Degree theory for generalized variational inequalities and applications

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Abstract

In this paper, a degree theory for finite dimensional generalized variational inequalities is built and employed to prove some results on solution existence and solution stability.

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1. Introduction

Many problems in analysis and in the application of analysis can be reduced to a study of the solution set of an equation \( \phi(x) = p \) in an appropriate space. Degree theory has developed as means of examining the solution existence and their number of the solution.

Suppose that \( D \) is a open bounded set in \( \mathbb{R}^n \) with the closure \( \overline{D} \) and the boundary \( \partial D \). Let \( \phi : \overline{D} \to \mathbb{R}^n \) be an continuous map and \( p \in \mathbb{R}^n \) such that \( p \notin \phi(\partial D) \). The aim of degree theory is to define an integer \( d(\phi,D,p) \), the degree of \( \phi \) at \( p \) respect to \( D \) (see [7,10,20] for the definition) with the properties that \( d(\phi,D,p) \) is an estimate of the number of solution of \( \phi(x) = p \) in \( D \), \( d \) is continuous in \( \phi \) and \( p \) and \( d \) is additive in the domain \( D \). The following list summarizes some properties most frequently used (see, for instance [7,10,18,26]).

Theorem 1.1. Suppose that \( p \notin \phi(\partial D) \). Then the following properties hold:

1. (Normalization) If \( p \in D \) then \( d(I,D,p) = 1 \), where \( I \) is the identity mapping.
2. (Existence) If \( d(\phi,D,p) \neq 0 \) then there is \( x \in D \) such that \( \phi(x) = p \).
3. (Additivity) Suppose that \( D_1 \) and \( D_2 \) are disjoint open sets of \( D \). If \( p \notin \phi(D \setminus (D_1 \cup D_2)) \) then
   \[ d(D,f,p) = d(\phi,D_1,p) + d(\phi,D,p) \]
4. (Homotopy invariance) Suppose that \( H : [0,1] \times D \to \mathbb{R}^n \) is continuous. If \( p \notin H(t,\partial D) \) for all \( t \in [0,1] \) then
   \[ d(H(t,\cdot),D,p) \text{ is independent of } t \]
5. (Excision) If \( D_0 \) is a closed set of \( D \) and \( p \notin \phi(D_0) \) then
   \[ d(\phi,D,p) = d(\phi,D \setminus D_0,p) \]

Recently, in the two-volume book [9] dedicated entirely to finite dimensional variational inequalities (VI, for brevity), Facchinei and Pang have used degree theory to obtain existence theorems for variational inequalities (see [9, Proposition 2.2.3 and Theorem 2.3.4]). These results gave a necessary and sufficient condition for a pseudomonotone VI on a general closed convex set to have a solution. In particular, Pang [19] used degree theory to obtain interesting results on sensitivity of a parametric nonsmooth equation with multivalued perturbed solution sets. This paper has been very influential for the optimization community. Also, based on degree theory, Robinson [22] provided a strong
conclusion on the solution stability of variational conditions; Gowda [11] proved inverse and implicit function theorems for H-differentiable functions, thereby giving a unified treatment of such theorems for $C^k$-functions and for locally Lipschitzian function. In order to obtain these results, the authors have used degree theory as a bridge to marry nonlinear analysis and variational inequality theory under which we can study problems via nonlinear equations.

Let us assume that $\mathbb{R}^n$ is a finite dimensional space with the Euclidian norm and $K$ is a closed convex set in $\mathbb{R}^n$. Let $f : K \to \mathbb{R}^n$ be a continuous mapping. The variational inequality defined by $K$ and $f$ denoted by VI$(f,K)$, is the problem of finding a vector $x \in K$ such that it satisfies the inclusion

$$0 \in f(x) + N_K(x),$$

(1.1)

where $N_K(x)$ is the normal cone of $K$ at $x$ defined by the formula

$$N_K(x) = \begin{cases} \{ x^* \in \mathbb{R}^n : \langle x^*, y-x \rangle \leq 0 \ \forall y \in K \} & \text{if } x \in K, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We denote by $\Pi_K(x)$ the metric projection of $x$ onto $K$ and put

$$\Phi(x) = x - \Pi_K(x - f(x)).$$

(1.2)

$\Phi$ is called the natural map. It is clear that $x$ is a solution of (1.1) if and only if $x$ is a solution of the equation $\Phi(x) = 0$. Let $\Omega$ be an open bounded set in $\mathbb{R}^n$ such that $\Omega \cap K \neq \emptyset$. We wish to investigate the number of solutions of (1.1) in $\Omega$. Since (1.1) is equivalent to the equation $\Phi(x) = 0$, it suggests us to compute the degree $d(\Phi, \Omega, 0)$. By this way, as it mentioned above, [9,22] obtained interesting results on solution existence and solution stability of VIs.

