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Disjointness preservers of AW^*-algebras

Jung-Hui Liu [a], Chun-Yen Chou [b,*], Ching-Jou Liao [c], Ngai-Ching Wong [d]

[a] School of Information Engineering, Sanming University, No. 25, Jing Dong Road, San Yuan District, Sanming City, Fujian 365004, China
[b] Department of Applied Mathematics, A330 Science and Engineering Building I, National Dong Hwa University, Hualien 97401, Taiwan
[c] Department of Mathematics, Hong Kong Baptist University, Hong Kong
[d] Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan

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ABSTRACT

In this paper, we study bijective linear maps $\theta : A \to B$ between AW^*-algebras preserving either zero products or range orthogonality. Such a map is automatically continuous, and provides an algebra or a $*$-algebra isomorphism $\pi(\cdot) = \theta(\cdot)\theta^{**}(1)^{-1}$ from A onto B. Our results extend previous works for the case of W^*-algebras, and also cover such maps between C^*-algebras satisfying an extra assumption on preserving abelian C^*-subalgebras.

* Corresponding author.
E-mail addresses: 18760259501@163.com (J.-H. Liu), choucy@mail.ndhu.edu.tw (C.-Y. Chou), cjliao@hkbu.edu.hk (C.-J. Liao), wong@math.nsysu.edu.tw (N.-C. Wong).

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1. Introduction

A C^*-algebra A carries many structures. Indeed, if A is abelian, then A is (complex) $*$-algebra isomorphic to the algebra $C_0(X)$ of continuous functions on a locally compact space X vanishing at infinity. In general, A is $*$-algebra isomorphic to a norm closed $*$-subalgebra of $B(H)$ of bounded linear operators on a complex Hilbert space H. We note that every $*$-algebra isomorphism of C^*-algebras is an isometry. Conversely, every bijective complex linear isometry Ψ between C^*-algebras also provides a Jordan $*$-algebra isomorphism $\Psi(\cdot)\Psi^{**}(1)^{-1}$, where Ψ^{**} is the double dual map of Ψ. Indeed, any one of the metric structure, the algebra structure, and the order structure, determines A. See, e.g., [12,21,18,19,10,26,27].

In this paper, we are looking for minimum conditions to ensure a bijective map between C^*-algebras being a ($*$-)algebra, or a Jordan ($*$-)algebra isomorphism. These minimum conditions we consider are the disjointness structures of operator algebras. In the context of operator algebras (on Hilbert spaces) there are at least four versions of disjointness: zero product ($ab = 0$), range orthogonality ($a^*b = 0$), domain orthogonality ($ab^* = 0$), and double orthogonality ($a^*b = ab^* = 0$). Of course, the range and the domain orthogonality are symmetric, and we do have only three different variants. We call a map θ between C^*-algebras a *disjointness preserver* if θ preserves any one of these four disjointness.

If the algebra is abelian, then all these concepts coincide. Let $\theta : C_0(X) \to C_0(Y)$ be a bijective linear map between abelian C^*-algebras preserving zero products, i.e.,

$$ab = 0 \quad\Longrightarrow\quad \theta(a)\theta(b) = 0.$$

Then θ is automatically continuous and assumes a weighted composition form $\theta(f) = h \cdot f \circ \sigma$ with a continuous function h on Y, which is bounded and away from zero, and a homeomorphism σ from Y onto X. See Proposition 2.3 below.

Zero product and orthogonality complex linear preservers $\theta : A \to B$ between general C^*-algebras are also well studied. In this case, all disjointness coincide on the set of positive elements. Suppose θ sends positive elements with zero products to (not necessarily self-adjoint) elements with zero products, i.e.,

$$ab = 0 \quad\Longrightarrow\quad \theta(a)\theta(b) = 0, \quad \forall a, b \in A_+.$$

Assume that θ is bijective and *continuous*. Then A and B are isomorphic as Jordan algebras. If θ preserves zero products (resp. range orthogonality) of arbitrary elements in A, then A and B are isomorphic as algebras (resp. $*$-algebras). In both cases,

$$\pi(\cdot) = \theta(\cdot)\theta^{**}(1)^{-1}$$

is a Jordan isomorphism and an algebra isomorphism (resp. $*$-algebra isomorphism), respectively. For the double orthogonality preservers, we also have a similar result. Let

