LOCATING COMMON FIXED POINTS OF NONLINEAR REPRESENTATIONS OF SEMIGROUPS

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Abstract. This paper is concerned with the problem of finding common fixed points for a family of Bregman relatively weak nonexpansive mappings. The motivation is due to our finding of some gaps in a paper of K. S. Kim (Nonlinear Analysis, 73 (2010), 3413-3419), where the author was developing a hybrid iterative scheme for locating common fixed points of a nonlinear representation of a left reversible semigroup. After a brief discussion about the gaps and why they are fatal, we present a new approach by using Bergman type nonexpansive mappings. A correct version of Kim's convergence theorem is given as a consequence of our new results, which also improve and extend some recent results in the literature.

1. Introduction

Let S be a semigroup. Let C be a nonempty closed and convex subset of a (real) Banach space E with dual space E^*. Let $\mathcal{T} := \{T(s) : s \in S\}$ be a representation of S as mappings from C into C such that

$$T(st) = T(s)T(t), \quad \forall s, t \in S.$$

Assume the set $F(\mathcal{T})$ of common fixed points of all $T(s)$ in $\mathcal{T}$ is nonempty. The question is to establish an algorithm to locate the elements in $F(\mathcal{T})$. Note that S can be uncountable, while an "effective" algorithm is expected to finish in almost finite, i.e., countably, many steps.

A translation invariant subspace X of $l^\infty(S)$ is called *rich for* $\mathcal{T}$ if X contains the constant functions and all the "matrix entries" of the representation $\mathcal{T}$, namely, the functions $s \mapsto \langle T(s)x, x^*\rangle$ with $x \in C$ and $x^* \in E^*$. Assume also that every point x in C is weakly almost periodic for $\mathcal{T}$, i.e., the set $\{T(s)x : s \in S\}$ is relatively weakly compact in E. Then, as in [7], for each x in C and each mean μ on X, there exists a unique point $T_\mu x$ in E, called the *barycenter* of $T(\cdot)x$ with

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respect to μ, in the sense that

$$\mu\langle T(\cdot)x, x^*\rangle = \langle T_\mu x, x^*\rangle, \quad \forall x^* \in E^*.$$

It follows from the strong separation theorem that $T_\mu x$ is contained in the closure of the convex hull of $\{T(s)x : s \in S\}$ for each x in C. In particular, $F(\mathcal{T}) \subseteq F(T_\mu)$, the set of fixed points of T_μ. Conversely, we consider an asymptotically left invariant sequence $\{\mu_n\}$ of means on X; i.e.,

$$\lim_n (\mu_n(l_s f) - \mu_n(f)) = 0, \quad \forall s \in S,\ f \in X.$$

Here, l_s denotes the left translation by s defined by

$$l_s(f)(x) = f(sx), \quad \forall f \in X,\ x \in S.$$

It follows from [9, Lemma 3.5] (see also [13]) that

$$\liminf_{n\to\infty} \|T_{\mu_n} z - z\| = 0 \quad \Longrightarrow \quad z \in F(\mathcal{T}). \tag{1.1}$$

This implies

$$F(\mathcal{T}) = \bigcap_n F(T_{\mu_n}) \tag{1.2}$$

Consequently, the question of finding common fixed points of $\mathcal{T}$ reduces to that of finding those z in C satisfying (1.1), or finding common fixed points of the sequence $\{T_{\mu_n}\}$.

In 2010, K. S. Kim [9] provided the following plausible strong convergence theorem for a class of representations for left reversible semigroups. Recall that a topological semigroup S with an identity is *left reversible* if every two closed right ideals of S intersect, i.e., $\overline{aS} \cap \overline{bS} \neq \emptyset$ for all a, b in S.

(False) Assertion 1.1 (Kim, [9, Theorem 4.1])**.** *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $\mathcal{T} = \{T(s) : s \in S\}$ be a representation of a left reversible semigroup S as relatively nonexpansive maps from C into C with $F(\mathcal{T}) \neq \emptyset$.*

Let X be a rich subspace of $\ell^\infty(S)$ for $\mathcal{T}$, and let $\{\mu_n\}_{n\in\mathbb{N}}$ be an asymptotically left invariant sequence of means on X. Let T_{μ_n} be the barycenter representation of $\mathcal{T}$ associated to each μ_n. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the following algorithm

$$\begin{cases} x_0 = x \in C \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n = J^{-1}[\alpha_n J x_1 + (1-\alpha_n) J T_{\mu_n} x_n)], \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \alpha_n \phi(z, x_1) + (1-\alpha_n)\phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1 \ \text{for } n \in \mathbb{N}. \end{cases} \tag{1.3}$$

Here, $J : E \to E^*$ is the normalized duality map and Π_D is the generalized projection from C onto a nonempty closed convex subset D of C.

Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to the fixed point $\Pi_{F(\mathcal{T})}x_1$ of $\mathcal{T}$.

Unfortunately, there are some gaps in the original proof of Assertion 1.1. For example, in [9, line -11, p. 3416], the author derived that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence after he showed $\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0$ for all fixed $m = 1, 2, \ldots$. It is not a tautology, however, as $x_n = \sum_{k=1}^{n} 1/k$ verifies.

After some preparations, we will provide in §2 a concrete counter example to demonstrate that the original plan proving Assertion 1.1 in [9] does not work.

In §3, we collect some necessary definitions and preliminary results for introducing the recent developed notions of Bregman type nonexpansive mappings. As an extension of nonexpansive mappings, the class of Bregman type nonexpansive mappings appears in many applications. The theory of fixed points involving Bregman distances and Bregman type nonexpansive mappings are studied in, e.g., [1, 2, 17].

In §4, we present a correct version of Assertion 1.1. In a more general setting, we will study the problem of finding common fixed points for an arbitrary family of Bregman relatively weak nonexpansive mappings, and obtain strong convergence theorems by hybrid schemes of Halpern types. The method of the present paper is different from the original one proposed by Kim in [9] and our results improve and extend some recent results in the literature, for example, [14, 15].

Finally, we mention that the hybrid projection method was first introduced by Hangazeau in [6]. In [8, 10, 25], the authors investigated hybrid projection method. As a generalization of the hybrid projection method, the shrinking projection method was first introduced by Takahashi *et al.* in [25]. Our approach in this paper follows this line.

2. A counter example

In the following, we let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive (real) Banach space E. We denote by $x_n \to x$ and $x_n \rightharpoonup x$, respectively, the strong and weak convergence of a sequence $\{x_n\}_{n\in\mathbb{N}}$ to x in E. For any x in E, the value of a bounded linear functional x^* in the Banach dual space E^* of E at x is denoted by $\langle x, x^*\rangle$. When E^* is strictly convex, one can define a single-valued normalized duality map $J : E \to E^*$ such that Jx is the unique functional satisfying

$$\langle x, Jx\rangle = \|x\|^2 = \|Jx\|^2.$$

When E is uniformly smooth, J is uniformly norm-to-norm continuous on bounded subsets of E.

The generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \operatorname{argmin}_{y\in C}\ \phi(y,x),$$

where

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2.$$

When E is a Hilbert space, we have $\phi(x,y) = \|x-y\|^2$.