It is natural to try to study generalized variational inequalities (GVI, for short) which is also known in the literature as set-valued variational inequalities in this direction. Namely, we consider the problem of finding $x \in K$ such that

$$0 \in F(x) + N_K(x),$$

(1.3)

where $F : K \to 2^\mathbb{R}^n$ is a multifunction. We consider the so-called generalized natural map which defined by

$$\Phi_F(x) = x - \Pi_K(x - F(x)).$$

(1.4)

In this case we will meet some difficulties for applications of degree theory to our problem. Namely, we can not apply degree theory to $\Phi_F$ directly because $\Phi_F$ has no convex values and so the degree of $\Phi_F$ is undefined generally.

The aim of the present paper is to build a degree theory for multifunctions in the infinite dimensional setting so far (see [3–5,12,13]). We emphasize that degree theory for GVIs in the present paper is somewhat different from degree theory for upper semicontinuous multifunctions with convex and compact values. It is built via the map $\Phi_F$ which does not necessarily have convex values.

The rest of the paper contains two sections. In Section 2 we build a degree theory for GVIs. Section 3 is devote to applications of obtained results. In this section we shall prove some facts on the solution existence and solution stability of GVIs.

2. Degree theory for GVIs

Throughout the paper, $K$ is a closed convex set in $\mathbb{R}^n$, $\Omega$ is an open bounded set in $\mathbb{R}^n$ such that $\Omega \cap K \neq \emptyset$. Let $F : K \to 2^\mathbb{R}^n$ be a multifunction which is upper semicontinuous with compact convex values.

Recall that a map $F : K \to 2^\mathbb{R}^n$ is upper semicontinuous (u.s.c., for brevity) if for all $x \in K$ and for any open set $W \subset \mathbb{R}^n$ satisfying $F(x) \subset W$ there exists an open neighborhood $U$ of $x$ such that $F(y) \subset W$ for all $y \in U \cap K$. If $F(x) \neq \emptyset$ for all $x \in K$ and for any open set $W \subset \mathbb{R}^n$ satisfying $F(x) \cap W \neq \emptyset$, there exists an open neighborhood $U$ of $x$ such that $F(y) \cap W \neq \emptyset$ for all $y \in U \cap K$ then $F$ is said to be lower semicontinuous (l.s.c., for brevity).

The following lemma plays an essential role for building a degree theory of GVIs.

**Lemma 2.1.** Suppose that $F : K \to 2^\mathbb{R}^n$ is u.s.c. with closed convex values. Then for any $\epsilon > 0$ there exists a continuous map $f_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ such that for every $x \in K$ it holds

$$f_\epsilon(x) \in F((x + \epsilon B) \cap K) + \epsilon B,$$

(2.1)

where $B$ is the unit ball in $\mathbb{R}^n$.

**Proof.** By our assumptions and the approximate selection theorem due to Cellina (see [1, p. 84]), for every $\epsilon > 0$ there exists a continuous map $g_\epsilon : K \to \mathbb{R}^n$ such that

$$g_\epsilon(x) \in F((x + \epsilon B) \cap K) + \epsilon B \ \forall x \in K.$$

By Tietze-Urysohn’s theorem (see [8, Theorem 5.1, p. 149]), for each $\epsilon > 0$, there exists a continuous extension $f_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ of $g_\epsilon$. As $f_\epsilon$ and $g_\epsilon$ agree on $K$, $f_\epsilon$ satisfies the conclusion of the theorem. The proof is complete. □

We now consider GVI$(F,K)$. For each $\epsilon > 0$ we define a map $\Phi_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ by the formula

$$\Phi_\epsilon(x) = x - \Pi_K(x - f_\epsilon(x)),$$

(2.2)

where $f_\epsilon$ is a approximate continuous selection of $F$ which satisfies (2.1). By the continuity of the metric projection and Lemma 2.1, $\Phi_\epsilon$ is continuous on $\mathbb{R}^n$ and hence on $\overline{K}$.

We have the following lemma on properties of $\Phi_\epsilon$.