$\theta : A \to B$ be a bounded bijective linear map between C^*-algebras preserving double orthogonality on positive elements. Then $\theta = \theta^{**}(1)\pi$, where $u = \theta^{**}(1)$ is an invertible multiplier of B with $u^*u = uu^*$ being central, and $\pi : A \to B$ is a Jordan $*$-isomorphism. The proofs make heavy uses of functional calculus, thanks to the continuity of θ. See, e.g., [37,34,7,3,22,38,5,36,23].

Without assuming continuity, we can only utilize pure algebraic technique. A few partial results exist in literature, e.g., for properly infinite unital C^*-algebras [28] and CCR C^*-algebras with Hausdorff spectrum [25,35]. In [24], it is showed that every bijective linear zero product (resp. range orthogonality) preserver θ between W^*-algebras provides an algebra isomorphism (resp. a $*$-algebra isomorphism) $\theta(\cdot)\theta(1)^{-1}$. Systematic approaches can be found in [7,5].

In this paper, we study disjointness preservers of AW^*-algebras. Recall that an AW^*-algebra A is a C^*-algebra such that the right annihilator of every subset S of A is a (norm closed) left ideal arising from some projection p in A, i.e.,

$$S_r^\perp = \{a \in A : aS = 0\} = Ap.$$

It is equivalent to saying that the left annihilator $S_l^\perp$ of every subset S of A is a right ideal qA for some projection q in A. In particular, every AW^*-algebra has an identity. Moreover, every abelian AW^*-algebra carries the form $C(X)$ for a compact Stonian space X, i.e., the closure of any open subset of X is open. It is also plain that every W^*-algebra is an AW^*-algebra.

Our main results extend those in [24] and state

Theorem 1.1. *Let $\theta : A \to B$ be a bijective additive map between AW^*-algebras.*

(a) *If θ preserves zero products in both directions, i.e.,*

$$ab = 0 \text{ in } A \iff \theta(a)\theta(b) = 0 \text{ in } B,$$

then A, B are ring isomorphic. If θ is also assumed complex linear, then $\theta(1)$ is a central invertible element, and $\theta(\cdot)\theta(1)^{-1}$ is an algebra isomorphism from A onto B.

(b) *If θ preserves range orthogonality in both directions, i.e.,*

$$a^*b = 0 \text{ in } A \iff \theta(a)^*\theta(b) = 0 \text{ in } B,$$

then A, B are $*$-ring isomorphic. If θ is also assumed complex linear, then $\theta(1)$ is an invertible element, and $\theta(\cdot)\theta(1)^{-1}$ is a $*$-algebra isomorphism from A onto B.

It is expected that we would have a result for double orthogonality preservers of AW^*-algebras as those stated in Theorem 1.1. The W^*-algebra version is done in [6]. We make a conjecture to respond to this concern.

Conjecture 1.2. Let $\theta : A \to B$ be a bijective additive map between AW^*-algebras preserving double orthogonality in both directions, i.e.,

$$a^*b = ab^* = 0 \text{ in } A \iff \theta(a)^*\theta(b) = \theta(a)\theta(b)^* = 0 \text{ in } B.$$

Then A, B are Jordan $*$-ring isomorphic. If θ is also assumed complex linear, then $\theta(1)$ is invertible such that $\theta(1)^*\theta(1) = \theta(1)\theta(1)^*$ is central, and $\theta(\cdot)\theta(1)^{-1}$ is a Jordan $*$-algebra isomorphism from A onto B.

Our ultimate goal is to verify the

Conjecture 1.3. Every bijective disjointness complex linear preserver between C^*-algebras is automatically continuous, and arises from an algebra, a $*$-algebra, or a Jordan $*$-algebra isomorphism.