Let $T : C \to C$ be a map. The set of fixed points of T is denoted by

$$F(T) = \{x \in C : Tx = x\}.$$

A point $p \in C$ is said to be an *asymptotic fixed point* [19] of T if there exists a sequence $x_n \rightharpoonup p$ in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. If we have $x_n \to p$ instead, we call p a *strong asymptotic fixed point* of T. The set of all asymptotic and strong asymptotic fixed points of T are denoted by $F_a(T)$ and $F_{sa}(T)$, respectively. Clearly,

$$F(T) \subseteq F_{sa}(T) \subseteq F_a(T).$$

Following Matsushita and Takahashi [14] and Kim [9], we call T a *relatively nonexpansive* (resp. *relatively weak nonexpansive*) map if $F_a(T) = F(T) \neq \emptyset$ (resp. $F_{sa}(T) = F(T) \neq \emptyset$) and

$$\phi(u,Tx) \leq \phi(u,x), \quad \forall u \in F(T),\ x \in C.$$

Let us return to the promised counterexample to Assertion 1.1, i.e., [9, Theorem 4.1]. In [9] the proof of its Theorem 4.1 is divided into three parts.

Step 1. $\{x_n\}_{n\in\mathbb{N}}$ is well defined.

Step 2. $\lim_{n\to\infty} \|x_n - T_{\mu_n} x_n\| = 0$, and based on this assertion, $x_n \in F(\mathcal{T}), \forall n = 1, 2, \ldots$.

Step 3. $p = \lim_{n\to\infty} x_n = \Pi_{F(\mathcal{T})} x_1$.

Unfortunately, we discovered gaps and errors there. Beside the false statement $\lim_{n\to\infty} \|x_{n+m} - x_n\| = 0$, $\forall m = 1, 2, \ldots$, implying that $\{x_n\}$ converged (to p) as mentioned before, we also find that the conclusion in Step 2 does not hold either. More precisely, we do not see the validity of using [9, Lemma 3.5], i.e. (1.1), to conclude $x_n \in F(\mathcal{T})$. Indeed, $x_n \notin F(\mathcal{T})$ in the following example.

Example 2.1. Let S be the left reversible additive semigroup of nonnegative integers in discrete topology. Define $T_n : \mathbb{R} \to \mathbb{R}$ (so $C = \mathbb{R}$ here) by

$$T_n(x) = e^{-n}x,$$

for $n = 0, 1, 2, \ldots$. It is plain that $\mathcal{T} = \{T_n\}_{n\in S}$ is a representation of the additive semigroup S as relatively nonexpansive (indeed, contractive and linear) mappings with the common fixed point set $F = \{0\}$.

Let $X \subset \ell^\infty(S)$ be the Banach space of all convergent real sequences, and let μ_n be the point evaluation at $n = 1, 2, \ldots$. Then $\{\mu_n\}$ is an asymptotically left invariant sequence of means on X, and $T_{\mu_n} = T_n$ is the barycenter representation of $\mathcal{T}$ associated to each μ_n. If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence defined by (1.3) above with $x_0 \neq 0$, then $x_n \notin \bigcap_{s\in S} F(T_s) = \{0\}$ for each $n \geq 0$.

However, it follows from $\|x_n - T_n x_n\| \to 0$ that $(1 - e^{-n})x_n \to 0$ and hence $x_n \to 0$. Therefore, the implication from the first part of Step 2 to Step 3 still holds. One shall see our new Theorem 4.3 below applies to this example. A numerical demonstration is given in §5.

3. Bregman distance and Bregman type nonexpansive mappings

Let E be a Banach space, and let $g : E \to (-\infty, +\infty]$ be a convex function. Denote by $\operatorname{dom} g = \{x \in E : g(x) < +\infty\}$ the *domain* of g. For any point x in the interior of $\operatorname{dom} g$, the *right-hand derivative* $g^o(x, y)$ of g at x in the direction y is defined as

$$g^o(x, y) = \lim_{t\downarrow 0} \frac{g(x + ty) - g(x)}{t}. \tag{3.1}$$

The function g is said to be *Gâteaux differentiable* at x if $\lim_{t\to 0} \frac{g(x+ty)-g(x)}{t}$ exists for any $y \neq 0$. In this case, $g^o(x, y)$ coincides with $\langle y, \nabla g(x)\rangle$. Here, the vector $\nabla g(x)$ in E^* is the value of the *gradient* ∇g of g at x. The function g is said to be *Fréchet differentiable* at x if the limit in (3.1) is attained uniformly wherever $\|y\| = 1$. The function g is said to be Gâteaux differentiable or Fréchet differentiable if it is Gâteaux differentiable or Fréchet differentiable everywhere. Finally, g is said to be *uniformly Fréchet differentiable* on a subset X of E if the limit is attained uniformly for all x in X and $\|y\| = 1$.

It is well known that if a continuous convex function $g : E \to \mathbb{R}$ is Gâteaux differentiable, then ∇g is norm-to-weak* continuous (see, e.g., [4]). It is also known that if g is Fréchet differentiable, then ∇g is norm-to-norm continuous (see, e.g., [12]).

Let $S_E = \{z \in E : \|z\| = 1\}$ and $B_r := \{z \in E : \|z\| \leq r\}$ for all $r > 0$. Define the gauge $\rho_r : [0, +\infty) \to [0, +\infty]$ of uniform convexity of g by

$$\rho_r(t) = \inf_{x,y\in B_r, \|x-y\|=t, \alpha\in(0,1)} \frac{\alpha g(x) + (1-\alpha)g(y) - g(\alpha x + (1-\alpha)y)}{\alpha(1-\alpha)}, \quad \forall t \geq 0.$$

Define $\sigma_r : [0, +\infty) \to [0, +\infty]$ by

$$\sigma_r(t) = \sup_{x\in B_r, y\in S_E, \alpha\in(0,1)} \frac{\alpha g(x + (1-\alpha)ty) + (1-\alpha)g(x - \alpha ty) - g(x)}{\alpha(1-\alpha)}, \quad \forall t \geq 0.$$

We call the function g *strongly coercive* if

$$\lim_{\|x_n\|\to+\infty} \frac{g(x_n)}{\|x_n\|} = +\infty.$$

We call g *bounded on bounded subsets of* E if $g(B_r)$ is bounded for each $r > 0$. We call g *uniformly convex on bounded subsets of* E ([27], pp. 203, 221) if $\rho_r(t) > 0$ for all $r, t > 0$. Finally, we call g *uniformly smooth on bounded subsets of* E ([27], pp. 207, 221) if $\lim_{t\downarrow 0} \frac{\sigma_r(t)}{t} = 0$ for all $r > 0$.

Let E be a Banach space. Let $g : E \to \mathbb{R}$ be a convex and Gâteaux differentiable function. The *Bregman distance* [3] corresponding to g is the function $D_g : E \times E \to \mathbb{R}$ defined by

$$D_g(x, y) = g(x) - g(y) - \langle x - y, \nabla g(y)\rangle, \ \forall x, y \in E.$$

It is clear that $D_g(x, y) \geq 0$ for all $x, y \in E$. When E is a smooth Banach space, setting $g(x) = \|x\|^2$, we obtain that $\nabla g(x) = 2Jx$ and hence

$$D_{\|\cdot\|^2}(x, y) = \phi(x, y), \quad \forall x, y \in E.$$

The following definition is slightly different from that in Butnariu and Iusem [4].

