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Research Article

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Generalized n-circular projections on JB*-triples and Hilbert $C_0(\Omega)$-modules

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Abstract: Being expected as a Banach space substitute of the orthogonal projections on Hilbert spaces, generalized n-circular projections also extend the notion of generalized bicontractive projections on JB*-triples. In this paper, we study some geometric properties of JB*-triples related to them. In particular, we provide some structure theorems of generalized n-circular projections on an often mentioned special case of JB*-triples, i.e., Hilbert C*-modules over abelian C*-algebras $C_0(\Omega)$.

Keywords: Projection, JB*-triple, Hilbert C*-module

MSC: 47A65, 17C65, 47A62

To the memory of our beloved Professor James E. Jamison.

1 Introduction

A projection $P : E \to E$ on a Banach space E is a bounded linear operator such that $P^2 = P$. The range space PE has a topological complementary subspace $(I - P)E$, in the sense that $E = PE + (I - P)E$ is a topological direct sum of closed linear subspaces. An orthogonal projection $q : H \to H$ on a Hilbert space H is a bounded *self-adjoint* projection, i.e., $q = q^2 = q^*$. In this case, the range space qH has an orthogonal complementary subspace $(I-q)H$, in the sense that $H = qH + (I-q)H$ with $qH \perp (I-q)H$, or equivalently, $\|qh + \alpha(I-q)k\| = \|qh - \alpha(I-q)k\|$ for all scalars α and all vectors h, k in H.

There are non-orthogonal projections on H. For example, $P(x, y) = (x-y/2, 0)$ is a non-orthogonal projection on the two-dimensional Hilbert space, with complementary projection $I - P$ sending (x, y) to $(y/2, y)$. Clearly, orthogonal projections carry richer structure than non-orthogonal projections.

Recently, there are some efforts in looking for suitable generalizations of orthogonal projections in the Banach space setting. See [1, 2, 7–16, 18, 23–25, 29–31, 34, 35, 38, 39, 42, 43]. The main task is to get rid of the involution in defining an orthogonal projection.

Observe that a bounded linear operator $P : E \to E$ on a Banach space is a projection if and only if $T = 2P - I$ is a symmetry, i.e., $T^2 = I$. For an orthogonal projection q on a Hilbert space, $U = 2q - I$ is a self-adjoint symmetry. It amounts to say that U is a self-adjoint unitary, or equivalently, a surjective (linear) isometry with spectrum $\sigma(U) = \{1, -1\}$. In this set up, q and $I - q$ are the eigenprojections of U associated to the eigenvalues 1

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and -1, respectively. This motivates us to call a (necessarily proper) projection P on a Banach space E a *generalized orthogonal projection* if P and $I - P$ are the eigenprojections of a surjective isometry T on E with $T^2 = I$ associated to its eigenvalues 1 and -1, respectively. In this setting,

$$T = P - (I - P) = 2P - I \quad \text{and} \quad P = \frac{I + T}{2}. \tag{1}$$

Generalized orthogonal projections give rise to *orthogonal* decompositions of Banach spaces into closed subspaces in the following sense. According to [41], two elements x, y in a Banach space E are said to be *Roberts orthogonal* if $\|x + \lambda y\| = \|x - \lambda y\|$ for all scalars λ. It is just by the definition that elements in PE are Roberts orthogonal to elements in $(I - P)E$ if and only if P is a generalized orthogonal projection. The Roberts orthogonality can be considered as one of the strongest orthogonalities among others commonly used in the general Banach space setting; see, e.g., [3, 4].

Note that the spectrum of a surjective isometry is contained in the unit circle $\mathbb{T}$ of the complex plane. We call a projection P on a Banach space E a *generalized bicircular projection* if there is a surjective isometry $T : E \to E$ with spectrum $\sigma(T) = \{e^{2\pi r \mathbf{i}}, e^{2\pi s \mathbf{i}}\}$ for some real numbers r, s such that P and $I - P$ are eigenprojections of T associated to $e^{2\pi r \mathbf{i}}$ and $e^{2\pi s \mathbf{i}}$, respectively. Replacing T with $e^{-2\pi r \mathbf{i}}T$, we can assume that $e^{2\pi r \mathbf{i}} = 1$. In this setting,

$$T = P + e^{2\pi s \mathbf{i}}(I - P) \quad \text{and} \quad P = \frac{T - e^{2\pi s \mathbf{i}}I}{1 - e^{2\pi s \mathbf{i}}}.$$

When $e^{2\pi s \mathbf{i}} = -1$, i.e., $2\pi s$ is the straight angle $(\operatorname{mod} 2\pi)$, we see that $P = \frac{I+T}{2}$ is a generalized orthogonal projection.

There is another way to make the generalization. Observe that a projection q on a Hilbert space is self-adjoint if $e^{2\pi t \mathbf{i} q}$ is a unitary for all real t. Note that

$$\begin{aligned} e^{2\pi t \mathbf{i} q} &= \sum_{n=0}^{\infty} \frac{(2\pi t\, \mathbf{i} q)^n}{n!} = I + q\left[\sum_{n=1}^{\infty} \frac{(2\pi t\, \mathbf{i})^n}{n!}\right] \\ &= I + q[e^{2\pi t \mathbf{i}} - 1] = e^{2\pi t \mathbf{i}} q + (I - q). \end{aligned}$$

Therefore, q is self-adjoint if and only if $q + e^{-2\pi t \mathbf{i}}(I - q)$ is a surjective isometry for all real t. We call a projection P on a Banach space a *hermitian projection* if $e^{2\pi t \mathbf{i} P}$ is a surjective isometry for all real t, and we call P a *bicircular projection* if $P + e^{2\pi s \mathbf{i}}(I - P)$ is a surjective isometry for all real s. Above arguments say that hermitian projections are exactly bicircular projections, see [34, Lemma 2.1].