**Lemma 2.2.** Suppose that $F$ is u.s.c. with compact convex values and $0 \notin (F + N_K)(\partial \Omega)$. Then the following assertions hold:

(a) there exists $\epsilon_1 > 0$ such that $0 \notin \Phi_\epsilon(\partial \Omega)$ for all $\epsilon \in (0, \epsilon_1]$

(b) there exists $\epsilon_2 > 0$ such that
By standard arguments as in the proof of (a) we get
\[ d(\Phi_\epsilon, \Omega, 0) = d(\Phi_{\epsilon'}, \Omega, 0) \quad \text{for all } \epsilon, \epsilon' \in (0, \epsilon_2]. \] (2.3)

**Proof.** (a) Suppose the assertion is false. Then there exists a sequence \( \epsilon_k \to 0^+ \) and a sequence \( x_k \in \partial \Omega \) such that \( \Phi_{\epsilon_k}(x_k) = 0 \). This means that
\[ \|y_k - x_k\| < \epsilon_k; \quad \|z_k - f_{\epsilon_k}(x_k)\| < \epsilon_k. \] (2.4)

By compactness of \( \partial \Omega \) we can assume that \( x_k \to x_0 \in \partial \Omega \).

Since \( x_k \in K \cap \partial \Omega \), by Lemma 2.1, there exist \( y_k \in K \) and \( z_k \in F(x_k) \) such that
\[ \|y_k - x_k\| < \epsilon_k; \quad \|z_k - f_{\epsilon_k}(x_k)\| < \epsilon_k. \]

Hence, \( y_k \to x_0 \). As \( F(x_0) \) is a compact set and \( F \) is upper semicontinuous at \( x_0 \), by taking a subsequence (if necessary) we can suppose furthermore that \( z_k \to z_0 \in F(x_0) \).

Hence, \( f_{\epsilon_k}(x_k) \to z_0 \). Letting \( k \to \infty \), from (2.4) we obtain
\[ x_0 = \Pi_K(x_0 - z_0) \quad \text{with } z_0 \in F(x_0). \]
By the property of the metric projection we have
\[ 0 \in z_0 + N_K(x_0) \subseteq F(x_0) + N_K(x_0), \]
which contradicts our assumptions. We obtain the proof part (a).

(b) On the contrary, suppose there exist sequences
\[ 0 < \epsilon_k < \epsilon_k' \to 0 \quad \text{such that} \]
\[ d(\Phi_{\epsilon_k'}, \Omega, 0) \neq d(\Phi_{\epsilon_k}, \Omega, 0). \] (2.5)

Put
\[ H(t, x) = x - \Pi_K(x - tf_{\epsilon_k}(x)) - (1 - t)f_{\epsilon_k}(x), \quad (t, x) \in [0, 1] \times \partial \Omega. \]

We have \( H(0, x) = \Phi_{\epsilon_k}(x) \) and \( H(1, x) = \Phi_{\epsilon_k'}(x) \). If
\[ 0 \notin H(t, \partial \Omega) \quad \text{for all } t \in [0, 1] \quad \text{then} \]
\[ d(H(0, \cdot), \Omega, 0) = d(H(1, \cdot), \Omega, 0), \]

because of (4) in Theorem 1.1. But the latter contradicts (2.6).

Hence, for each \( k \), there exists \( t_k \in [0, 1] \) such that
\[ 0 \in H(t_k, \partial \Omega). \]
This implies that, for each \( k \) there exists \( x_k \in \partial \Omega \) such that
\[ x_k = \Pi_K(x_k - t_k f_{\epsilon_k}(x_k) - (1 - t_k) f_{\epsilon_k'}(x_k)). \] (2.6)

Since \( x_k \in K \cap \partial \Omega \), by Lemma 2.1, there exist \( y_k, y_k' \in K \); \( z_k \in F(x_k) \) and \( z_k' \in F(y_k') \) such that
\[ \|y_k - x_k\| < \epsilon_k'; \quad \|z_k - f_{\epsilon_k}(x_k)\| < \epsilon_k'; \]
and
\[ \|y_k' - x_k\| < \epsilon_k'; \quad \|z_k' - f_{\epsilon_k}(x_k)\| < \epsilon_k'. \]

By compactness of \([0, 1] \times \partial \Omega\) we can assume that
\[ (t_k, x_k) \to (\bar{t}, \bar{x}) \in [0, 1] \times \partial \Omega. \]
Hence, \( y_k \to \bar{y} \) and \( y_k' \to \bar{y}' \).

By standard arguments as in the proof of (a) we get
\[ f_{\epsilon_k}(x_k) \to z_1 \quad \text{and} \quad f_{\epsilon_k'}(x_k) \to z_2 \quad \text{for some } z_1, z_2 \in F(\bar{x}). \]

By letting \( k \to \infty \), from (2.6) we get \( \bar{x} = \Pi_K(\bar{x} - \bar{z}_1 - (1 - \bar{t}) \bar{z}_2) \).

Put \( \bar{z} = \bar{z}_1 + (1 - \bar{t}) \bar{z}_2 \) then \( \bar{z} \in F(\bar{x}) \) and \( \bar{x} = \Pi_K(\bar{x} - \bar{z}) \).