Definition 3.1 ([12])**.** Let E be a Banach space. The function $g : E \to \mathbb{R}$ is said to be a *Bregman function* if the following conditions are satisfied.

(1) g is continuous, strictly convex and Gâteaux differentiable;
(2) the set $\{y \in E : D_g(x, y) \leq r\}$ is bounded for all x in E and $r > 0$.

Let C be a nonempty and convex subset of E. It follows from [16] that for x in E and x_0 in C we have

$$D_g(x_0, x) = \min_{y\in C} D_g(y, x) \quad \text{if and only if} \quad \langle y - x_0, \nabla g(x) - \nabla g(x_0)\rangle \leq 0, \quad \forall y \in C. \tag{3.2}$$

Furthermore, if C is a nonempty, closed and convex subset of a reflexive Banach space E and $g : E \to \mathbb{R}$ is a strongly coercive Bregman function, then for each x in E, there exists a unique x_0 in C such that

$$D_g(x_0, x) = \min_{y\in C} D_g(y, x).$$

In this case, the *Bregman projection* proj^g_C from E onto C is defined by $\mathrm{proj}^g_C(x) = x_0$. It is well known that

$$D_g\Big(y, \mathrm{proj}^g_C x\Big) + D_g\Big(\mathrm{proj}^g_C x, x\Big) \leq D_g(y, x), \quad \forall y \in C, x \in E. \tag{3.3}$$

See [4] for more details.

Lemma 3.2 ([18])**.** *Let E be a Banach space and $g : E \to \mathbb{R}$ a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then*

$$\lim_{n\to\infty} D_g(x_n, y_n) = 0 \quad \Longleftrightarrow \quad \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Let E be a reflexive Banach space. For any proper, lower semicontinuous and convex function $g : E \to (-\infty, +\infty]$, the *conjugate function* g^* of g is defined by

$$g^*(x^*) = \sup_{x \in E}\{\langle x, x^*\rangle - g(x)\}, \quad \forall x^* \in E^*.$$

It is well known that

$$g(x) + g^*(x^*) \geq \langle x, x^*\rangle, \quad \forall (x, x^*) \in E \times E^*,$$

and

$$(x, x^*) \in \partial g \quad \Longleftrightarrow \quad g(x) + g^*(x^*) = \langle x, x^*\rangle.$$

Here, ∂g is the subdifferential of g [22]. We also know that if $g : E \to (-\infty, +\infty]$ is a proper, lower semicontinuous and convex function, then $g^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semicontinuous and convex function; see [24] for more details.

The following lemma follows from Butnariu and Iusem [4] and Zălinscu [27].

Lemma 3.3. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function. Then*

(1) $\nabla g : E \to E^*$ is one-to-one, onto and norm-to-weak* continuous;
(2) $\langle x - y, \nabla g(x) - \nabla g(y)\rangle = 0$ if and only if $x = y$;
(3) $\{x \in E : D_g(x, y) \leq r\}$ is bounded for all $y \in E$ and $r > 0$;
(4) dom $g^* = E^*$, g^* is Gâteaux differentiable and $\nabla g^* = (\nabla g)^{-1}$.

The following result was first proved in [5] (see also [12]).

Lemma 3.4. *Let E be a reflexive Banach space, $g : E \to \mathbb{R}$ a strongly coercive Bregman function, and V the function defined by*

$$V(x, x^*) = g(x) - \langle x, x^*\rangle + g^*(x^*), \quad x \in E, \ x^* \in E^*.$$

Then the following assertions hold:

(1) $D_g(x, \nabla g^*(x^*)) = V(x, x^*)$ for all x in E and $x^* \in E^*$.
(2) $V(x, x^*) + \langle \nabla g^*(x^*) - x, y^*\rangle \leq V(x, x^* + y^*)$ for all x in E and $x^*, y^* \in E^*$.

We know the following two results from [27].

Theorem 3.5. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent:*

(1) g is strongly coercive and uniformly convex on bounded subsets of E;
(2) dom $g^* = E^*$, g^* is bounded and uniformly smooth on bounded subsets of E^*;
(3) dom $g^* = E^*$, g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*.

Theorem 3.6. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a continuous convex function which is strongly coercive. Then the following assertions are equivalent:*

(1) g is bounded and uniformly smooth on bounded subsets of E;
(2) g^* is Fréchet differentiable and ∇g^* is uniformly norm-to-norm continuous on bounded subsets of E^*;
(3) dom $g^* = E^*$, g^* is strongly coercive and uniformly convex on bounded subsets of E^*.

Definition 3.7. Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. A mapping $T : C \to C$ is said to be

(1) *Bregman quasi-nonexpansive*, if $F(T) \neq \emptyset$ and

$$D_g(p, Tx) \leq D_g(p, x), \quad \forall p \in F(T), x \in C.$$

(2) *Bregman relatively nonexpansive* (resp. *Bregman relatively weak nonexpansive*) if
  i. $F(T)$ is nonempty;
  ii. $D_g(p, Tx) \leq D_g(p, x)$, $\forall p \in F(T), x \in C$;
  iii. $F_a(T) = F(T)$ (resp. $F_{sa}(T) = F(T)$).

It is clear that quasi-nonexpansive (resp. relatively nonexpansive, relatively weak nonexpansive) maps are exactly Bregman quasi-nonexpansive (resp. Bregman relatively nonexpansive, Bregman weakly quasi-nonexpansive) with respect to the Bregman distance D_g with $g(x) = \|x\|^2$. It is also clear that every Bregman relatively nonexpansive mapping is Bregman weakly relatively nonexpansive, and every Bregman relatively weak nonexpansive mapping is Bregman quasi-nonexpansive. However, the converses are in general not true. For more details, we refer the readers to [18].

We call $T : C \to C$ a *closed map* if we have $Tx_0 = y_0$ whenever $x_n \to x_0$ in C with $Tx_n \to y_0$. It is easy to verify that any Bregman quasi-nonexpansive closed map $T : C \to C$ is a Bregman relatively weak nonexpansive mapping. To this end, let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in C such that $x_n \to x \in C$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. This implies that $Tx_n \to x \in C$ as $n \to \infty$. From the closedness of T we conclude that $x \in F(T)$. In Example 3.8 below, we see that there exists a Bregman relatively weak nonexpansive mapping which is neither a Bregman relatively nonexpansive mapping nor a closed mapping.