In [39, Theorem 1], one sees that a projection P on a Banach space E is hermitian if it is a generalized bicircular projection for some irrational angle s in $\mathbb{R} \setminus \mathbb{Q}$, i.e., $T = P + e^{2\pi s \mathbf{i}}(I - P)$ is a surjective isometry. Indeed, $T^n = P + e^{2ns\pi \mathbf{i}}(I - P)$ will be again surjective isometry on E for all $n = 1, 2, \ldots$. Since s is irrational, the set $\{e^{2ns\pi \mathbf{i}} : n = 1, 2, \ldots\}$ is dense in the complex unit circle $\mathbb{T}$. With a continuity argument, we see that $P + e^{2\pi t \mathbf{i}}(I - P)$ is a surjective isometry for all real t, and thus P is hermitian. Therefore, the study of generalized bicircular projections emphasis on those associated to rational angles. According to [39, Theorem 3], for each rational angle r there is a Banach space E_r and a non-hermitian generalized bicircular projection P on E_r such that $P + e^{2\pi r \mathbf{i}}(I - P)$ is a surjective isometry on E_r.

Obviously, all the notions of generalized orthogonal projections, generalized bicircular projections, bicircular projections, and hermitian projections coincide with that of orthogonal projections in the Hilbert space setting. On the other hand, only generalized orthogonal projections survive on the C*-algebra $C_0(\Omega)$ of continuous functions when the underlying locally compact Hausdorff space Ω is connected. If Ω has a proper component Y then the projection $Pf = \mathbf{1}_Y f$ is a nonzero hermitian projection on $C_0(\Omega)$, where $\mathbf{1}_Y$ is the indicator function of Y. See, e.g., [7, 12, 32].

After the efforts in studying generalized bicircular projections, people define the notion of generalized tricircular projections. We call a projection P_0 on a Banach space E a *generalized tricircular projection* if there is a surjective isometry $T : E \to E$ with spectrum $\sigma(T) = \{\lambda_0, \lambda_1, \lambda_2\}$ consisting of three distinct eigenvalues of modulus one, such that P_0 is the eigenprojection of T associated to λ_0. Replacing T with $\overline{\lambda_0}T$, we can assume $\lambda_0 = 1$. Let P_1 and P_2 be the nonzero eigenprojections of T associated to the other two modulus one eigenvalues λ_1 and λ_2,

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respectively. Then

$$P_0 + P_1 + P_2 = I \quad \text{and} \quad P_0 + \lambda_1 P_1 + \lambda_2 P_2 = T.$$

Generalized tricircular projections on $C(\Omega)$ for a connected compact Hausdorff space Ω were studied in [2].

Analogously, one can study projections P_0 on E which are eigenprojections of surjective isometries on E with spectrum consisting of n distinct eigenvalues, and without loss of generality it can be assumed that P_0 is associated to the eigenvalue 1. This leads to the notion of generalized n-circular projections.

In this paper, we study generalized n-circular projections on JB*-triples and Hilbert $C_0(\Omega)$-modules. As a special case of JB*-triples, Hilbert $C_0(\Omega)$-modules $\mathcal{M}$ are in one-to-one correspondence to Hilbert bundles $\mathcal{H}$ over Ω such that $\mathcal{M}$ is the Banach space of continuous sections $f : \Omega \to \mathcal{H}$ vanishing at infinity, and every surjective isometry between Hilbert $C_0(\Omega)$-modules arises exactly from an isometric isomorphism between the underlying Hilbert bundles. See, e.g., [26, 27]. In this sense, next to Hilbert spaces, Hilbert $C_0(\Omega)$-modules are the natural objects on which we consider generalized n-circular projections.

2 Generalized n-circular projections

Ever since the papers [20] and [42], various classes of projections on JB*-triples attract many attention in literature. According to [22, Theorem 4], see also [29, Theorem 2.1], every generalized bicircular projection on a JB*-triple is generalized orthogonal or hermitian (hence also generalized orthogonal). Below we give a geometric meaning of such projections. We shall say that a subtriple $\mathcal{J}_1$ is *complementary* to $\mathcal{J}_2$ if $\ker(\mathcal{J}_1) = \mathcal{J}_2$ and $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$. Here,

$$\ker(\mathcal{J}_1) := \{y \in \mathcal{J} : \{x, y, z\} = 0, \forall x, z \in \mathcal{J}_1\}$$

is an inner ideal of $\mathcal{J}$. Note that $\mathcal{J}_1 \cap \ker(\mathcal{J}_1) = \{0\}$. See [20] for details. We also refer the readers to [17] for the general theory of JB*-triples.

Theorem 2.1. *Let P be a generalized bicircular projection on a JB*-triple $\mathcal{J}$. Then P is a generalized orthogonal projection, and $\mathcal{J} = P\mathcal{J} + (I - P)\mathcal{J}$ is a direct sum of JB*-subtriples. Furthermore, P is hermitian if and only if $P\mathcal{J}$ and $(I - P)\mathcal{J}$ are complementary to each other.*

Proof. By [39, Corollary 2], every generalized bicircular projection P is bicontractive, i.e., both P and $I - P$ are contractive projections. It is pointed out in [22, Theorem 4] that a bicontractive projection of a JB*-triple is also a generalized orthogonal projection.

Since the range of a bicontractive projection of a JB*-triple is a JB*-subtriple ([22, Proposition 3.1]), P gives rise to a decomposition of the JB*-triple $\mathcal{J} = P\mathcal{J} + (I - P)\mathcal{J}$ into a direct sum of two JB*-subtriples.