By the property of the metric projection we have
\[ 0 \in \bar{z} + N_K(\bar{x}) \subseteq F(\bar{x}) + N_K(\bar{x}), \]
which is a contradiction. The proof of the lemma is complete. \( \square \)

From Lemma 2.2 it follows that there exists \( \bar{\epsilon} > 0 \) such that
\[ 0 \notin \Phi_{\bar{\epsilon}}(\partial \Omega) \quad \text{and} \quad d(\Phi_{\epsilon'}, \Omega, 0) = d(\Phi_{\epsilon}, \Omega, 0) \quad \text{for all } \epsilon, \epsilon' \in (0, \bar{\epsilon}). \]
It is a basis for the following definition.

**Definition 2.1.** Let \( F : K \to 2^{\mathbb{R}^n} \) be an u.s.c. multifunction with compact convex values and \( 0 \notin (F + N_K(\partial \Omega)). \)

The degree of generalized variational inequality defined by \( F \) and \( K \) with respect to \( \Omega \) at \( 0 \) is the common value \( d(\Phi_{\epsilon}, \partial \Omega, 0) \) for \( \epsilon > 0 \) sufficiently small and denoted by \( d(F + N_K(\partial \Omega), 0). \)

**Example 2.1.** Let
\[ F(x) = \begin{cases} 1 & \text{if } x > 0, \\ [-1, 1] & \text{if } x = 0, \\ \{-1\} & \text{if } x < 0. \end{cases} \]

Then \( F(\partial \Omega) = (-1, 2) \). Then \( d(F + N_K(\partial \Omega), 0) = 1. \)

Indeed, for each \( \epsilon > 0 \) we consider the following function
\[ f_{\epsilon}(x) = \begin{cases} 1 & \text{if } x \geq \epsilon, \\ x/\epsilon & \text{if } x \in (-\epsilon, \epsilon), \\ -1 & \text{if } x \leq -\epsilon. \end{cases} \]

We have the following cases:

1. If \( x \in (-\epsilon, \epsilon) \) then \( |f_{\epsilon}(x)| \leq 1 \). So \( F_K(x - f_{\epsilon}(x)) = 0 \).
2. If \( \epsilon \leq x \leq 2 \) then \( F_K(x - f_{\epsilon}(x)) = 0 \).
3. If \( x > 2 \) then \( F_K(x - f_{\epsilon}(x)) = x - 2 \).
4. If \( -2 \leq x \leq -\epsilon \) then \( F_K(x - f_{\epsilon}(x)) = 0 \).
5. If \( x < -2 \) then \( F_K(x - f_{\epsilon}(x)) = -x - 2 \).

From the above we obtain
\[ F_K(x - f_{\epsilon}(x)) = \begin{cases} 0 & \text{if } x \in [-2, 2], \\ x - 2 & \text{if } x > 2, \\ -x - 2 & \text{if } x < -2. \end{cases} \]

Hence,
\[ \Phi_{\epsilon}(x) = \begin{cases} x & \text{if } x \in [-2, 2], \\ 2x + 2 & \text{if } x < -2. \end{cases} \]

Note that \( \partial \Omega = \{-1/2, 2\} \); \( F(-1/2) + N_K(-1/2) = \{1\} \) and \( F(2) + N_K(2) = 0 \). Hence, \( 0 \notin (F + N_K)(\partial \Omega) \).
We now compute \( d(\Phi_{\epsilon}, \partial \Omega, 0) \). As \( \Phi_{\epsilon} \) is differentiable in \( \Omega \) we get
\[ d(\Phi_\epsilon, \Omega, 0) = \sum_{x \in \Phi_\epsilon \setminus \{0\}} \text{sign}\Phi_\epsilon(x) = 1. \]

Since the latter equality is true for all \( \epsilon \in (0, \bar{\epsilon}] \), we obtain \( d(F + N_K, \Omega, 0) = 1 \).

We have the following theorem on existence.

**Theorem 2.1.** Suppose that \( 0 \notin (F + N_K)(\partial \Omega) \). Then the following assertions hold:

(a) (Existence) If \( d(F + N_K, \Omega, 0) \neq 0 \) then there exists \( x \in \Omega \cap K \) such that
\[ 0 \in F(x) + N_K(x). \]

(b) If \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a continuous map such that \( f(x) \in F(x) \) for all \( x \in K \) then \( d(F + N_K, \Omega, 0) = d(\Phi, \Omega, 0) \), where \( \Phi(x) = x - \Pi_K(x - f(x)). \)