Example 3.8. Let $E = l^2$ be the infinite separable Hilbert space with the canonical orthonormal basis $\{e_1, e_2, \ldots\}$. Define

$$y_n = e_1 + e_n, \quad \forall n = 1, 2, \ldots.$$

Let k be an even number in $\mathbb{N}$ and let $g : E \to \mathbb{R}$ be defined by

$$g(y) = \frac{1}{k}\|y\|^k, \quad y \in E.$$

It is easy to show that $\nabla g(y) = J_k(y)$ for all $y \in E$, where

$$J_k(y) = \left\{ y^* \in E^* : \langle y, y^* \rangle = \|y\| \|y^*\|, \ \|y^*\| = \|y\|^{k-1} \right\}.$$

It is also obvious that

$$J_k(\lambda y) = \lambda^{k-1} J_k(y), \qquad \forall y \in E, \ \forall \lambda \in \mathbb{R}.$$

Let $S = (0, +\infty)$. For any $s \in S$, we define $T_s : E \to E$ by

$$T_s(y) = \begin{cases} \dfrac{n}{n+1} y, & \text{if } y = y_n \text{ for any } n = 1, 2, \ldots, \\ \dfrac{-s}{s+1} y, & \text{if } x \neq y_n \text{ for all } n = 1, 2, \ldots. \end{cases}$$

It is clear that $F(T_s) = \{0\}$ for all s in S.

Let $s \in S$. For any n in $\mathbb{N}$, we have

$$\begin{aligned} D_g(0, T_s y_n) &= g(0) - g(T_s y_n) - \langle 0 - T_s y_n, \nabla g(T_s y_n) \rangle \\ &= -\tfrac{n^k}{(n+1)^k} g(y_n) + \tfrac{n^k}{(n+1)^k} \langle y_n, \nabla g(y_n) \rangle \\ &= \tfrac{n^k}{(n+1)^k} [-g(y_n) + \langle y_n, \nabla g(y_n) \rangle] \\ &= \tfrac{n^k}{(n+1)^k} [D_g(0, y_n)] \\ &\leq D_g(0, y_n). \end{aligned}$$

If $y \neq y_n$ for all $n \geq 1$, then

$$\begin{aligned} D_g(0, T_s y) &= g(0) - g(T_s y) - \langle 0 - T_s y, \nabla g(T_s y) \rangle \\ &= -\tfrac{s^k}{(s+1)^k} g(y) - \tfrac{s^k}{(s+1)^k} \langle y, -\nabla g(y) \rangle \\ &= \tfrac{s^k}{(s+1)^k} [-g(y) - \langle -y, \nabla g(y) \rangle] \\ &\leq D_g(0, y). \end{aligned}$$

Therefore, T_s is a Bregman quasi-nonexpansive mapping.

We claim that T_s is a Bregman relatively weak nonexpansive mapping. Indeed, for any sequence $\{z_n\}_{n \in \mathbb{N}}$ in E such that $z_n \to z_0$ and $\|z_n - T_s z_n\| \to 0$ as $n \to \infty$, by passing to a subsequence we can assume that $z_n \neq y_m$ for any $n, m = 1, 2, \ldots$. This implies that $T_s z_n = -\frac{s}{s+1} z_n$ for all n. It follows from $\|z_n - T_s z_n\| = \frac{2s+1}{s+1} \|z_n\| \to 0$ that $z_n \to z_0 = 0 \in F(T_s)$. Thus, T_s is a Bregman relatively weak nonexpansive mapping.

However, T_s is not Bregman relatively nonexpansive. In fact, although $y_n \rightharpoonup e_1$ and

$$\|y_n - T_s y_n\| = \left\| y_n - \frac{n}{n+1} y_n \right\| = \frac{1}{n+1} \|y_n\| \to 0, \quad \text{as } n \to \infty,$$

we have $e_1 \notin F(T_s)$ for all s in S. Therefore, $F_a(T_s) \neq F(T_s)$ for all s in S.

Finally, we verify that T_s is not a closed map. Let $u_n = (1 + \frac{1}{n}) y_2$. Then $u_n \to y_2$ and $T_s u_n = \frac{-s}{1+s} u_n \to \frac{-s}{1+s} y_2$ as $n \to \infty$ (since $u_n \neq y_m$ for all n, m in $\mathbb{N}$). But $T_s y_2 = \frac{2}{3} y_2 \neq \frac{-s}{1+s} y_2$ for all s in S.

4. Strong convergence theorems

In this section, we prove strong convergence theorems in a reflexive Banach space. We start with the following simple lemma which has been proved in [20].

Lemma 4.1. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded and uniformly convex on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let $T : C \to C$ be a Bregman quasi-nonexpansive mapping. Then $F(T)$ is closed and convex.*

Theorem 4.2. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function which is bounded, uniformly convex and uniformly smooth on bounded subsets of E. Let C be a nonempty, closed and convex subset of E. Let $\{T_n\}_{n\in\mathbb{N}}$ be a family of Bregman relatively weak nonexpansive mappings from C into C such that $F := \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by

$$
\tag{4.1}
\begin{cases}
x_0 = x \in C \quad \textit{chosen arbitrarily},\\
C_1 = C,\\
x_1 = \mathrm{proj}^g_{C_1} x_0\\
y_{n,k} = \nabla g^*[\alpha_n \nabla g(x_1) + (1-\alpha_n)\nabla g(T_k x_n)], \quad k = 1,2,\ldots,n,\\
C_{n+1} = \{z \in C_n : \max_{1\le k\le n} D_g(z, y_{n,k}) \le \alpha_n D_g(z, x_1) + (1-\alpha_n) D_g(z, x_n)\},\\
x_{n+1} = \mathrm{proj}^g_{C_{n+1}} x_1.
\end{cases}
$$

Then, all $\{x_n\}_{n\in\mathbb{N}}$, $\{T_n x_n\}_{n\in\mathbb{N}}$, and $\{y_{n,k}\}_{n\in\mathbb{N}}$ converge strongly to $\mathrm{proj}^g_F x_1$, where k is any fixed positive integer.

Proof. We divide the proof into several steps.

Step 1. We show that C_n is closed and convex for each n in $\mathbb{N}$.

By assumption, $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for some m in $\mathbb{N}$. For $z \in C_{m+1}$, by definition, $z \in C_m$, and

$$D_g(z, y_{m,k}) \le \alpha_m D_g(z, x_1) + (1-\alpha_m) D_g(z, x_m), \quad \forall k = 1,2,\ldots,m.$$

This implies that

$$
\begin{aligned}
&g(z) - g(y_{m,k}) - \langle z - y_{m,k}, \nabla g(y_{m,k})\rangle\\
\le\ &\alpha_m[g(z) - g(x_1) - \langle z - x_1, \nabla g(x_1)\rangle]\\
&\quad + (1-\alpha_m)[g(z) - g(x_m) - \langle z - x_m, \nabla g(x_m)\rangle], \quad \forall k = 1,2,\ldots,m,
\end{aligned}
$$

which is equivalent to

$$\begin{aligned}&\langle z-y_{m,k},-\nabla g(y_{m,k})\rangle+\alpha_m\langle z-x_1,\nabla g(x_1)\rangle+(1-\alpha_m)\langle z-x_m,\nabla g(x_m)\rangle\\ \leq\ & g(y_{m,k})-\alpha_m g(x_1)-(1-\alpha_m)g(x_m),\quad \forall k=1,2,\ldots,m.\end{aligned}$$

Now, it is plain that the closedness and convexity of C_m ensure those of C_{m+1}. By the principle of induction, C_n is closed and convex for each n in $\mathbb{N}$.