It is indicated in [42] that a projection P on $\mathcal{J}$ is hermitian if and only if P is a *skew derivation*, which means

$$P\{x, y, z\} = \{Px, y, z\} - \{x, Py, z\} + \{x, y, Pz\}, \quad \forall x, y, z \in \mathcal{J}.$$

Suppose that P is hermitian. We claim that in this case the JB*-subtriples $P\mathcal{J}$ and $(I - P)\mathcal{J}$ are complementary to each other. Notice that for any modulus one complex scalar λ, the surjective isometry $T = P + \lambda(I - P)$ is a triple isomorphism by [36, Proposition 5.5], i.e., T preserves the triple products. Consequently, we have

$$\begin{aligned} T\{Px, (I - P)y, Pz\} &= \{TPx, T(I - P)y, TPz\} \\ &= \overline{\lambda}\{Px, (I - P)y, Pz\}, \quad \forall x, y, z \in \mathcal{J}. \end{aligned}$$

On the other hand, since both P and $I - P$ are skew derivations, we have

$$\begin{aligned} T\{Px, (I - P)y, Pz\} &= P\{Px, (I - P)y, Pz\} + \lambda(I - P)\{Px, (I - P)y, Pz\} \\ &= 2\{Px, (I - P)y, Pz\} - \lambda\{Px, (I - P)y, Pz\}. \end{aligned}$$

Therefore,

$$(2 - \lambda - \overline{\lambda})\{Px, (I - P)y, Pz\} = 0.$$

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Choosing any modulus one $\lambda \neq 1$, we have

$$\{Px, (I-P)y, Pz\} = 0.$$

It amounts to say that $(I-P)\mathcal{J} \subseteq \ker(P\mathcal{J})$. Conversely, if $\{Px, y, Pz\} = 0$ for all x, z in $\mathcal{J}$ then

$$\{Py, Py, Py\} = \{Py, y, Py\} - \{Py, (I-P)y, Py\} = 0.$$

This forces $Py = 0$, and thus $y = (I-P)y$. Therefore, $\ker(P\mathcal{J}) = (I-P)\mathcal{J}$. Analogously, $\ker((I-P)\mathcal{J}) = P\mathcal{J}$.

Let us prove the converse. Suppose that $\ker(P\mathcal{J}) = (I-P)\mathcal{J}$ and $\ker((I-P)\mathcal{J}) = P\mathcal{J}$. Then for all $x, y, z \in \mathcal{J}$ we have

$$\{Px, Py, Pz\} = \{Px, y, Pz\}, \tag{2}$$

$$\{x, Py, z\} - \{Px, Py, z\} - \{x, Py, Pz\} + \{Px, Py, Pz\} = 0. \tag{3}$$

Since P is generalized orthogonal, $2P - I$ is an isometry, and thus a triple isomorphism. Hence

$$(2P-I)\{x, y, z\} = \{(2P-I)x, (2P-I)y, (2P-I)z\}, \quad \forall x, y, z \in \mathcal{J}.$$

From the above, using (2) and (3), we get

$$\begin{aligned}(2P-I)\{x, y, z\} &= 4(\{Px, Py, Pz\} - \{x, Py, Pz\} - \{Px, Py, z\} + \{x, Py, z\}) \\ &\quad +2(\{Px, y, z\} - \{x, Py, z\} + \{x, y, Pz\}) - \{x, y, z\} \\ &= 2(\{Px, y, z\} - \{x, Py, z\} + \{x, y, Pz\}) - \{x, y, z\},\end{aligned}$$

which implies that P is a skew derivation, hence a hermitian projection. □

In [37, Theorem 5], it is shown that every isolated point in the spectrum $\sigma(T)$ of a surjective isometry T on a Banach space is an eigenvalue of T with a complemented eigenspace. In particular, if $\sigma(T) = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ then all λ_i's are eigenvalues, and the associated eigenprojections P_i's satisfy

$$P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I \quad \text{and} \quad T = P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}.$$

Here, we write $P \oplus Q$ to indicate that the Banach space projections P and Q disjoint from each other, i.e., $PQ = QP = 0$. By [37, Corollary 3], vectors x, y from different eigenspaces are *James orthogonal*, i.e., $\|x\| \leq \|x + \alpha y\|$ for all scalars α.

The following definition is equivalent to the one given in [31] but different from [1, 2, 7].

Definition 2.2. *Let P_0 be a nonzero projection on a complex Banach space E, and $n \geq 2$. We call P_0 a* generalized n-circular projection *if there exists a surjective isometry $T : E \to E$ with $\sigma(T) = \{1, \lambda_1, \ldots, \lambda_{n-1}\}$ consisting of n distinct modulus one eigenvalues such that P_0 is the eigenprojection of T associated to $\lambda_0 = 1$. In this case, there are nonzero projections $P_1, \ldots, P_{n-1}$ on E such that*

$$P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I \quad \textit{and} \quad T = P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}.$$

We also say that P_0 is a generalized n-circular projection associated with $(\lambda_1, \ldots, \lambda_{n-1}, P_1, \ldots, P_{n-1})$. Moreover, we call P_0 a proper *generalized n-circular projection if it is not a generalized k-circular projection for any integer $1 < k < n$.*

It follows from the spectral theory that the surjective isometry T in Definition 2.2 is *periodic*, i.e., $T^m = I$ for some integer $m \geq n$, if and only if all its eigenvalues $1, \lambda_1, \ldots, \lambda_{n-1}$ are mth roots of unity. In this case, we can write

$$P_i = \frac{I + \overline{\lambda_i} T + \cdots + \overline{\lambda_i^{m-1}} T^{m-1}}{m}, \quad i = 0, 1, \ldots, n-1.$$

We call the generalized n-circular projection P_0 *periodic* (resp. *primitive*) if it is an eigenprojection of a periodic surjective isometry T of period $m \geq n$ (resp. of period $m = n$). In [1, 7] the authors assume generalized n-circular projections to be primitive.

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Lemma 2.3. *Let P_0 be a generalized n-circular projection ($n \geq 3$) on a Banach space E, associated with $(\lambda_1, \dots, \lambda_{n-1}, P_1, \dots, P_{n-1})$. If P_0 is a proper generalized n-circular projection then all $\lambda_1, \dots, \lambda_{n-1}$ are of finite order. In particular, the surjective isometry*

$$T = P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$$

is periodic, i.e., $T^m = I$ for some positive integer m.

