UNIFORMLY CONDITIONED BASES OF SPECTRAL SUBSPACES

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Abstract. The condition number of an ordered basis of a finite dimensional normed space is defined in an intrinsic manner. The concept is extended to a sequence of bases of finite dimensional normed spaces, and is used to determine uniform conditioning of such a sequence. The problem of finding a sequence of uniformly conditioned bases of spectral subspaces of operators of the form $T_n = S_n + U_n$, where $S_n$ is a finite-rank operator on a Banach space and $U_n$ is an operator which is ‘well-behaved’ with $S_n$, is reduced to constructing a sequence of uniformly conditioned bases of spectral subspaces of operators on $\mathbb{C}^{n \times 1}$. The applicability of these considerations in spectral approximation is illustrated.

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1. Introduction

Orthonormal bases play a crucial role in the setting of a Hilbert space, both in the theoretical development as well as in computational reliability. In the setting of a Banach space $X$, an element $x$ in $X$ is said to be orthogonal to a subspace $M$ of $X$ if the norm of $x$ is equal to its distance from $M$. (See, for example, [18, p. 215].) A subset $\{x_1, \ldots, x_m\}$ of $X$ is called orthogonal if the norm of each $x_j$ is equal to its distance from the span of the remaining $x_i$'s. In any event, it is important to keep the norm of each element of a basis of $X$ not too large or not too small in comparison with other elements of the basis (in order to avoid overflow and/or underflow) and to keep its distance from the span of the remaining elements of the basis not much smaller than its norm (in order to avoid almost linear dependence). This leads us to the concept of a condition number of a basis of a finite dimensional normed space. Further, this concept can be extended in a natural manner to a sequence of finite dimensional subspaces of a Banach space, yielding a measure of ‘uniform conditioning’ of such a sequence.

Our focus in this article is on finite dimensional spectral subspaces of a bounded operator $F$ on a Banach space $X$. Since it is very difficult to
construct any ordered basis of a spectral subspace of $F$, let alone one which is well-conditioned, a recourse is often made to approximate the operator $F$ by an operator $T$ for which such a basis can be constructed. Usually the operator $T$ is of finite rank, and then the problem is reduced to constructing an ordered basis of a subspace of $C^{n \times 1}$ and transferring it back to $X$. In a recent article, the author has shown how this can be achieved for an operator $T$ which is the sum of a finite-rank operator $S$ and another operator $U$ which is ‘well-behaved’ with $S$ in a precise sense. Even when an ordered basis of a subspace of $C^{n \times 1}$ is chosen to be well-conditioned, it is not clear why the transferred basis would be well-conditioned in $X$. The problem becomes even more acute when a sequence $(T_n)$, where $T_n := S_n + U_n$ for each $n \in \mathbb{N}$, approximates the operator $F$, and a sequence of ordered bases is transferred from $C^{n \times 1}$ to the Banach space $X$ as $n$ becomes large. The purpose of this article is to address this problem and find conditions under which uniform conditioning of the bases is preserved under such a transfer from $C^{n \times 1}$ to $X$ for $n = 1, 2, \ldots$. The special case where $U_n = 0$, that is, where $T_n$ is of finite-rank, for each $n \in \mathbb{N}$, was considered earlier in [6, §4] and in [1, §5.1.2].

Our results are also applicable to various cases given in [14, Examples 2.1], where the operator $T$ is not of finite rank, and in particular, to the sequences of approximating operators $(T_n)$ considered in [4] and [16].

The question of uniform conditioning of bases is crucial in spectral approximation for several reasons apart from the considerations of numerical reliability. Let $\Lambda$ be a spectral set of $F$ of finite type, $M$ denote the corresponding spectral subspace, and let $P$ be the associated spectral projection. Let $(T_n)$ be a suitable approximation of $F$ and let $\phi_n := [\varphi_{n,1}, \ldots, \varphi_{n,m}]$ be an ordered basis of the spectral subspace $M_n$ associated with $T_n$ and an approximate spectral set $\Lambda_n$. Then $P_n \phi_n$ is an (exact) ordered basis of $M$ and the error norm $\|\phi_n - P_n \phi_n\|$ is of the order of $\|(T_n - F) \phi_n\|$, provided the sequence $(\phi_n)$ is uniformly conditioned. (See [1, Theorem 2.21].) Further, there are two major methods for improving accuracy of approximate spectral values and of approximate bases of spectral subspaces, namely (i) iterative refinement and (ii) acceleration technique. Improved error bounds can be established for both these methods under the assumption of uniform conditioning of the approximate ordered bases. (See [1, Theorems 3.7, 3.8 and 3.15 (b)].) The present article complements a recent article on iterative refinement for weakly singular operators, where uniform conditioning of ordered bases of spectral subspaces was postulated, but their construction was not assured. (See [2, Theorem 2.5].)