Summarizing the beautiful extensions of the Dye theorem [9] from Hamhalter [13, 15,14], Lindenhovius states in his thesis [29] the following structure theorem about AW^*-algebras. Here we write $C(A)$ for the set of all commutative C^*-subalgebras of A, and order its members by set inclusion.

Proposition 1.4 *(see [29, Corollary 9.2.9]). Let A, B be AW^*-algebras. Then the following statements are equivalent:*

(a) *A and B are Jordan $*$-algebra isomorphic.*
(b) *$C(A)$ and $C(B)$ are order isomorphic.*
(c) *The lattice of projections in A and that in B are orthoisomorphic.*

Motivated by Proposition 1.4, we provide the following partial answer to Conjecture 1.3. Here, we write $C^*(h)$ for the abelian C^*-algebra generated by any self-adjoint element h.

Theorem 1.5. *Let $\theta : A \to B$ be a bijective complex linear disjointness preserver between C^*-algebras. Suppose that $\theta(C^*(h)) \in C(B)$, $\forall h \in A_{sa}$. Then θ is continuous. Moreover, if θ preserves zero products, or preserves range/domain (resp. double) orthogonality, then $\theta^{**}(1)$ is an invertible central multiplier of B, and $\theta(\cdot)\theta^{**}(1)^{-1}$ is a $*$-algebra (resp. Jordan $*$-algebra) isomorphism from A onto B.*

2. Preliminaries

The following results will be used in our arguments.

Proposition 2.1 *(Berberian [4, Theorem 1]). An AW^*-algebra is the ring generated by its projections if and only if it has no abelian summand.*

Proposition 2.2 *(Chebotar, Ke, Lee and Wong [7, Theorem 2.6]). Let $\mathcal{M}$ be a unital ring generated by its idempotents and θ a zero-product preserving additive map from $\mathcal{M}$ into a ring $\mathcal{N}$. Denote by $\mathcal{N}'$ the subring of $\mathcal{N}$ generated by $\theta(\mathcal{M})$. Then*

(i) $\theta(a)\theta(bc) = \theta(ab)\theta(c)$ *for all* $a, b, c \in \mathcal{M}$*;*
(ii) $\theta(1)\theta(a) = \theta(a)\theta(1)$ *for all* $a \in \mathcal{M}$*;*
(iii) $\theta(1)\theta(ab) = \theta(a)\theta(b)$ *for all* $a, b \in \mathcal{M}$*;*
(iv) θ *preserves commutativity;*
(v) *if* $\theta(1) = 0$ *then* $\theta(a)\theta(b) = 0$ *for all* $a, b \in \mathcal{M}$*;*
(vi) *if* $\theta(1)$ *is invertible in* $\mathcal{N}$*, or* $\mathcal{N}'$ *contains an identity, then*

$$\theta(a) = \theta(1)\varphi(a) \quad \text{for all } a \in \mathcal{M},$$

where φ *is a ring homomorphism from* $\mathcal{M}$ *into* $\mathcal{N}$*.*

For the abelian parts, we have the following results. Note that the spectrum X of any abelian AW^*-algebra $C(X)$ is compact, and thus realcompact. On the other hand, Araujo, Beckenstein and Narici provide in [2, Remarks 2] an example of an additive bijective zero product preserver of the one-dimensional abelian C^*-algebra $\mathbb{C}$ which is not either linear or a ring isomorphism.

Proposition 2.3. *If $\theta : C_0(X) \to C_0(Y)$ is an additive bijective map preserving zero products in both directions, then the realcompactifications of the locally compact spaces X and Y are homeomorphic. Moreover, θ is real linear if and only if θ is continuous; in this case, there exist a homeomorphism $\sigma : Y \to X$, a partition $Y = Y_1 \cup Y_2$ of clopen subsets of Y, and a bounded continuous scalar function h on Y away from zero such that*

$$\theta(f)(y) = \begin{cases} h(y) \cdot f(\sigma(y)) & \text{on } Y_1, \\ h(y) \cdot \overline{f(\sigma(y))} & \text{on } Y_2, \end{cases} \qquad \forall f \in C_0(X).$$

If θ is complex linear then $Y = Y_1$, and $\theta(f) = h \cdot f \circ \sigma$ for all f in $C_0(X)$.