Step 2. We claim that $F\subset C_n$ for all n in $\mathbb{N}$.

Noticing that $F\subset C_1=C$, we assume $F\subset C_m$ for some m in $\mathbb{N}$. Owing to Lemma 3.4, for any $w\in F\subset C_m$ and $k=1,2,\ldots,m$ we obtain

$$\begin{aligned}D_g(w,y_{m,k}) &= D_g(w,\nabla g^*[\alpha_m\nabla g(x_1)+(1-\alpha_m)\nabla g(T_kx_m)])\\ &= V(w,\alpha_m\nabla g(x_1)+(1-\alpha_m)\nabla g(T_kx_m))\\ &= g(w)-\langle w,\alpha_m\nabla g(x_1)+(1-\alpha_m)\nabla g(T_kx_m)\rangle\\ &\quad +g^*(\alpha_m\nabla g(x_1)+(1-\alpha_m)\nabla g(T_kx_m))\\ &\leq \alpha_m g(w)+(1-\alpha_m)g(w)-\alpha_m\langle w,\nabla g(x_1)\rangle-(1-\alpha_m)\langle w,\nabla g(T_kx_m)\rangle\\ &\quad +\alpha_m g^*(\nabla g(x_1))+(1-\alpha_m)g^*(\nabla g(T_kx_m))\\ &= \alpha_m V(w,\nabla g(x_1))+(1-\alpha_m)V(w,\nabla g(T_kx_m))\\ &= \alpha_m D_g(w,x_1)+(1-\alpha_m)D_g(w,T_kx_m)\\ &\leq \alpha_m D_g(w,x_1)+(1-\alpha_m)D_g(w,x_m).\end{aligned}\tag{4.2}$$

Thus we have $w\in C_{m+1}$. The assertion follows from induction.

Step 3. We shall show that $\{x_n\}_{n\in\mathbb{N}}$, $\{T_kx_n\}_{n\in\mathbb{N}}$ and $\{y_{n,k}\}_{n\in\mathbb{N}}$ are bounded sequences in C.

Using (3.3), we get

$$\begin{aligned}D_g(x_n,x_1)=D_g\left(\mathrm{proj}^g_{C_n}x_1,x_1\right)&\leq D_g(w,x_1)-D_g(w,x_n)\\ &\leq D_g(w,x_1),\quad \forall w\in F\subset C_n,\ n\in\mathbb{N}.\end{aligned}$$

This entails the boundedness of the sequence $\{D_g(x_n,x_1)\}_{n\in\mathbb{N}}$ and hence there exists $M_1>0$ such that

$$D_g(x_n,x_1)\leq M_1,\qquad \forall n\in\mathbb{N}.\tag{4.3}$$

In view of Lemma 3.3(3), we conclude that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded. Since T_k is Bregman relatively weak nonexpansive, for any q in F one has

$$D_g(q,T_kx_n)\leq D_g(q,x_n),\qquad \forall k,n\in\mathbb{N}.$$

This, together with Definition 3.1(2) and the boundedness of $\{x_n\}_{n\in\mathbb{N}}$ implies that the sequence $\{T_kx_n\}_{n\in\mathbb{N}}$ is bounded for any fixed $k=1,2,\ldots$. Indeed, from the boundedness of $\{x_n\}_{n\in\mathbb{N}}$ we conclude that $\{\nabla g(x_n)\}_{n\in\mathbb{N}}$ is bounded (see, e.g., [4]). Also $\{g(x_n)\}_{n\in\mathbb{N}}$ is bounded too by the assumption. On the other hand, from the definition of Bregman distance, we know that

$$D_g(q,x_n)=g(q)-g(x_n)-\langle q-x_n,\nabla g(x_n)\rangle\leq|g(q)|+|g(x_n)|+\|g(q)\|\|\nabla g(x_n)\|,$$

which ensures the boundedness of $D_g(q,x_n)$.

It follows from Lemma 3.3 and (4.2) that the sequence $\{y_{n,k}\}_{n\in\mathbb{N}}$ is bounded.

Step 4. We show that $x_n \to u$ for some u in F, and $u = \mathrm{proj}^g_F x_1$.
By the construction of C_n, we conclude that $C_m \subset C_n$ and $x_m = \mathrm{proj}^g_{C_m} x_1 \in C_m \subset C_n$ for any positive integer $m \geq n$. This, together with (3.3), implies that

$$
\begin{aligned}
D_g(x_m, x_n) &= D_g\Big(x_m, \mathrm{proj}^g_{C_n} x_1\Big) \leq D_g(x_m, x_1) - D_g\Big(\mathrm{proj}^g_{C_n} x_1, x_1\Big) \\
&= D_g(x_m, x_1) - D_g(x_n, x_1). \qquad (4.4)
\end{aligned}
$$

In view of (3.3) again, we conclude that

$$D_g(x_n, x_1) \leq D_g(x_m, x_n) + D_g(x_n, x_1) \leq D_g(x_m, x_1), \quad \forall m \geq n.$$

This proves that $\{D_g(x_n, x_1)\}_{n\in\mathbb{N}}$ is an increasing sequence in $\mathbb{R}$ and hence by (4.3) the limit $\lim_{n\to\infty} D_g(x_n, x_1)$ exists. Letting $m, n \to \infty$ in (4.4), we deduce that $D_g(x_m, x_n) \to 0$. Since $\{x_n\}_{n\in\mathbb{N}}$ is bounded, Lemma 3.2 ensures that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. In other words, $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since C is complete, there exists u in C such that

$$\lim_{n\to\infty} \|x_n - u\| = 0. \qquad (4.5)$$

Let us show that $u \in F$. As $x_{n+1} \in C_{n+1}$, we are led to

$$D_g(x_{n+1}, y_{n,k}) \leq \alpha_n D_g(x_{n+1}, x_1) + (1 - \alpha_n) D_g(x_{n+1}, x_n), \quad \forall k = 1, 2, \ldots, n.$$

It follows from (4.4) that

$$\lim_{n\to\infty} D_g(x_{n+1}, x_n) = 0. \qquad (4.6)$$

Hence,

$$\lim_{n\to\infty} D_g(x_{n+1}, y_{n,k}) = 0, \quad \forall k = 1, 2, \ldots. \qquad (4.7)$$

Employing Lemma 3.2 and (4.6)-(4.7), we deduce that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_{n+1} - y_{n,k}\| = 0, \quad \forall k = 1, 2, \ldots.$$

Consequently, it turns out from (4.5) that for any fixed $k = 1, 2, \ldots$ we have

$$\lim_{n\to\infty} \|y_{n,k} - u\| = 0.$$

Also, in view of (4.1), for any fixed $k = 1, 2, \ldots$, we have

$$\nabla g(y_{n,k}) - \nabla g(T_k x_n) = \alpha_n (\nabla g(x_1) - \nabla g(T_k x_n)).$$

Because $\{T_k x_n\}$ is bounded and $\alpha_n \to 0$, we have

$$\lim_{n\to\infty} \|\nabla g(y_{n,k}) - \nabla g(T_k x_n)\| = 0, \quad \forall k = 1, 2, \ldots.$$