Proof. Let $\lambda_0 = 1$ and $T = \lambda_0 P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$. Suppose that there is some modulus one λ_i that is not of finite order. Without loss of generality we assume that it is λ_{n-1}. Let us define $S = \overline{\lambda_{n-1}} T$, and $\nu_i = \lambda_i \overline{\lambda_{n-1}}$, $i = 0, \dots, n-1$. Then $S = \nu_0 P_0 + \cdots + \nu_{n-2} P_{n-2} + \nu_{n-1} P_{n-1}$. Notice that ν_0 has infinite order, and $\nu_{n-1} = 1$.

Suppose that $\nu_0, \dots, \nu_{k-1}$ have infinite order, and $\nu_k, \dots, \nu_{n-1}$ have finite order, $1 \leq k \leq n-1$. For some positive integer l we have

$$S^l = \nu_0^l P_0 + \cdots + \nu_{k-1}^l P_{k-1} + P_k + \cdots + P_{n-1}.$$

For simplicity of notations, we write $Q = P_k + \cdots + P_{n-1}$, and $\mu_i = \nu_i^l$, $i = 0, \dots, k-1$. Then we have

$$S^l = \mu_0 P_0 + \cdots + \mu_{k-1} P_{k-1} + Q.$$

If $k = 1$ then P_0 is generalized bicircular, which is a contradiction. Thus in the sequel we assume that $k \geq 2$.

Write $\mu_j = e^{2 r_j \pi \mathbf{i}}$ for $j = 0, \dots, k-1$. By assumption, all r_j are irrational numbers in [0,1]. Assume first that the set $\{r_0, \dots, r_{k-1}\}$ is rational linearly independent, i.e., any rational linear relationship $\sum_{j=0}^{k-1} m_j r_j = 0 \pmod 1$ with integers $m_0, \dots, m_{k-1}$ is trivial; namely, $m_0 = \cdots = m_{k-1} = 0$. For any distinct modulus one scalars $\alpha_0, \dots, \alpha_{k-1}$, with $\alpha_{k-1} = 1$, by Kronecker's theorem (see, e.g., [5, Theorem 7.10]), we have a sequence $\{q_i\}_i$ of positive integers such that

$$\lim_i \mu_j^{q_i} = \lim_i e^{2 q_i r_j \pi \mathbf{i}} = \alpha_j, \quad \forall j = 0, 1, \dots, k-1.$$

By continuity, we see that

$$\alpha_0 P_0 + \cdots + \alpha_{k-2} P_{k-2} + P_{k-1} + Q$$

is a surjective isometry, which conflicts with the properness assumption on P_0.

Suppose now the set $\{r_0, \dots, r_{k-1}\}$ is rational linearly dependent. We fix a maximal rational linearly independent subset of it, $\{r_0, \dots, r_{h-1}\}$, say. Then all $r_h, \dots, r_{k-1}$ are rational linearly dependent on members in this subset.

If $h = 1$, then $r_1, \dots, r_{k-1}$ are all rational multiples of r_0. Let $c_0, \dots, c_{k-1}$ be positive integers such that

$$c_0 r_i = c_i r_0, \quad \forall i = 0, 1, \dots, k-1.$$

Changing the indices if necessary, we can assume further that $c_0 < c_1 < \cdots < c_{k-1}$. Consider the surjective isometry

$$\mu_{k-1}^{c_0} P_0 + \mu_{k-1}^{c_1} P_1 + \cdots + \mu_{k-1}^{c_{k-1}} P_{k-1} + Q.$$

Using Kronecker's theorem, we have a sequence $\{h_n\}_n$ of positive integers such that

$$\lim_n \mu_{k-1}^{h_n} = e^{2\pi \mathbf{i}/c_{k-1}}.$$

Then we will have a surjective isometry

$$e^{2 c_0 \pi \mathbf{i}/c_{k-1}} P_0 + \cdots + e^{2 c_{k-2} \pi \mathbf{i}/c_{k-1}} P_{k-2} + (P_{k-1} + Q).$$

This provides a contradiction since P_0 is a proper generalized n-circular projection.

Suppose $h \geq 2$. Let

$$f_j(r_0, \dots, r_{h-1}) = m r_j,$$

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where m is a positive integer and f_j are polynomials in h variables with integral coefficients for all $j = h, \ldots, k-1$. Choose distinct modulus one scalars $\alpha_0, \ldots, \alpha_{h-1}$, with $\alpha_{h-1} = 1$, with arguments $2s_0\pi\mathbf{i}/m, \ldots, 2s_{h-1}\pi\mathbf{i}/m$ such that the set $\{s_0, \ldots, s_{h-1}\}$ is rational linearly independent. Notice that $s_{h-1} = m$, and $f_j(s_0, \ldots, s_{h-1})$ does not belong to the $\mathbb{Q}$-span of the set $\{s_0, \ldots, s_{h-1}\}$ for $j = h, \ldots, k-1$ by the independence. The arguments as above will bring us a surjective isometry

$$\alpha_0 P_0 + \alpha_1 P_1 + \cdots + \alpha_{h-2} P_{h-2} + \alpha_h P_h + \cdots + \alpha_{k-1} P_{k-1} + (P_{h-1} + Q)$$

to establish a contradiction again. □

3 Generalized n-circular projections on Hilbert $C_0(\Omega)$-modules

Let Ω be a locally compact Hausdorff space, and let $C_0(\Omega)$ denote the algebra (with usual pointwise operations) of all continuous complex-valued functions on Ω vanishing at infinity. Equipped with the involution $f^*(w) = \overline{f(w)}$ and the supremum norm, $C_0(\Omega)$ is a commutative C*-algebra, thus a JB*-triple. According to [22, Theorem 4], every generalized bicircular projection on $C_0(\Omega)$ is hermitian, or generalized orthogonal.

Surjective isometries $T\colon C_0(\Omega) \to C_0(\Omega)$ are weighted composition operators. By the Banach-Stone theorem (see, e.g., [6, Theorem 7.1]), there exist a homeomorphism $\phi\colon \Omega \to \Omega$, and a continuous unimodular function $u\colon \Omega \to \mathbb{C}$ (that is, a continuous function satisfying $|u(w)| = 1$ for every w in Ω), such that

$$Tf(w) = u(w) f(\phi(w)), \qquad \forall f \in C_0(\Omega), \forall w \in \Omega.$$

Using this, we can write down explicitly the structure of a generalized bicircular projection on $C_0(\Omega)$. See, e.g., [7, 12]. We want to extend this line to generalized n-circular projections on Hilbert C*-modules over $C_0(\Omega)$.