**Proof.** (a) By definition, there exists \( \bar{\epsilon} > 0 \) such that \( d(F + N_K, \Omega, 0) = d(\Phi_\epsilon, \Omega, 0) \) for all \( \epsilon \in (0, \bar{\epsilon}] \). Let \( \{\epsilon_k\} \) be a sequence such that \( \epsilon_k \to 0^+ \). Then \( d(\Phi_{\epsilon_k}, \Omega, 0) \neq 0 \) for \( k \) sufficiently large. By (2) in Theorem 1.1, there exists \( x_k \in \Omega \) such that \( \Phi_{\epsilon_k}(x_k) = 0 \). This is equivalent to
\[ x_k = \Pi_K(x_k - f_{\epsilon_k}(x_k)). \] (2.7)

Since \( x_k \in K \cap \Omega \), by Lemma 2.1, there exists \( y_k \in K \) and \( z_k \in F(y_k) \) such that
\[ \|y_k - x_k\| < \epsilon_k; \|z_k - f_{\epsilon_k}(x_k)\| < \epsilon_k. \]

By compactness of \( K \cap \Omega \) we can assume that \( x_k \to x_0 \in K \cap \Omega \). Hence, \( y_k \to y_0 \). By standard arguments we get \( z_k \to z_0 \) and \( f_{\epsilon_k}(x_k) \to z_0 \) for some \( z_0 \in F(x_0) \). Letting \( k \to \infty \), from (2.7) we obtain \( x_0 = \Pi_K(x_0 - z_0) \). The property of the metric projection yields
\[ 0 \in z_0 + N_K(x_0) \subseteq F(x_0) + N_K(x_0). \]

Since \( 0 \notin (F + N_K)(\partial \Omega) \) we have \( x_0 \in K \cap \Omega \). (b) By putting \( f_\epsilon = f \) for all \( \epsilon > 0 \) we get the desired property. The proof of the theorem is complete. □

**Example 2.2.** Consider Example 2.1 we have \( d(F + N_K, \Omega, 0) = 1 \). By the above theorem, \( GV(\theta, K) \) has a solution \( x \in \Omega \cap K \). In this case, \( x = 0 \) is a solution.

The following theorem contains most usual properties of degree theory.

**Theorem 2.2.** Assume that \( 0 \notin (F + N_K)(\partial \Omega) \). The following assertions hold:

(a) (Homotopy invariance) If \( F_1, F_2 : K \to 2^{\mathbb{R}^n} \) are u.s.c. multifunctions with compact convex values and
\[ 0 \notin (tF_1 + (1-t)F_2)(\partial \Omega) \] for all \( t \in [0, 1] \) then
\[ d(F_1 + N_K, \Omega, 0) = d(F_2 + N_K, \Omega, 0). \]

(b) (Additivity) If \( \Omega_1, \Omega_2 \) are disjoint open subsets of \( \Omega \) such that \( 0 \notin (F + N_K)(\partial \Omega \setminus (\Omega_1 \cup \Omega_2)) \) then
\[ d(F + N_K, \Omega, 0) = d(F + N_K, \Omega_1, 0) + d(F + N_K, \Omega_2, 0). \]

(c) (Excision) If \( D \subseteq \Omega \) is a closed set such that
\[ 0 \notin (F + N_K)(\partial D) \] then
\[ d(F + N_K, \Omega, 0) = d(F + N_K, \Omega \setminus D, 0). \]

**Proof.** (a) Let \( f_\epsilon, g_\epsilon : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) be approximate selections of \( F_1 \) and \( F_2 \), respectively satisfying the conclusion of Lemma 2.1. Put
\[ \Phi_\epsilon(x) = x - \Pi_K(x - tF_1(x) - (1-t)g_\epsilon(x)). \]

We claim that there is \( \bar{\epsilon} > 0 \) such that \( 0 \notin \Phi_\epsilon(\partial \Omega) \) for all \( \epsilon \in (0, \bar{\epsilon}] \) and \( t \in [0, 1] \). In fact, if the claim is false, then there exist a sequence \( t_k \in [0, 1] \) and a sequence \( \epsilon_k \to 0^+ \) such that \( 0 \notin \Phi_{\epsilon_k}(\partial \Omega) \). Hence, for each \( k \), there exists \( x_k \in \partial \Omega \) such that
\[ x_k = \Pi_K(x_k - t_kf_{\epsilon_k}(x_k) - (1-t_k)g_{\epsilon_k}(x_k)). \]

By compactness of \( [0, 1] \times \partial \Omega \) we can assume that \( (t_k, x_k) \to (t_0, x_0) \in [0, 1] \times \partial \Omega \). By standard arguments we can show that \( f_{\epsilon_k}(x_k) \to z_1 \) for some \( z_1 \in F_1(x_0) \) and \( g_{\epsilon_k}(x_k) \to z_2 \) for some \( z_2 \in F_2(x_0) \). Letting \( k \to \infty \) from the above we obtain
\[ x_0 = \Pi_K(x_0 - t_0z_1 - (1-t_0)z_2). \]