Proof. The first assertion is [2, Proposition 2]. It is plain that the additive map θ is real linear when θ is continuous. When θ is real linear, the remaining assertions can be obtained from the results in [16] for the compact case, and those in [11,17] for the locally compact case. $\square$

In the process, we also need

Proposition 2.4 *(Yen [39, Lemma 2.1]). Any a in an AW^*-algebra A is of the form $a = w(a^*a)^{1/2}$, where w is a partial isometry in A with $s_l(a) = ww^*$ being the left support projection of a, and $s_r(a) = w^*w$ being the right support projection of a.*

Proposition 2.5 *(Cuntz [8]). Every linear functional of a C^*-algebra A, which is continuous on every abelian C^*-subalgebra $C^*(h)$ generated by a self-adjoint element h in A, is continuous on A.*

3. The results and proofs

The following proof of Theorem 1.1 is modeled on that for the corresponding results for W^*-algebras in [24, Theorem 1.3]. In [24], one observes that the assertions hold for finite type I W^*-algebras, and the non finite type I summands of W^*-algebras are linear spans of their projections. However, we do not know if a similar property holds for AW^*-algebras with no finite type I summand. Instead, we decompose an AW^*-algebra into a direct sum of an abelian ideal and another ideal with no abelian summand. The latter is generated by its projections as a ring.

Proof of Theorem 1.1. Let z be a central projection in A such that the ideal $A_1 = (1-z)A$ is abelian, and the ideal $A_2 = zA$ contains no abelian summand. Similarly, we write $B = B_1 + B_2$ with $B_1 = (1-z')B$ and $B_2 = z'B$. Note that as norm closed two-sided ideals of a C^*-algebra, all A_1, A_2, B_1, B_2 are self-adjoint.

(a) As $A_1A_2 = A_2A_1 = \{0\}$, we have $\theta(A_1)\theta(A_2) = \theta(A_2)\theta(A_1) = 0$. Let L_i, R_i be the norm closed left and right ideals of B generated by $\theta(A_i)$, for $i = 1, 2$, respectively. It is clear that $L_1R_2 = L_2R_1 = \{0\}$. As θ^{-1} also preserves zero products, we have $\theta^{-1}(L_1)A_2 = A_1\theta^{-1}(R_2) = A_2\theta^{-1}(R_1) = \theta^{-1}(L_2)A_1 = 0$. Therefore, $\theta^{-1}(L_i), \theta^{-1}(R_i) \subseteq A_i$ for $i = 1, 2$, respectively. It follows that $\theta(A_i) = L_i = R_i$ is a norm closed two-sided ideal of B, for $i = 1, 2$. Since $B = \theta(A) = \theta(A_1) + \theta(A_2)$, we see that $\theta(A_1) = \theta(A_2)_l^{\perp} = \theta(A_2)_r^{\perp}$, and thus there is a central projection q in B such that $\theta(A_1) = (1-q)B$ and $\theta(A_2) = qB$.

By Proposition 2.1, the AW^*-algebra A_2 is generated by its projections as a ring. It follows from Proposition 2.2(vi), there is a ring isomorphism $\varphi : A_2 \to qB$ such that

$$\theta(a) = \theta(z)\varphi(a), \quad \forall a \in A_2.$$

It follows from Proposition 2.2(ii) that $\theta(z)$ is a central element in the ideal qB, and thus in B. Let $d \in A_2$ such that $\theta(d) = q$. Then Proposition 2.2(iii) gives

$$\theta(z)\theta(d^2) = \theta(d)^2 = q.$$

It follows $\theta(z)$ is invertible in qB, and its inverse $\theta(d^2)$ is also a central element.