Recall that a (right complex) *Hilbert C^*-module* $\mathcal{M}$ over a C^*-algebra $\mathcal{A}$, also called a *Hilbert $\mathcal{A}$-module*, is a right $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle\cdot,\cdot\rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ such that

1. $\langle x, ya\rangle = \langle x, y\rangle a$ for all x, y in $\mathcal{M}$ and a in $\mathcal{A}$;
2. $\langle x, y\rangle^* = \langle y, x\rangle$ for all x, y in $\mathcal{M}$;
3. $\langle x, x\rangle \geq 0$, and $\langle x, x\rangle = 0$ exactly when $x = 0$.

Moreover, $\mathcal{M}$ is a Banach space equipped with the norm $\|x\| = \|\langle x, x\rangle\|^{1/2}$. We note that any C*-algebra $\mathcal{A}$ is itself a Hilbert $\mathcal{A}$-module with the inner product $\langle a, b\rangle = a^*b$. See e.g. [40] for more information about Hilbert C*-modules.

We know from [33, Theorem 1.4] that every Hilbert C^*-module is a JB*-triple with respect to the Jordan triple product given by

$$\{x, y, z\} = \frac{1}{2}(x\langle y, z\rangle + z\langle y, x\rangle).$$

Therefore, by [22, Theorem 4], we see that all generalized bicircular projections on Hilbert C*-modules are hermitian or generalized orthogonal.

Remark 3.1. *Suppose that P is a hermitian projection on a Hilbert C^*-module $\mathcal{M}$ over a C^*-algebra $\mathcal{A}$. Then $T_\lambda \overset{def}{=} P + \lambda(I - P)$ is a surjective linear isometry on $\mathcal{M}$ for every modulus one λ in $\mathbb{C}$. We call P a* complete hermitian projection *if all T_λ are complete surjective isometries with respect to the operator space structure of $\mathcal{M}$. By [28, Theorem 1.1], in this case, there exists a $*$-isomorphism $\varphi_\lambda\colon \mathcal{A} \to \mathcal{A}$ such that*

$$\langle T_\lambda x, T_\lambda y\rangle = \varphi_\lambda(\langle x, y\rangle) \quad \text{and} \quad T_\lambda(xa) = T_\lambda(x)\varphi_\lambda(a), \quad \forall x, y \in \mathcal{M}, \forall a \in \mathcal{A}.$$

Let us define a new module operation $\circ$ on $\mathcal{M}$ by $x \circ a = x\varphi_\lambda^{-1}(a)$ for all x in $\mathcal{M}$ and a in $\mathcal{A}$, and a new $\mathcal{A}$-valued inner product $[\cdot,\cdot]_\lambda$ by $[x, y]_\lambda = \varphi_\lambda(\langle x, y\rangle)$ for all x, y in $\mathcal{M}$. Let us denote the obtained Hilbert C-module by $\mathcal{M}_\lambda$. Notice that $T_\lambda\colon \mathcal{M}_\lambda \to \mathcal{M}$ is $\mathcal{A}$-linear since $T_\lambda(x \circ a) = T_\lambda(x\varphi_\lambda^{-1}(a)) = T_\lambda(x)a$ for all $x \in \mathcal{M}_\lambda$, $a \in \mathcal{A}$.*

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By [40, Theorem 3.5], T_λ is a unitary, that is, $T_\lambda^* T_\lambda = I_{\mathcal{M}_\lambda}$ and $T_\lambda T_\lambda^* = I_{\mathcal{M}}$. Since T_λ is adjointable, P is also adjointable. Now for every modulus one scalar λ, we have

$$\begin{aligned} I_{\mathcal{M}} &= T_\lambda T_\lambda^* = (P + \lambda(I-P))(P+\lambda(I-P))^* \\ &= I_{\mathcal{M}} + 2PP^* - P - P^* + \lambda(P^* - PP^*) + \overline{\lambda}(P - PP^*), \end{aligned}$$

which implies $PP^* = P = P^*$. Hence, $P : \mathcal{M} \to \mathcal{M}$ is a self-adjoint idempotent.

If the underlying C-algebra $\mathcal{A}$ is abelian, i.e., $\mathcal{A} = C_0(\Omega)$ for some locally compact Hausdorff space Ω, every surjective (complex linear) isometry is a complete isometry ([27, Theorem 1]), therefore every hermitian projection on a Hilbert $C_0(\Omega)$-module $\mathcal{M}$ is complete hermitian, and thus corresponds to a self-adjoint idempotent on $\mathcal{M}$.*

We continue to study the structure of non-hermitian n-circular projections on Hilbert $C_0(\Omega)$-modules. To this end, we recall that every Hilbert $C_0(\Omega)$-module is exactly the continuous section space of a Hilbert bundle based on Ω. We sketch briefly the construction below.

A *Hilbert bundle* $\mathcal{H}$ over a locally compact Hausdorff space Ω is a pair $\langle \mathcal{H}_\Omega, \pi_\Omega \rangle$. Here $\mathcal{H}_\Omega$ is a topological space and $\pi_\Omega : \mathcal{H}_\Omega \to \Omega$ is a continuous open surjective map. For all ω in Ω, the fiber $H_\omega = \pi_\Omega^{-1}(\omega)$ carries a nonzero complex Hilbert space structure. Moreover, we assume:

(HB1) Scalar multiplication, addition and the norm on $\mathcal{H}_\Omega$ are all continuous wherever they are defined.
(HB2) If $\omega \in \Omega$ and $\{h_i\}$ is any net in $\mathcal{H}_\Omega$ such that $\|h_i\| \to 0$ and $\pi(h_i) \to \omega$ in Ω, then $h_i \to 0_\omega$ (the zero element of H_ω) in $\mathcal{H}_\Omega$.