Since ∇g^* is uniformly norm-to-norm continuous on any bounded subset of E by Theorem 3.6, we obtain form Lemma 3.3 that

$$\lim_{n\to\infty} \|y_{n,k} - T_k x_n\| = 0, \quad \forall k = 1, 2, \ldots.$$

Moreover, the triangle inequality

$$\|x_n - T_k x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_{n,k}\| + \|y_{n,k} - T_k x_n\|$$

implies that

$$\lim_{n\to\infty} \|x_n - T_k x_n\| = 0, \quad \forall k = 1, 2, \ldots.$$

Therefore, u is the strong limit of all sequences $\{x_n\}$, $\{y_{n,k}\}$ and $\{T_k x_n\}$, for all fixed $k = 1, 2, \ldots$. In particular, u is a strong asymptotic fixed point of the Bregman relatively weak nonexpansive mapping T_k. Therefore, $T_k u = u$, for all $k = 1, 2, \ldots$, and thus u in F.

Finally, we show that $u = \mathrm{proj}_F^g x_1$. From $x_n = \mathrm{proj}_{C_n}^g x_1$, we conclude that

$$\langle z - x_n, \nabla g(x_n) - \nabla g(x_1)\rangle \ge 0, \quad \forall z \in C_n.$$

Since $F \subset C_n$, for each n in $\mathbb{N}$, we have

$$\langle z - x_n, \nabla g(x_n) - \nabla g(x_1)\rangle \ge 0, \quad \forall z \in F. \tag{4.8}$$

Letting $n \to \infty$ in (4.8), we deduce that

$$\langle z - u, \nabla g(u) - \nabla g(x_1)\rangle \ge 0, \quad \forall z \in F.$$

In view of (3.2), we have $u = \mathrm{proj}_F^g x_1$, which completes the proof. □

Here is the correct version of Assertion 1.1. Note that the construction of the closed convex sets C_n is a bit different from those in [9, Theorem 4.1]. Moreover, we can now deal with the more general case of *weakly* relative nonexpansive representations than that of relative nonexpansive representations in [9].

Theorem 4.3. *Let C be a nonempty, closed and convex subset of a uniformly convex and uniformly smooth Banach space E. Let $\mathcal{T} = \{T(s) : s \in S\}$ be a representation of a left reversible semigroup S as maps from C into C with common fixed point set $F(\mathcal{T}) \neq \emptyset$. Assume that every point in C is almost periodic for $\mathcal{T}$. Let X be a rich subspace of $\ell^\infty(S)$ for $\mathcal{T}$, and let $\{\mu_n\}_{n\in\mathbb{N}}$ be an asymptotically left invariant sequence of means on X. Let T_{μ_n} be the barycenter representation of $\mathcal{T}$ associated to each μ_n. Assume one of the following conditions holds.*

(a) all T_{μ_n} are relatively weak nonexpansive.
(b) all $T(s)$ are nonexpansive.
(c) all $T(s)$ are norm-to-weak continuous and quasi-nonexpansive.

Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty}\alpha_n = 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by the following algorithm

$$
\tag{4.9}
\begin{cases}
x_0 = x \in C \quad \textit{chosen arbitrarily}, \\
C_1 = C, \\
x_1 = \Pi_{C_1} x_0, \\
y_{n,k} = J^{-1}[\alpha_n Jx_1 + (1-\alpha_n)JT_{\mu_k}x_n)], \quad \forall k = 1,2,\dots n, \\
C_{n+1} = \{z \in C_n : \max_{1\le k\le n}\phi(z,y_{n,k}) \le \alpha_n\phi(z,x_1) + (1-\alpha_n)\phi(z,x_n)]\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_1.
\end{cases}
$$

Then $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to the common fixed point $\Pi_{F(\mathcal{T})}x_1$ of $\mathcal{T}$.

Proof. (a) Assume that all T_{μ_n} are relatively weak nonexpansive. We consider here the Bregman distance $D_g(x,y) = \phi(x,y)$ with $g(x) = \|x\|^2$. Then T_μ is Bregman relatively weak nonexpansive mappings from C into C for D_g. Applying Theorem 4.2 to the family $\{T_{\mu_n}\}$, we get a strong limit $u = \lim_n x_n = \Pi_{\bigcap_{n=1}^\infty F(T_{\mu_n})}x_1$, which is a common fixed point of all T_{μ_n}. It follows from (1.2) that $F(\mathcal{T}) = \bigcap_{n=1}^\infty F(T_{\mu_n})$. Hence, we have $u = \Pi_{F(\mathcal{T})}x_1$.

(b) Assume to start with all $T(s)$ being nonexpansive, and thus quasi-nonexpansive. Let T_μ be a barycenter of $\{T(s) : s \in S\}$ for a mean μ on X. We consider μ as a norm one functional of functions in s. Let $x_n \to u$ and $\lim_n \|T_\mu x_n - x_n\| = 0$. As in [9, p. 3416], we have $\|T_\mu x\| \le \mu\|T(\cdot)x\|$ for all x in C. Thus,

$$\|T_\mu u - u\| = \lim_n \|T_\mu u - T_\mu x_n\| \le \lim_n \mu\|T(\cdot)u - T(\cdot)x_n\| \le \lim_n \mu\|u - x_n\| = 0.$$

Therefore, $T_\mu u = u$. This says that all barycenters T_{μ_k} are weak relatively nonexpansive. We apply case (a).

(c) Assume in the beginning that all $T(s)$ are quasi-nonexpansive maps from C into C. The arguments in [9, p. 3417] shows that the barycenter representation T_μ of the family $\mathcal{T}$ is also quasi-nonexpansive for any mean μ on X. Now suppose further that all $T(s)$ are norm-to-weak continuous, and thus so are their barycenters T_{μ_k}. If $x_n \to u$ and $\lim_n \|T_{\mu_k}x_n - x_n\| = 0$, then by the norm-to-weak continuity of T_μ we have $T_\mu x_n \rightharpoonup T_\mu u$, and thus $T_{\mu_k}u = u$. Therefore all T_{μ_k} are relatively weak nonexpansive. We apply case (a) to finish the proof. $\square$

Remark 4.4. (1) As been pointed out earlier, closed quasi-nonexpansive maps are relatively weak nonexpansive, and thus so are norm-to-weak continuous quasi-nonexpansive maps. On the other hand, nonexpansive maps are norm-to-norm continuous, and thus (b) is indeed a special case of (c).

(2) Suppose instead all $T(s)$ are relatively weak nonexpansive. We do not know, however, if the barycenter T_μ for a mean μ on X is relatively weak nonexpansive as well.

Remark 4.5. Theorems 4.2 and 4.3 improve Assertion 1.1 in the following aspects.

(1) We extend the duality mapping J to the more general case, that is, the gradient ∇g of a convex, continuous and strongly coercive Bregman function g which is bounded, uniformly convex and uniformly smooth on bounded subsets.