Define $\pi_2 : A_2 \to qB$ by

$$\pi_2(a) = \theta(d^2)\theta(a), \quad \forall a \in A_2.$$

Then π_2 is additive and bijective. Moreover,

$$\pi_2(z) = \theta(d^2)\theta(z) = q,$$

and by Proposition 2.2(iii) again,

$$\pi_2(ab) = \theta(d^2)\theta(z)\theta(ab)\theta(d^2) = \theta(d^2)\theta(a)\theta(b)\theta(d^2) = \pi_2(a)\pi_2(b), \quad \forall a, b \in A_2.$$

Therefore, π_2 is a ring isomorphism from A_2 onto qB. If qB has an abelian summand I, then $\pi_2^{-1}(I)$ will be an abelian direct summand in A_2. As A_2 has no nonzero abelian summand, we have $I = \{0\}$. In particular, $\theta(A_2) = qB \subseteq B_2$. Applying the same arguments to θ^{-1}, we see that $\theta^{-1}(B_2) \subseteq A_2$. Consequently, $\theta(A_2) = B_2$, and hence $\theta(A_1) = B_1$.

We have already seen that π_2 is a ring isomorphism from A_2 onto B_2. On the other hand, Proposition 2.3 says that the abelian AW^*-algebras A_1 and B_1 have homeomorphic compact spectrums, and thus they are algebra $*$-isomorphic. Therefore, A and B are ring isomorphic.

If θ is also assumed to be linear, then π_2 is an algebra isomorphism from A_2 onto B_2. On the other hand, Proposition 2.3 ensures that $\theta(1 - z)$ has an inverse w in the abelian AW^*-algebra B_1, and $\pi_1 := w\theta|_{A_1}$ is an algebra isomorphism from A_1 onto B_1. Consequently, $\theta(1) = \theta(1 - z) + \theta(z)$ is a central invertible element in B, and $\pi(\cdot) := \theta(1)^{-1}\theta(\cdot) = \theta(\cdot)\theta(1)^{-1}$ is an algebra isomorphism from A onto B.

(b) As $A_1^* A_2 = \{0\}$, we have $\theta(A_1)^*\theta(A_2) = 0$. Let R_1, R_2 be the norm closed right ideals of B generated by $\theta(A_1), \theta(A_2)$, respectively. It is clear that $R_1^* R_2 = \{0\}$. Moreover, the identity $B = \theta(A) = \theta(A_1) + \theta(A_2)$ forces $R_1 = \theta(A_1) = (R_2^*)_l^\perp$ and $R_2 = \theta(A_2) = (R_1^*)_l^\perp$, respectively. Let q be the projection in B such that $\theta(A_1) = (1 - q)B$ and $\theta(A_2) = qB$.

Consider any projection e and any arbitrary elements b, c in A. Since

$$(ec)^*(b - eb) = (c - ec)^* eb = 0,$$

we have

$$\theta(ec)^*(\theta(b) - \theta(eb)) = (\theta(c)^* - \theta(ec)^*)\theta(eb) = 0.$$

It follows

$$\theta(c)^*\theta(eb) = \theta(ec)^*\theta(eb) = \theta(ec)^*\theta(b).$$

By Proposition 2.1, we have

$$\theta(c)^*\theta(a^* b) = \theta(ac)^*\theta(b), \quad \forall a \in A_2, \ \forall b, c \in A. \tag{3.1}$$

Put $c = z$ in (3.1), we have

$$\theta(z)^*\theta(a^*b) = \theta(a)^*\theta(b), \quad \forall a \in A_2, \forall b \in A. \tag{3.2}$$

Let $d \in A_2$ such that $\theta(d) = q$. By (3.2), we have

$$\theta(z)^*\theta(d^*d) = \theta(d)^*\theta(d) = q. \tag{3.3}$$

Setting $b = z$ in (3.2), we have

$$\theta(z)^*\theta(a^*) = \theta(a)^*\theta(z), \quad \forall a \in A_2.$$

In particular, as $\theta(z) = q\theta(z) \in \theta(A_2) = qB$, it follows

$$\theta(z)^*B = \theta(z)^*qB = Bq\theta(z) = B\theta(z)$$

is a two-sided self-adjoint ideal of B. Let w be the central projection in B such that wB is the annihilator ideal $(B\theta(z))_{ll}^{\perp\perp}$ generated by $B\theta(z)$.