A *continuous section* f of a Hilbert bundle $\langle \mathcal{H}_\Omega, \pi_\Omega \rangle$ is a continuous function from Ω into $\mathcal{H}_\Omega$ such that $\pi_\Omega(f(\omega)) = \omega$, i.e., $f(\omega) \in H_\omega$ for all ω in Ω. Denote by $C_0(\Omega; \mathcal{H}_\Omega)$ the Banach space of all C_0-sections of $\langle \mathcal{H}_\Omega, \pi_\Omega \rangle$, i.e., those continuous sections f with $\lim_{\omega \to \infty} \|f(\omega)\| = 0$. Note that the space $C_0(\Omega; \mathcal{H}_\Omega)$ is a Hilbert $C_0(\Omega)$-module with pointwise module action and inner product

$$(f\phi)(\omega) = f(\omega)\phi(\omega), \quad \langle f, g\rangle(\omega) = (f(\omega), g(\omega)), \quad \forall f, g \in C_0(\Omega; \mathcal{H}_\Omega), \forall \phi \in C_0(\Omega), \forall \omega \in \Omega.$$

Conversely, every Hilbert $C_0(\Omega)$-module $\mathcal{M}$ can be represented as $C_0(\Omega; \mathcal{H}_\Omega)$ arising from some Hilbert bundle $\langle \mathcal{H}_\Omega, \pi_\Omega \rangle$. Here, we outline the construction in [26, 28], which is based on [19]. For each ω in Ω, let

$$I_\omega = \{f \in C_0(\Omega) : f(\omega) = 0\}.$$

By Cohen's factorization theorem,

$$\mathcal{M}I_\omega = \{vf : v \in \mathcal{M}, f \in I_\omega\}$$

is norm closed in $\mathcal{M}$. In particular, if $u \in \mathcal{M}I_\omega$ and $w \in \mathcal{M}$, we have $\langle u, w\rangle(\omega) = \langle w, u\rangle(\omega) = 0$. Then $\mathcal{M}/\mathcal{M}I_\omega$ is a pre-Hilbert space with inner product

$$\langle u + \mathcal{M}I_\omega, v + \mathcal{M}I_\omega \rangle := \langle u, v\rangle(\omega).$$

Denote by H_ω the completion of $\mathcal{M}/\mathcal{M}I_\omega$. Let

$$\mathcal{H}_\Omega := \coprod_{\omega \in \Omega} H_\omega = \Big\{(z_\omega)_{\omega \in \Omega} : z_\omega \in H_\omega\Big\}.$$

Each element u in $\mathcal{M}$ can be regarded as a section from Ω into $\mathcal{H}_\Omega$ by

$$u(\omega) := u + \mathcal{M}I_\omega.$$

Let $\pi_\Omega : \mathcal{H}_\Omega \to \Omega$ be the canonical projection. By [21, Theorem 13.18], there is a unique topology on $\mathcal{H}_\Omega$ such that $\langle \mathcal{H}_\Omega, \pi_\Omega \rangle$ is a Hilbert bundle over Ω with C_0-section space $C_0(\Omega; \mathcal{H}_\Omega) = \mathcal{M}$.

We assume in below that all Hilbert $C_0(X)$-modules $\mathcal{M} = C_0(\Omega; \mathcal{H}_\Omega)$ are full, i.e., the *-subalgebra $\langle \mathcal{M}, \mathcal{M}\rangle = \{\langle u, v\rangle \in C_0(\Omega) : u, v \in \mathcal{M}\}$ is dense in $C_0(\Omega)$. In this case, all fiber Hilbert spaces H_ω are nonzero,

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and for any vectors a_1 in H_{ω_1} and a_2 in H_{ω_2}, there exists a continuous section a in $\mathcal{M}$ such that $a(\omega_1) = a_1$ and $a(\omega_2) = a_2$.

The following is a consequence of [26, Theorem 3.1], which asserts the more general case of into isometries of Banach $C_0(\Omega)$-modules; see also [28] for the real Hilbert C*-module case.

Theorem 3.2. *Let Ω be a locally compact Hausdorff space and $\mathcal{M} = C_0(\Omega; \mathcal{H}_\Omega)$ be a Hilbert $C_0(\Omega)$-module. Let $T : C_0(\Omega; \mathcal{H}_\Omega) \to C_0(\Omega; \mathcal{H}_\Omega)$ be a surjective isometry. Then there exist a homeomorphism $\phi : \Omega \to \Omega$ and, for each ω in Ω, a surjective isometry $u(\omega) : H_{\phi(\omega)} \to H_\omega$ such that*

$$T(f)(\omega) = u(\omega)(f(\phi(\omega))), \quad \forall f \in C_0(\Omega; \mathcal{H}_\Omega).$$

The examples below can be considered as extensions of the product Hilbert bundle case given in [12], i.e., the case $\mathcal{H}_\Omega = \Omega \times H$ for a fixed Hilbert space H with $C_0(\Omega; \mathcal{H}_\Omega) = C_0(\Omega, H)$.

Example 3.3. *Let Ω be a locally compact Hausdorff space and $\mathcal{M} = C_0(\Omega; \mathcal{H}_\Omega)$ be a Hilbert $C_0(\Omega)$-module. Let $T : \mathcal{M} \to \mathcal{M}$ be a surjective isometry. Assume that the spectrum $\sigma(T) = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ consists of n distinct unimodular eigenvalues of finite orders. Thus there exists $m \geq 2$ such that $T^m = I$ and all λ_i's are distinct mth roots of unity. Note it is necessary that $m \geq n$. Write*

$$T(f)(\omega) = u(\omega)(f(\phi(\omega))), \quad \forall f \in C_0(\Omega; \mathcal{H}_\Omega), \ \forall \omega \in \Omega,$$

as in Theorem 3.2.