By the property of the metric projection we get
\[ 0 \notin t_0z_1 + (1-t_0)z_2 + N_K(x_0) \]
\[ \subset t_0F_1(x_0) + (1-t_0)F_2(x_0) + N_K(x_0). \]

This contradicts the assumption and so our claim is proved. We now can apply (4) of Theorem 1.1 to \( \Phi_\epsilon \) to get
\[ d(\Phi_\epsilon, \Omega, 0) = d(\Phi_\epsilon, \Omega, 0) \] for all \( \epsilon \in (0, \bar{\epsilon}] \). Hence, \( d(F_1 + N_K, \Omega, 0) = d(F_2 + N_K, \Omega, 0) \).

(b) We will show that there exists \( \bar{\epsilon} > 0 \) such that \( 0 \notin \Phi_\epsilon(\partial \Omega \setminus (\Omega_1 \cup \Omega_2)) \) for all \( \epsilon \in (0, \bar{\epsilon}] \). Indeed, if the assertion is false then there exists a sequence \( \epsilon_k \to 0^+ \) and \( x_k \in \partial \Omega \setminus (\Omega_1 \cup \Omega_2) \) such that \( x_k = \Pi_K(x_k - f_{\epsilon_k}(x_k)) \). By compactness of \( \partial \Omega \) we can assume that \( x_k \to x_0 \in \partial \Omega \). If \( x_0 \in \Omega_1 \cup \Omega_2 \) then \( x_k \in \Omega_1 \cup \Omega_2 \) for \( k \) sufficiently large. This contradicts the fact that \( x_k \in \partial \Omega \setminus (\Omega_1 \cup \Omega_2) \). Hence, \( x_0 \in \partial \Omega \setminus (\Omega_1 \cup \Omega_2) \). By standard arguments we have \( f_{\epsilon_k}(x_k) \to z_0 \) for some \( z_0 \in F(x_0) \). Letting \( k \to \infty \) from the above we obtain \( x_0 = \Pi_K(x_0 - z_0) \). This implies that \( 0 \notin F(x_0) + N_K(x_0) \) for some \( x_0 \in \partial \Omega \setminus (\Omega_1 \cup \Omega_2) \), which is a contradiction.

Thus, we have \( 0 \notin \Phi_\epsilon(\partial \Omega \setminus (\Omega_1 \cup \Omega_2)) \) for all \( \epsilon \in (0, \bar{\epsilon}] \). By (3) of Theorem 1.1, we get
\[ d(\Phi_\epsilon, \Omega, 0) = d(\Phi_\epsilon, \Omega_1, 0) + d(\Phi_\epsilon, \Omega_2, 0); \]

It follows that
\[ d(F + N_K, \Omega, 0) = d(F + N_K, \Omega_1, 0) + d(F + N_K, \Omega_2, 0). \]

(c) By standard arguments we show that \( 0 \notin \Phi_\epsilon(D) \) for all \( \epsilon \in (0, \bar{\epsilon}] \). Applying (5) of Theorem 1.1 to \( \Phi_\epsilon \) we obtain the desired conclusion. The proof of the theorem is complete. □
**Theorem 3.1.** Let \( K \subset \mathbb{R}^n \) be a nonempty closed convex set and \( F : K \to 2^{\mathbb{R}^n} \) be an u.s.c. multifunction with nonempty compact convex values. Assume that there exists a vector \( \hat{x} \in K \) such that the set
\[
L_{\hat{x}}(\epsilon) := \{ x \in K : \inf_{x \in F(\hat{x})} \langle x^*, x - \hat{x} \rangle \leq 0 \}
\]
is bounded (possibly empty).

Then \( \text{GVG}(F,K) \) has a solution.

**Proof.** Let \( \Omega \) be an open ball containing \( L_{\hat{x}}(\epsilon) \cup \{ \hat{x} \} \). We must have \( L_{\hat{x}}(\epsilon) \cap \partial \Omega = \emptyset \) and hence,
\[
\inf_{x \in F(\hat{x})} \langle x^*, x - \hat{x} \rangle > 0 \quad \forall x \in K \cap \partial \Omega. \tag{3.1}
\]

If \( 0 \in (F + N_{K_\epsilon})(\partial \Omega) \), then \( \text{GVG}(F,K) \) has a solution. Otherwise, the degree \( d(F + N_{K_\epsilon}, \partial \Omega, 0) \) is well defined.

