Ngai-Ching Wong
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, 80424, Taiwan
E-mail: hsumh@math.nsysu.edu.tw
wong@math.nsysu.edu.tw