Let $s_l(\theta(z))$ and $s_r(\theta(z))$ be the left and right support projections of $\theta(z)$ respectively, which are in B by Proposition 2.4. Observe

$$\theta(a)^*\theta(z) = \theta(a)^*\theta(z)s_r(\theta(z)), \quad \forall a \in A_2.$$

Consequently,

$$w \leq s_r(\theta(z)). \tag{3.4}$$

Since $\theta(z) = q\theta(z)$, and by (3.3), $q = \theta(z)^*\theta(d^*d) \in wB$, we also have

$$s_l(\theta(z)) \leq q \leq w. \tag{3.5}$$

Because $s_r(\theta(z))$ is equivalent to $s_l(\theta(z))$, they have the same central support. It then follows from (3.4) and (3.5) that

$$w = s_r(\theta(z)) \geq q \geq s_l(\theta(z)).$$

Let

$$q_1 = (1 - z')q \in B_1 \quad \text{and} \quad w_1 = (1 - z')w \in B_1.$$

Since B_1 is an abelian AW*-algebra, we have $q_1 = w_1$ is a central projection in B. Note that the norm closed two-sided ideal $q_1B \subseteq qB_1 = \theta(A_2) \cap B_1$, and $q_1\theta(z) = \theta(z)q_1$.

Argue similarly with $\Psi = \theta^{-1} : B \to A$, we have

$$\Psi(1)^*\Psi(ry) = \Psi(r)^*\Psi(y)$$

for every projection r and for every element y in B. Putting $y = \theta(z)$, we get

$$\Psi(1)^*\Psi(r\theta(z)) = \Psi(r)^* z.$$

If r is a projection in B with $r \leq q$ then $r \in qB = \theta(A_2)$, and thus

$$\Psi(1)^*\Psi(r\theta(z)) = \Psi(r)^*. \tag{3.6}$$

Since Ψ is one-to-one, $r\theta(z) = 0$ implies $r = 0$ by (3.6). Let $x \in B$ such that $x\theta(z) = xq\theta(z) = 0$. Then, $\theta(z)^* qx^* xq\theta(z) = 0$. This implies $\theta(z)^* r\theta(z) = 0$, and hence $r = 0$, for every spectral projection r of qx^*xq. Thus, $xq = 0$. As a result, the right multiplication operator $R_{\theta(z)} : q_1B \to q_1B$, sending xq_1 to $xq_1\theta(z)$, is one-to-one. Moreover, by (3.3) we have

$$q_1B\theta(z) = B\theta(z)q_1 \supseteq B\theta(d^*d)^*\theta(z)q_1 = q_1B.$$

So $R_{\theta(z)}$ is a bounded bijective linear map from q_1B onto itself. Consider also the right multiplication operator $R_{\theta(d^*d)^*} : q_1B \to q_1B$ sending xq_1 to $xq_1\theta(d^*d)^*$. The identity (3.3) says that

$$R_{\theta(z)}R_{\theta(d^*d)^*} = R_{q_1}.$$

Here, R_{q_1} is the identity map from q_1B onto q_1B. Since $R_{\theta(z)}$ is bijective, we have

$$R_{\theta(d^*d)^*}R_{\theta(z)} = R_{q_1}.$$

In particular,

$$q_1\theta(z)\theta(d^*d)^* = q_1. \tag{3.7}$$

Define $\pi_{21} : A_2 \to q_1B$ by

$$\pi_{21}(a) = q_1\theta(a)\theta(d^*d)^*, \quad \forall a \in A_2.$$

It is easy to see that π_{21} is onto, $\pi_2(z) = q_1$, the identity of the AW*-algebra q_1B, and by (3.7) and (3.2),

$$\begin{aligned}
\pi_{21}(a^*b) &= q_1\theta(a^*b)\theta(d^*d)^* \\
&= q_1\theta(d^*d)\theta(z)^*\theta(a^*b)\theta(d^*d)^* \\
&= q_1\theta(d^*d)\theta(a)^*\theta(b)\theta(d^*d)^* \\
&= \pi_{21}(a)^*\pi_{21}(b), \quad \forall a, b \in A_2.
\end{aligned}$$

In other words, π_{21} is a surjective additive $*$-homomorphism. It then follows that $q_1 B \subseteq B_1$ contains no abelian summand, as A_2 does not either. This forces $q_1 = 0$, and thus $\theta(A_2) = qB \subseteq B_2$. Dealing with $\Psi = \theta^{-1}$, we see also that $\Psi(B_2) \subseteq A_2$. It follows $\theta(A_2) = B_2$, and thus $\theta(A_1) = B_1$.