Hence, there exists \( \epsilon_1 > 0 \) such that \( 0 \notin \Phi_\epsilon(\partial \Omega) \) for all \( \epsilon \in [0, \epsilon_1] \). Recall that \( \Phi_\epsilon(x) = x - \Pi_K(x - f_\epsilon(x)) \), where \( f_\epsilon \) is approximative selection of \( F \) which is continuous on \( \mathbb{R}^n \).

We claim that there exists \( \epsilon_2 > 0 \) such that for every \( \epsilon \in (0, \epsilon_2] \) it holds
\[
\langle f_\epsilon(x), x - \hat{x} \rangle \geq 0 \quad \forall x \in K \cap \partial \Omega. \tag{3.2}
\]

Indeed, if the assertion is false then there exist sequences \( \epsilon_k \to 0^+ \) and \( x_k \in K \cap \partial \Omega \) such that
\[
\langle f_{\epsilon_k}(x_k), x_k - \hat{x} \rangle < 0 \quad \forall k \in \mathbb{N}. \tag{3.3}
\]

By Lemma 2.1, there exists \( (y_k, z_k) \in \text{Graph} F \) such that
\[
||y_k - x_k|| < \epsilon_k \quad \text{and} \quad ||f_{\epsilon_k}(x_k) - z_k|| < \epsilon_k.
\]

By compactness of \( K \cap \partial \Omega \) we may suppose that there exists \( \hat{x} \in K \cap \partial \Omega \) such that \( x_k \to \hat{x} \). Then \( y_k \to \hat{x} \). As \( F(\hat{x}) \) is a compact set and \( F \) is upper semicontinuous at \( \hat{x} \), by taking a subsequence (if necessary) we can suppose furthermore that \( z_k \to z \in F(\hat{x}) \). Then \( f_{\epsilon_k}(x_k) \to z \). Letting \( k \to \infty \), from (3.3) we obtain \( ||z - x - \hat{x}|| \leq 0 \), hence
\[
\inf_{x \in F(\hat{x})} \langle x^*, x - \hat{x} \rangle \leq 0.
\]

This contradicts (3.1) and our claim is obtained.

Put \( \epsilon = \min(\epsilon_1, \epsilon_2) \). We now show that \( d(\Phi_\epsilon, \partial \Omega, 0) = 1 \) for all \( \epsilon \in (0, \epsilon] \) and so \( d(F + N_{K_\epsilon}, \partial \Omega, 0) = 1 \). For this we build a homotopy as in [9].

Fix any \( \epsilon \in (0, \epsilon] \) and put
\[
H(t,x) = x - \Pi_K(t(x - f_\epsilon(x) + (1 - t)\hat{x})), \quad (t,x) \in [0,1] \times \partial \Omega.
\]

We have \( H(0,x) = x - \hat{x} \) and \( H(1,x) = \Phi_\epsilon(x) \). Note that
\[
d(H(0,\cdot), \partial \Omega, 0) = 1.
\]

If \( 0 \notin H(t, \partial \Omega) \) for all \( t \in [0,1] \), then it is obvious that \( 0 \notin H(0, \partial \Omega) \) and
\[
d(H(1,\cdot), \partial \Omega, 0) = 1.
\]

Thus, there exist \( t \in (0,1) \) and \( x \in \partial \Omega \) such that \( 0 = H(t,x) \). By the property of the metric projection we have
\[
\langle x - t(x - f_\epsilon(x) + (1 - t)\hat{x}), x - y \rangle \geq 0 \quad \forall y \in K.
\]

In particular, for \( y = \hat{x} \) we get
\[
\langle t(x) + (1 - t)(x - \hat{x}), x - \hat{x} \rangle \geq 0.
\]

This implies
\[
\langle f_\epsilon(x), x - \hat{x} \rangle \geq \frac{1 - t}{t} ||x - \hat{x}||^2 > 0,
\]

where the last inequality holds because \( t \in (0,1) \) and \( x \neq \hat{x} \).

But then it follows that \( \{ f_\epsilon(x), x - \hat{x} \} < 0 \) which contradicts (3.2). Thus, \( 0 \notin H(t, \partial \Omega) \) for all \( t \in [0,1] \). By the homotopy invariance (4) in Theorem 1.1 we obtain
\[
d(H(0,\cdot), \partial \Omega, 0) = d(H(1,\cdot), \partial \Omega, 0) = 1.
\]

In summary, we have proved that \( d(\Phi_\epsilon, \partial \Omega, 0) = 1 \) for all \( \epsilon \in (0, \epsilon] \). By the degree definition of GVIs we have
\[
d(F + N_{K_\epsilon}, \partial \Omega, 0) = 1.
\]

According to Theorem 2.1, there exists \( x_0 \in \Omega \cap K \) such that \( 0 \in F(x_0) + N_{K_\epsilon}(x_0) \). The proof of the theorem is complete. \( \square \)