(2) We extend our discussion from relatively nonexpansive mappings to Bregman weakly relatively nonexpansive mappings. We replace the assumption $F_a(T) = F(T)$ with the weaker one $F_{sa}(T) = F(T)$. Here, $F_a(T)$ and $F_{sa}(T)$ are the set of asymptotic fixed points and the set of strong asymptotic fixed points of T, respectively.

Remark 4.6. The main result of [26] gave a strong convergence theorem to approximate common fixed points of a family of closed relatively nonexpansive mappings, while the present paper give a strong convergence theorem to approximate common fixed points of a family of Bregman relatively weak nonexpansive mappings. We note that the proof of [26, Theorem 3.2], more precisely, line 15 where the authors used the closedness of the mappings S_λ, is not valid in our discussion, as Example 3.8 demonstrates. We note also that the proof of [11, Theorem 3.2], where the authors used the relatively nonexpansivity of the mappings S_λ, is not valid in our discussion, either. In fact, our result extends and improves the corresponding results of [11, 26].

Let E be a reflexive Banach space with the dual space E^*. Let $A : E \to 2^{E^*}$ be a set-valued mapping with $\operatorname{dom} A = \{x \in E : Ax \neq \emptyset\}$. The graph of A is $G(A) = \{(x, x^*) \in E \times E^* : x^* \in Ax\}$. The mapping $A \subset E \times E^*$ is said to be *monotone* if $\langle x - y, x^* - y^* \rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in A$. It is said to be *maximal monotone* if its graph is not contained in the graph of any other monotone operator on E. If $A \subset E \times E^*$ is maximal monotone, then the set $A^{-1}0 = \{z \in E : 0 \in Az\}$ is closed and convex. See [21] for details.

Let $g : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function. Let A be a maximal monotone operator from E to 2^{E^*}. For any $r > 0$, define the *g-resolvent* $\operatorname{Res}^g_{rA} : E \to \operatorname{dom} A$ by

$$\operatorname{Res}^g_{rA} = (\nabla g + rA)^{-1} \nabla g.$$

It is known that $\operatorname{Res}^g_{rA}$ is Bregman relatively weak nonexpansive and $A^{-1}(0) = F\left(\operatorname{Res}^g_{rA}\right)$ for each $r > 0$. Examples and some important properties of such operators are discussed in [1, 2, 23].

An application of Theorem 4.2 gives the following.

Theorem 4.7. *Let E be a reflexive Banach space and $g : E \to \mathbb{R}$ a strongly coercive Bregman function which is bounded, uniformly convex and uniformly smooth on bounded subsets of E. Let A be a maximal monotone operator from E to E^* such that $A^{-1}(0) \neq \emptyset$. Let $\{r_n\}_{n\in\mathbb{N}} \subset (0, +\infty)$ be a sequence of positive real numbers. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence generated by*

$$\begin{cases} x_0 = x \in E \quad \textit{chosen arbitrarily,} \\ C_1 = E, \\ x_1 = \operatorname{proj}^g_{C_1} x_0, \\ y_{n,k} = \nabla g^* \left[\alpha_n \nabla g(x_1) + (1 - \alpha_n) \nabla g \left(\operatorname{Res}^g_{r_k A} x_n\right)\right], \quad \forall k = 1, 2, \dots n, \\ C_{n+1} = \{z \in C_n : \max_{1\leq k\leq n} D_g(z, y_{n,k}) \leq \alpha_n D_g(z, x_1) + (1 - \alpha_n) D_g(z, x_n)\}, \\ x_{n+1} = \operatorname{proj}^g_{C_{n+1}} x_1, \quad \forall n = 1, 2, \dots . \end{cases}$$

Then the sequence $\{x_n\}_{n\in\mathbb{N}}$ *converges strongly to* $\mathrm{proj}^g_{A^{-1}(0)}x_1$ *as* $n \to \infty$.

5. A numerical example

In this section, in order to demonstrate the effectiveness, realization and convergence of Algorithm (4.1) in Theorem 4.2, we consider the following simple example.

Example 5.1. Let $E = \mathbb{R}$, $C = [0, +\infty)$ and $T_k : C \to C$ be defined by

$$T_k(x) = \begin{cases} 0, & \text{if} \quad x \in [0, 2], \\ e^{-k}x, & \text{otherwise.} \end{cases}$$

Then $\{T_k\}_{k\in\mathbb{N}}$ is a family of quasi-nonexpansive mapping from C into C such that $F = \bigcap_{k=1}^{+\infty} F(T_k) = \{0\}$. Indeed, for any $x \in (2, +\infty)$, we have

$$|T_k x - 0| = e^{-k}x \le |x - 0|, \quad \forall k \ge 1.$$

It is worth mentioning that T_k is neither nonexpansive nor continuous for all k in $\mathbb{N}$. Let $g(t) = t^2$ be the Bregman function on $\mathbb{R}$.

In this case, Algorithm (4.1) in Theorem 4.2 states as follows:

$$\begin{cases} x_0 = x \in (0, +\infty) \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}x_0, \\ y_{n,k} = \alpha_n x_1 + (1 - \alpha_n)T_k x_n, \ k = 1, 2, \cdots, n, \\ C_{n+1} = \{z \in C_n : \max_{1\le k\le n} |z - y_{n,k}|^2 \le \alpha_n|z - x_1|^2 + (1 - \alpha_n)|z - x_n|^2\}, \\ x_{n+1} = P_{C_{n+1}}x. \end{cases} \tag{5.1}$$

We set

$$H_{n,k} = \{z \in E : |z - y_{n,k}|^2 \le \alpha_n|z - x_1|^2 + (1 - \alpha_n)|z - x_n|^2\}.$$

Observe that

$$\begin{aligned} |z - y_{n,k}|^2 =& |\alpha_n(z - x_1) + (1 - \alpha_n)(z - T_k x_n)|^2 \\ =& \alpha_n(z - x_1)^2 + (1 - \alpha_n)(z - T_k x_n)^2 - \alpha_n(1 - \alpha_n)(T_k x_n - x_1)^2. \end{aligned}$$

It follows

$$H_{n,k} = \{z \in E : z \le \frac{\alpha_n(T_k x_n - x_1)^2}{2(x_n - T_k x_n)} + \frac{x_n + T_k x_n}{2}\}.$$

Note that $x_n - T_k x_n > 0$ if $x_n > 0$. Hence, $C_{n+1} = C_n \cap (\bigcap_{k=1}^n H_{n,k})$ is a closed interval for all $n = 0, 1, 2, \ldots$. Write $C_{n+1} = [a_{n+1}, b_{n+1}]$. Then

$$x_{n+1} = P_{C_{n+1}}x = \begin{cases} x, & \text{if } x \in [a_{n+1}, b_{n+1}]; \\ b_{n+1}, & \text{if } x > b_{n+1}; \\ a_{n+1}, & \text{if } x < a_{n+1}. \end{cases}$$