Since $T^m = I$, we have

$$f(\omega) = u(\omega) \cdots u(\phi^{m-1}(\omega)) f(\phi^m(\omega)), \quad \forall f \in C_0(\Omega; \mathcal{H}_\Omega), \ \forall \omega \in \Omega.$$

If there exists an ω in Ω such that $\phi^m(\omega) \neq \omega$ then we can choose an f from $\mathcal{M}$ such that $f(\omega) = 0$ and $f(\phi^m(\omega))$ is any nonzero vector in $H_{\phi^m(\omega)}$ to yield a contradiction. Hence $\phi^m(\omega) = \omega$, and thus $u(\omega) \ldots u(\phi^{m-1}(\omega)) = I_\omega$, the identity operator on the fiber Hilbert space H_ω for every ω in Ω. Let

$$P_i = \frac{I + \overline{\lambda_i} T + \cdots + \overline{\lambda_i}^{m-1} T^{m-1}}{m}, \qquad i = 0, 1, \ldots, n-1.$$

Being the eigenprojections of the surjective isometry T, all P_i, $i = 0, 1, \ldots, n-1$, are generalized n-circular projections. In particular, each P_i is a nonzero projection,

$$P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I, \quad \text{and} \quad \lambda_0 P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1} = T.$$

Example 3.4. *If the locally compact Hausdorff space Ω has a proper component Y, then the indicator function $\mathbf{1}_Y$ of Y gives rise to a hermitian projection $Pf = \mathbf{1}_Y f$ on $C_0(\Omega; \mathcal{H}_\Omega)$. If Ω is connected, any hermitian projection P of $C_0(\Omega; \mathcal{H}_\Omega)$ is of the form*

$$Pf(\omega) = p(\omega) f(\omega), \quad \forall \omega \in \Omega,$$

where $\omega \mapsto p(\omega)$ is a field of projections on the fiber Hilbert space H_ω such that it is continuous in the sense that the section $\omega \mapsto p(\omega) f(\omega)$ is continuous whenever $f \in C_0(\Omega; \mathcal{H}_\Omega)$.

Example 3.5. *Assume Ω is a connected locally compact Hausdorff space. Let P be a generalized bicircular projection on $C_0(\Omega; \mathcal{H}_\Omega)$ associated with a surjective isometry $T = P + \lambda(I - P)$. Then*

$$Pf(\omega) = \frac{Tf(\omega) - \lambda f(\omega)}{1 - \lambda} = \frac{u(\omega) f(\phi(\omega)) - \lambda f(\omega)}{1 - \lambda},$$

for a homeomorphism $\phi : \Omega \to \Omega$, and a surjective isometry $u(\omega) : H_{\phi(\omega)} \to H_\omega$ between the fiber Hilbert spaces of the Hilbert bundle $\mathcal{H}$. Since $P^2 = P$, we have

$$(1 - \lambda)(u(\omega) f(\phi(\omega)) - \lambda f(\omega)) = u(\omega) u(\phi(\omega)) f(\phi^2(\omega)) - 2\lambda u(\omega) f(\phi(\omega)) + \lambda^2 f(\omega),$$

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or

$$u(\omega)u(\phi(\omega))f(\phi^2(\omega)) - (1+\lambda)u(\omega)f(\phi(\omega)) + \lambda f(\omega) = 0, \quad \forall f \in C_0(\Omega; \mathcal{H}_\Omega), \forall \omega \in \Omega.$$

If $\phi^2(\omega) \neq \omega$ then $\phi(\omega) \neq \omega$ either. Choosing any f from $C_0(\Omega; \mathcal{H}_\Omega)$ such that $f(\omega)$ is nonzero and $f(\phi(\omega)) = f(\phi^2(\omega)) = 0$, we will arrive at a contradiction as $\lambda \neq 0$. Hence, $\phi^2(\omega) = \omega$ on Ω.

If there is an ω such that $\phi(\omega) \neq \omega$ then by considering a continuous section f with $f(\omega) = 0$ and $f(\phi(\omega)) \neq 0$, we have

Case 1. *$\lambda = -1$, and $P = \dfrac{I+T}{2}$ is a generalized orthogonal projection.*

If $\phi(\omega) = \omega$ for all ω in Ω then the unitary operator $u(\omega)$ on the fiber Hilbert space H_ω satisfies the operator equation

$$u(\omega)^2 - (1+\lambda)u(\omega) + \lambda I_\omega = 0, \quad \forall \omega \in \Omega,$$

where I_ω is the identity operator on H_ω. By spectral theory, the spectrum $\sigma(u(\omega))$ can contain only 1 and λ. Because Ω is connected, the unitary operator field $\omega \mapsto u(\omega)$ is a multiple of the identity whenever it is so at any one point ω. Since P and $I - P$ are both (proper) eigenprojections for $Tf(\omega) = u(\omega)f(\omega)$, we see that $\sigma(u(\omega)) = \{1, \lambda\}$ for all ω in Ω. Let $u(\omega) = p(\omega) + \lambda(I_\omega - p(\omega))$ be the diagonal decomposition of $u(\omega)$ into the orthogonal sum of its two proper eigenprojections. In this setting, we have

Case 2. *$Pf(\omega) = p(\omega)f(\omega)$ arises from a nonvanishing continuous field of Hilbert space projections $\omega \mapsto P(\omega)$.*

We end the paper with a structure theorem about generalized n-circular projections on an abelian C*-algebra $C_0(\Omega)$. The general Hilbert $C_0(\Omega)$-module case looks far away to us at this moment.

Recall that a modulus one complex scalar λ is called a *primitive mth root of unity* if $\lambda^m = 1$ but $\lambda^k \neq 1$ for $k = 1, \dots, m-1$. We also note that the scalars $\lambda_1, \dots, \lambda_{n-1}$ associated to the proper generalized n-circular projection P_0 below are automatically of finite orders by Lemma 2.3.