At this stage, one have already seen that $q = z'$ is a central projection in B. Repeating some of the above arguments with q_1 replaced by q, one can see that

$$\theta(z)^*\theta(d^*d) = \theta(z)\theta(d^*d)^* = q.$$

Similarly, the map $\pi_2 : A_2 \to B_2$, defined by

$$\pi_2(a) = \theta(a)\theta(d^*d)^*, \quad \forall a \in A_2,$$

is a $*$-ring isomorphism. On the other hand, it follows from Proposition 2.3 that the abelian AW*-algebras A_1 and B_1 have homeomorphic compact spectrum, and thus they are $*$-algebra isomorphic. Consequently, $A = A_1 + A_2$ is $*$-ring isomorphic to $B = B_1 + B_2$.

If θ is also assumed linear, we see that π_2 is a $*$-algebra isomorphism. Moreover, by Proposition 2.3 again, we see that $\theta(1-z)$ is invertible in $B_1 = (1-q)B$ and there is a $*$-algebra isomorphism $\pi_1 : A_1 \to B_1$ such that $\theta(a) = \pi_1(a)\theta(1-z), \forall a \in A_1$. In conclusion, $\theta(1)$ is invertible in B and the map $\pi : A \to B$ defined by $\pi(a) = \theta(a)\theta(1)^{-1}$ is a $*$-algebra isomorphism. $\square$

As in [24], the following two results follow easily from Propositions 2.1 and 2.2, and the arguments in the proof of Theorem 1.1.

Proposition 3.1. *Let A be an AW*-algebra containing no abelian summand. Let B be a unital algebra. Let $\theta : A \to B$ be an additive map satisfying the condition:*

$$ab = 0 \text{ in } A \quad \Longrightarrow \quad \theta(a)\theta(b) = 0 \text{ in } B. \tag{3.8}$$

Consider the following conditions. We have (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

(1) *θ is surjective.*
(2) *$\theta(1)$ is a central invertible element in B.*
(3) *There exists a ring homomorphism π from A into B such that*

$$\theta(a) = \theta(1)\pi(a) = \pi(a)\theta(1), \quad \forall a \in A.$$

Proposition 3.2. *Let A, B be two AW*-algebras. Suppose A contains no abelian summand. Let $\theta : A \to B$ be an additive map satisfying the condition:*

$$a^*b = 0 \text{ in } A \quad \Longrightarrow \quad \theta(a)^*\theta(b) = 0 \text{ in } B. \tag{3.9}$$

Consider the following conditions. We have (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

(1) *θ is bijective, and the reverse implication in* (3.9) *also holds.*
(2) *$\theta(1)$ is invertible.*
(3) *There exists a $*$-ring homomorphism π from A into B such that*

$$\theta(a) = \pi(a)\theta(1), \quad \forall a \in A.$$

Remark 3.3. In [20], it states the observations of Marcoux [31–33] that the following simple unital C^*-algebras are the linear spans of projections.

- Simple purely infinite ones.
- AF-algebras with finitely many extremal tracial states.
- AT-algebras with real rank zero and finitely many extremal tracial states.
- Certain AH-algebras with real rank zero, bounded dimension growth, and finitely many extremal tracial states.

It is then a routine matter to apply the arguments in the proof of Theorem 1.1 to obtain similar results as stated in Propositions 3.1 and 3.2 for these types of C^*-algebras.