In the rest of the paper we will present a result on solution stability of GVIs. Let us assume that \( M \) and \( \Lambda \) are subsets of \( \mathbb{R}^p \) and \( \mathbb{R}^n \), respectively; \( F : M \times \mathbb{R}^n \to 2^{\mathbb{R}^n} \) and \( K : \Lambda \to 2^{\mathbb{R}^n} \) be multifunctions. Consider the parametric generalized variational inequality
\[
0 \in F(\mu, x) + N_{K(\mu)}(x), \tag{3.4}
\]

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where $N_{K,l}(x)$ is the value at $x$ of the normal cone operator associated with the set $K_{l}$ and $(\mu,l) \in M \times A$ are parameters. We denote by $S(\mu,l)$ the solution set of the problem (3.4) corresponding to $(\mu,l)$ and suppose that $x_{0} \in S(\mu_{0},l_{0})$ for a given $(\mu_{0},l_{0}) \in M \times A$.

Our main concern is now to investigate the behaviour of $S(\mu,l)$ when $(\mu,l)$ vary around $(\mu_{0},l_{0})$. This problem has been addressed by many authors in the last two decades. For the relevant literature of the problem we refer the reader to [14–17,22–25] and several references given therein.

The following result gives a sufficient condition for the lowercontinuity of the solution map of (3.4). It is an extension of results in [14,22] for the case of GVI's.

**Theorem 3.2.** Assume that $X_{0}, A_{0}$ and $M_{0}$ are neighborhoods of $x_{0}, l_{0}$ and $\mu_{0}$, respectively and the following conditions are satisfied:

(i) $F(\cdot,\cdot)$ is l.s.c. on $M_{0} \times X_{0}$ with closed convex values and $F(\mu_{0},\cdot)$ is u.s.c. with compact convex values;

(ii) $K : A_{0} \rightarrow 2^{\mathbb{R}}$ is closed convex valued and pseudo-Lipschitz continuous around $(\mu_{0},l_{0})$, i.e., there exist neighborhoods $V_{l}, W_{0}$ of $l_{0}$, $W_{0}$ of $x_{0}$ and a constant $k > 0$ such that

$$K(\mu,l) \cap W \subseteq K(\mu,l) + k\|\mu - \hat{\mu}\|B(0,1) \quad \forall \mu,l \in V \cap A;$$

(iii) $x_{0}$ is an isolated solution and there exists $\sigma > 0$ such that

$$\sigma F(\mu_{0},\cdot) + N_{K,l}(\mu_{0},x_{0}) \cap x_{0} = 0 \quad \forall \sigma \in (0,\sigma].$$

Then there exist a neighborhood $U_{0}$ of $\mu_{0}$, a neighborhood $V_{0}$ of $l_{0}$ and an open bounded neighborhood $Q_{0}$ of $x_{0}$ such that the solution map $S : U_{0} \times X_{0} \rightarrow 2^{\mathbb{R}}$ of (3.4) defined by

$$S(\mu,l) = S(\mu,l) \cap Q_{0}$$

is nonempty valued and lower semi-continuous at $(\mu_{0},l_{0})$.

**Proof.** By (i) and the continuous selection theorem due to Michael (see [26, Theorem 9G, p. 466]), there exists a continuous mapping $f : M_{0} \times X_{0} \rightarrow \mathbb{R}^{n}$ such that $f(\mu,x) \in F(\mu,x)$ for all $(\mu,x) \in M_{0} \times X_{0}$. By Tietze-Urysohn's theorem (see [8, Theorem 5.1, p. 149]) we can assume that $f$ is continuous on $\mathbb{R}^{n} \times \mathbb{R}^{n}$.

According to Lemma 1.1 in [21,4], it follows from (ii) that there exist a neighborhood $A_{0} \subseteq A_{0} \cap V$ of $l_{0}$, a neighborhood $X_{0} \subseteq X_{0} \cap W$ of $x_{0}$ and a constant $k > 0$ such that

$$\|\Pi_{K,l}(x) - z|| \leq k\|\mu - \hat{\mu}\|^{rac{1}{2}}$$

for all $\mu,l \in A_{0}$ and $z \in X_{0}$. Hence, for any $z,z' \in X_{0}$ and $\lambda, \lambda' \in A_{0}$ we have

$$\|\pi(\lambda,z) - \pi(\lambda',z')\| \leq \|\Pi_{K,l}(x)z - \Pi_{K,l}(x)z'\| \leq k\|\mu - \hat{\mu}\|^{rac{1}{2}}\|\lambda - \lambda'\|^{rac{1}{2}}.$$
References