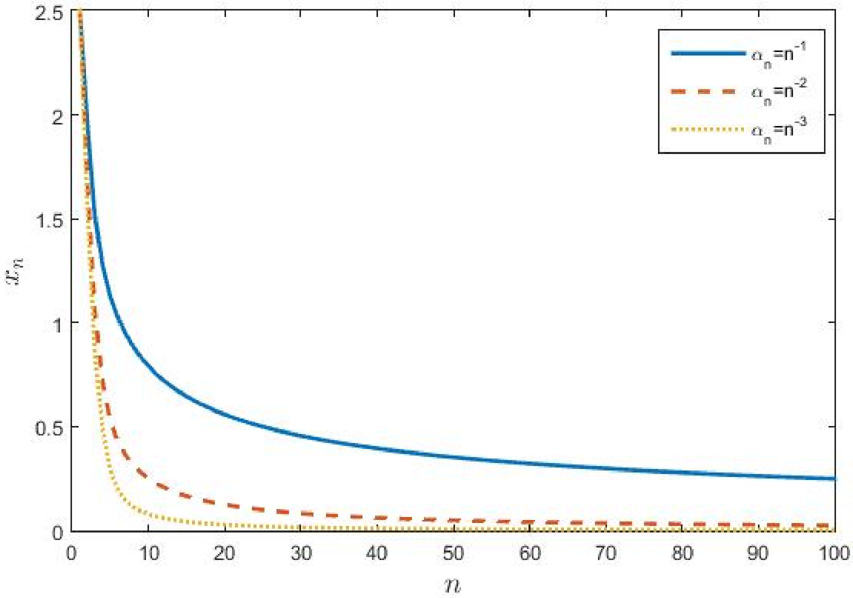

FIGURE 1. The plots of the sequence $\{x_n\}_{n\in\mathbb{N}}$ in Example 5.1 with initial value $x_0 = 1$ under different weight parameters α_n.

Choose $x_0 = x = 2.5$. The iteration process (5.1) produces

$$x_{n+1} = \min_{1\leq k\leq n} \left\{ \frac{\alpha_n(T_k x_n - x_1)^2}{2(x_n - T_k x_n)} + \frac{x_n + T_k x_n}{2} \right\}. \tag{5.2}$$

With different choices of the weights $\alpha_n = n^{-1}, n^{-2}, n^{-3}$, we demonstrate in Figure 1 the convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ generated by (5.2) to the unique common fixed point 0. Note that using smaller values of α_n means that the effect of x_1 in producing C_n is weakening. In this easy example, the efficiency of the algorithm is improved drastically.

References

[1] H. H. Bauschke, J. M. Borwein, P. L. Combettes, *Bregman monotone optimization algorithms*, SIAM J. Control Optim., 42 (2003) 596-636.

[2] M. J. Borwein, S. Reich, S. Sabach, *A charactrization of Bregman firmly nonexpansive operators using a new monotonicity concept*, Journal of Nonlinear and Convex Analysis, 12 (2011) 161-184.

[3] L. M. Bregman, *The relation method of finding the common point of convex sets and its application to the solution of problems in convex programming*, USSR Comput. Math. Math. Phys. 7 (1967) 200-217.

[4] D. Butnariu, A. N. Iusem, *Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization*, Kluwer Academic Publishers, Dordrecht 2000.

[5] D. Butnariu, E. Resmerita, *Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces*, Abstract and Applied Analysis, 2006 (2006) Art. ID 84919, 1-39.

[6] Y. Haugazeau, *Sur les inéquations variationnelles et la minimisation de fonctionnelles convexes*, Thése, Université de Paris, Paris, France, 1968.

[7] N. Hirano, K. Kido, W. Takahashi, *Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces*, Nonlinear Analysis, 12 (1998) 1269-1281.

[8] T. Ibaraki, W. Takahashi, *Strong convergence theorems for a finite family of nonlinear operators of firmly nonexpansive type in Banach spaces*, Nonlinear Analysis and Convex Analysis, S.-B. Hsu., H.-C. Lai, L.-J. Lin, W. Takahashi, T. Tanaka, J.-C. Yao (eds), Yokahama Publishers, Yokahama, 2009, pp. 49-62.

[9] K. S. Kim, *Convergence of a hybrid algorithm for a reversible semigroup of nonlinear operators in Banach space*, Nonlinear Analysis, 73 (2010) 3413-3419.

[10] T.-H. Kim, W. Takahashi, *Strong convergence of modified iterarion processes for relatively asymptotically nonexpansive mappings*, Taiwanese Journal of mathematics, Vol. 14, No. 6 (2010) 2163-2180.

[11] Y. Kimura, W. Takahashi, *On hybrid method for a family of reversible nonexpansive mappings in a Banach space*, Journal of Mathematical Analysis and Applications, 357 (2009) 356-363.

[12] F. Kohsaka, W. Takahashi, *Proximal point algorithms with Bregman functions in Banach spaces*, Journal of Nonlinear and Convex Analysis, Vol. 6, No. 3 (2005) 505-523.

[13] A. T.-M. Lau, H. Miyake, W. Takahashi, *Approximation of fixed points for amenable semigroups of nonexpansive mappings in Banach spaces*, Nonlinear Analysis, 67 (2007), 1211-1225. 39-57.

[14] S. Matsushita, W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, J. Approx. Theory (2005) 257-266.

[15] K. Nakajo, W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. 279 (2003) 372-379.

[16] E. Naraghirad, W. Takahashi, J.-C. Yao, *Generalized retraction and fixed point theorems using Bregman functions in Banach spaces*, Journal of Nonlinear and Convex Analysis, Vol. 13, No. 1 (2012) 141-156.

[17] E. Naraghirad, N.-C. Wong, J.-C. Yao, *Applications of the Bregman-Opial property to Bregman nonspreading mappings*, Abstract and Applied Analysis, Vol. 2014 (2014), Article ID 272867, 14 pages.

[18] E. Naraghirad, J.-C. Yao, *Bregman relatively weak nonexpansive mappings in Banach spaces*, Fixed Point Theory and Applications 2013, 2013:141 pp. 43.

[19] S. Reich, *A weak convergence theorem for the altering method with Bregman distances*, in: Theory and Applications of Nolinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, 1996, pp. 313-318.

[20] S. Reich, S. Sabach, *Existence and approximation of fixed points of Bregman firmly nonexpansive mappings in reflexive Banach spaces*, Fixed-Point Algorithms for Inverse Problems in Science and Engineering, Springer, New York, 2010, 299-314.

[21] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., 149 (1970) 75-88.

[22] R. T. Rockafellar, *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math., 33 (1970) 209-216.

[23] W. Takahashi, *Nonlinear Functional Analysis, Fixed Point Theory and Its Applications*, Yokahama Publishers, Yokahama, 2000.

[24] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokahama Publishers, Yokahama, 2000.

[25] W. Takahashi, Y. Takeuchi, R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, Journal of Mathematical Analysis and Applications, 341 (2008) 276-286.

[26] Z. Wang, Y. Su, D. Wang, Y. Dong, *A modified Halpern-type iteration algorithm for a family of hemi-relatively nonexoansive mappings and systems of equilibrium problems in Banch spaces*, Journal of Computational and Applied Mathematics, 235 (2011) 2364-2371.

[27] C. Zălinescu, *Convex Analysis in General Vector Spaces*, World Scientific Publishing Co. Inc., River Edge NJ, 2002.

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