Proof of Theorem 1.5. We first show that θ is continuous on A. Let h be a self-adjoint element in A and $C^*(h)$ be the abelian C^*-subalgebra of A generated by h. By assumption $\theta(C^*(h))$ is an abelian C^*-subalgebra of B. Now the restriction map $\theta\mid_{C^*(h)}$ of θ is a bijective disjointness preserving (complex) linear map between abelian C^*-algebras, and thus it is continuous by Proposition 2.3. Let ψ be any bounded linear functional of B. Then $\psi \circ \theta$ is continuous on $C^*(h)$. It follows from Proposition 2.5, and the closed graph theorem, that θ is continuous on A.

Suppose now that θ preserves zero products, range orthogonality or double orthogonality. By the respective established results for bounded linear disjointness preservers of C^*-algebras as in [37,34,7,38,5,23] (see also [30]), we see that $\pi(\cdot) = \theta(\cdot)\theta^{**}(1)^{-1}$ is an algebra (resp. $*$-algebra, Jordan $*$-algebra) isomorphism from A onto B and $\theta^{**}(1)$ is an invertible central multiplier (resp. invertible multiplier) of B when θ preserves zero products (resp. range, double orthogonality). The case for domain orthogonality preservers is similar.

Assume θ preserves zero products. We want to show that π is a $*$-algebra isomorphism. The goal is to show that $\pi(a)$ is self-adjoint whenever a is. Since $\theta(C^*(a))$ is an abelian C^*-algebra, $\theta(a)$, and thus $\pi(a) = \theta(a)\theta^{**}(1)^{-1}$, is normal. It follows from the observation

$$\|e^{it\pi(a)}\| = \|\pi(e^{ita})\| \leq \|\pi\|, \quad \forall t \in \mathbb{R},$$

that the spectrum of $\pi(a)$ consists of real numbers. Thus $\pi(a)$ is self-adjoint. Consequently, π is a $*$-algebra isomorphism, as asserted.

Finally, we show that the invertible element $u = \theta^{**}(1)$ is a central multiplier of B when θ preserves range, domain, or double orthogonality. In this case, π is a Jordan

$*$-algebra isomorphism, and thus $\pi(C^*(h)) = C^*(\pi(h))$ for every self-adjoint h in A. We see that the multiplication $b \mapsto bu$ sends abelian C^*-subalgebras $C^*(k)$ to abelian C^*-subalgebras of B for every self-adjoint k in B.

Recall that a projection p in B^{**} is called *open* by Akemann [1], if there is a net $\{a_i\}$ in B such that $0 \leq a_i \uparrow p$. Let p, q be any open projections in B^{**} such that $pq = 0$. Let $\{a_i\}$ and $\{b_j\}$ be nets in B such that $0 \leq a_i \uparrow p$ and $0 \leq b_j \uparrow q$. In particular, $a_i b_j = 0$ for all indices i, j. Being respectively the positive and negative parts of the self-adjoint element $a_i - b_j$, both a_i and b_j belong to $C^*(a_i - b_j)$. Since $C^*(a_i - b_j)u$ is commutative and u is invertible, we have

$$a_i u b_j = b_j u a_i$$

for all i, j. It follows

$$puq = qup = 0. \tag{3.10}$$

Let r be the largest open projection in B^{**} dominated by $1 - p$. In other words, the closure $\overline{r} = 1 - p$. Since (3.10) holds for all open projections q in B^{**} orthogonal to p, we see that

$$pur = rup = 0.$$

By the proof of [1, Proposition II.12], we see that $ar = 0$ implies $a\overline{r} = 0$ for any a in B. Recall that $\{a_i\}$ is a net in B such that $0 \leq a_i \uparrow p$. Observe that for all index i we have

$$0 \leq ru^* a_i^2 ur \leq ru^* pur = 0.$$

Therefore, all $a_i ur = 0$. Since u is a multiplier of B, we have $a_i u \in B$, and hence all $a_i u\overline{r} = 0$. This gives eventually

$$pu(1 - p) = pu\overline{r} = 0.$$

Similarly, we have

$$(1 - p)up = \overline{r}up = 0.$$

Consequently,

$$pu = pup = up$$

for every open projection p in B^{**}. Thus, u is central as asserted. $\square$

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