Theorem 3.6. *Let Ω be a locally compact Hausdorff space with at most $n-1$ connected components. Let P_0 be a proper generalized n-circular projection on $C_0(\Omega)$ associated with $(\lambda_1, \dots, \lambda_{n-1}, P_1, \dots, P_{n-1})$, where $n \geq 3$. In other words,*

$$I = P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1}, \quad \text{and} \quad T = P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1} \text{ is a surjective isometry.}$$

Assume that all $\lambda_1, \dots, \lambda_{n-1}$ are primitive mth roots. Then $m = n$, $\{1, \lambda_1, \dots, \lambda_{n-1}\}$ is the complete set of nth roots of unity, and $T^n = I$. Thus all $P_0, P_1, \dots, P_{n-1}$ are primitive generalized n-circular projections.

Moreover, there exist a homeomorphism $\phi \colon \Omega \to \Omega$, and a continuous unimodular scalar function u on Ω such that

$$\phi^n(w) = w, \qquad u(w) \cdots u(\phi^{n-1}(w)) = 1, \qquad Tf(\omega) = u(\omega)f(\phi(\omega)),$$

and

$$\begin{aligned} P_i f(w) &= \frac{I + \overline{\lambda_i} T + \cdots + \overline{\lambda_i}^{n-1} T^{n-1}}{n} \\ &= \frac{f(w) + \overline{\lambda_i} u(w) f(\phi(w)) + \cdots + \overline{\lambda_i}^{n-1} u(w) \dots u(\phi^{n-2}(w)) f(\phi^{n-1}(w))}{n} \end{aligned}$$

for all $f \in C_0(\Omega)$, $w \in \Omega$, and $i = 0, 1, \dots, n-1$.

Proof. Assume all $\lambda_1, \dots, \lambda_{n-1}$ have the same (minimum) order $m \geq n$. Consequently, $T^m = I$.

Suppose $m = rs$ and $r \geq 2$. Consider the surjective isometry

$$S := T^s = P_0 + \lambda_1^s P_1 + \cdots + \lambda_{n-1}^s P_{n-1}.$$

Note that none of the eigenvalues $\lambda_1^s, \dots, \lambda_{n-1}^s$ is 1. By the properness assumption on P_0, we see that all coefficients are distinct. Since they are the rth roots of unity, we conclude that $n \leq r$.

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Write

$$Tf(\omega) = u(\omega) f(\phi(\omega)) \tag{4}$$

for a homeomorphism $\phi : \Omega \to \Omega$ and a continuous unimodular scalar function u on Ω. Since $T^m = I$, we see that $\phi^m(\omega) = \omega$ for all ω in Ω. It also follows from the spectral theory that

$$\prod_{i=0}^{n-1} (T - \lambda_i I) = 0. \tag{5}$$

Putting (4) into (5) we get

$$\mu_0 f(w) + \sum_{i=1}^{n} \mu_i \prod_{j=0}^{i-1} u(\phi^j(w)) f(\phi^i(w)) = 0, \qquad \forall f \in C_0(\Omega),\ w \in \Omega, \tag{6}$$

with some scalars μ_i's in which $\mu_0 = \lambda_0 \lambda_1 \cdots \lambda_{n-1} \neq 0$.

Suppose an w in Ω is not equal to any of $\phi^i(w)$ for $i = 1, \ldots, n$. We choose f from $C_0(\Omega)$ such that $f(w) = 1$ and $f(\phi^i(w)) = 0$ for all $i = 1, \ldots, n$. Then (6) provides a contradiction that $\mu_0 = 0$. Hence there exists a smallest positive integer k_ω such that $1 \leq k_\omega \leq n$ and $\phi^{k_\omega}(\omega) = \omega$. Since $\phi^m(\omega) = \omega$, we can assume that k_ω is a factor of m by the division algorithm. As n is not greater than any factor of m other than one, we see that $k_\omega = 1$ or $k_\omega = n$. Suppose all $k_\omega = 1$. Since $T^m = I$, we have

$$f(\omega) = T^m f(\omega) = u(\omega) \cdots u(\phi^{m-1}(\omega)) f(\phi^m(\omega)) = u(\omega)^m f(\omega), \quad \forall f \in C_0(\Omega), \forall \omega \in \Omega.$$

It amounts to say that the continuous scalar function u assumes values from the finite set of mth roots of unity on Ω. Hence, u assumes constant values on each component of Ω, and thus T is a sum of at most $n-1$ scalar multiples of the canonical hermitian projections on components (cf. the Case 2 part of Example 3.5). But T has n distinct eigenvalues. With this contradiction we know that some $k_\omega = n$, and thus n is a factor of m. Moreover,

$$\phi^n(\omega) = \omega, \quad \forall \omega \in \Omega.$$

Write $m = nb$ for some positive integer b. Since $T^m = I$, we have

$$\begin{aligned} f(\omega) = T^m f(\omega) &= u(\omega) u(\phi(\omega)) \cdots u(\phi^{m-1}(\omega)) f(\phi^m(\omega)) \\ &= \left[u(\omega) u(\phi(\omega)) \cdots u(\phi^{n-1}(\omega)) \right]^b f(\omega), \quad \forall f \in C_0(\Omega), \forall \omega \in \Omega. \end{aligned}$$

Hence the continuous unimodular scalar function $u(\omega) u(\phi(\omega)) \cdots u(\phi^{n-1}(\omega))$ assumes values from the discrete set of bth roots of unity. Since Ω has at most $n-1$ connected components, it assumes at most $n-1$ different values u_0 with $u_0^b = 1$. Consequently, on each component of Ω, with some constant u_0 from them, we can write

$$u_0 f(\omega) = T^n f(\omega) = P_0 f(\omega) + \lambda_1^n P_1 f(\omega) + \cdots + \lambda_{n-1}^n P_{n-1} f(\omega), \quad \forall f \in C_0(\Omega), \forall \omega \in \Omega.$$

Since each projection above is nonzero on some of the at most $n-1$ components of Ω, and they sum up to the identity, we have all such

$$u_0 = \lambda_1^n = \ldots = \lambda_{n-1}^n = 1.$$

It follows that $m = n$ and $\{1, \lambda_1, \ldots, \lambda_{n-1}\}$ is the complete set of nth roots of unity. The asserted representations of P_i's then follows (see [31], or Example 3.3). □

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