This article can be considered as a sequel of [14]. It is organized as follows. In Section 2, the condition number of an ordered basis of a finite-dimensional normed space is defined, and an estimate is given for the possible increase in the condition number when an ordered basis of a spectral subspace of $C^{n \times 1}$ is transferred to an ordered basis of a spectral subspace of a Banach space $X$. Specific bounds are obtained for two finite-rank operators, one based to a modified projection method and another based on a degenerate
kernel method involving interpolation in both variables of a kernel. Section 3 is devoted to finding the block resolvent for a multiplication operator on a Banach algebra with an identity element. An upper bound for the block resolvent is needed for implementing the main result of Section 2. The case of a function algebra is treated in detail. In Section 4, the condition number of a sequence of ordered bases of finite dimensional normed spaces is defined. Further, the results of Section 3 are used to obtain a uniformly conditioned sequence of ordered bases of spectral subspaces of a Banach space $X$. Section 5 deals with applications to spectral approximation. Two major classes of operators are treated: (i) Integral operators on $C([0,1])$ with weakly singular kernels and (ii) Operators on $L^\infty([a,b])$, each of which is a sum of an integral operator with a continuous kernel and a multiplication operator by a continuous function. (See [4, 16] and references therein.)

2. Condition Number of a Finite Basis

Let $M$ be a normed space over $\mathbb{C}$ of dimension $m < \infty$ with a norm $\| \cdot \|$. For an ordered basis $\underline{x} := [x_1, \ldots, x_m]$ of $M$, let $\underline{x}' := [x_1', \ldots, x_m']$ denote the basis of $M'$ which is dual to $\underline{x}$. We define the condition number of $\underline{x}$ by

$$\text{cond} (\underline{x}) := \max \{\|x_j\| : j = 1, \ldots, m\} \max \{\|x_j'\|' : j = 1, \ldots, m\},$$

where $\| \cdot \|'$ is the induced norm on the dual space $M'$. Since $\|x_j'\|' = 1/\text{dist}(x_j, \hat{M}_j)$, where $\hat{M}_j := \text{span}\{x_i : i = 1, \ldots, m, i \neq j\}$ for $j = 1, \ldots, m$, it follows that

$$\text{cond} (\underline{x}) = \frac{\max \{\|x_j\| : j = 1, \ldots, m\}}{\min \{\text{dist}(x_j, \hat{M}_j) : j = 1, \ldots, m\}}.$$

By convention, $\hat{M}_1 := \{0\}$ when $m = 1$. Note that cond $(\underline{x})$ does not change if either the order of the $x_j$’s in $[x_1, \ldots, x_m]$ is altered or if all $x_j$’s are multiplied by a nonzero $\alpha \in \mathbb{C}$. Since $\|x_j\| \geq \text{dist}(x_j, \hat{M}_j)$ for each $j = 1, \ldots, m$, we observe that cond $(\underline{x}) \geq 1$. If $\underline{x}$ is orthogonal, that is, if $\|x_j\| = \text{dist}(x_j, \hat{M}_j)$ for each $j = 1, \ldots, m$, then cond $(\underline{x}) = \max \{\|x_j\| : j = 1, \ldots, m\}/\min \{\|x_j\| : j = 1, \ldots, m\}$. If, in addition, the norms of all $x_j$’s are equal, then cond $(\underline{x}) = 1$. For feasibility and reliability of computations, we seek to construct an ordered basis of $M$ whose condition number is not too large. Such a basis may be called ‘well-conditioned’.

Various properties and estimates for the condition number defined above, and also a justification for this nomenclature, are given in [5]. In particular, if the norm $\| \cdot \|$ on $M$ is induced by an inner product $\langle \cdot, \cdot \rangle$, and $G := [{\langle x_j, x_i \rangle}]$ denotes the Gram matrix of $\underline{x}$, then it is shown that cond $(\underline{x})$ is equal to the square root of the product of the largest diagonal entry of $G$ and the largest diagonal entry of $G^{-1}$.

Let a norm $\| \cdot \|$ on $\mathbb{C}^{m \times 1} := \{[u(1), \ldots, u(m)]^t : u(1), \ldots, u(m) \in \mathbb{C}\}$ be prescribed. The (relative) condition number of an ordered basis $\underline{x}$ of $M$ with
respect to the norm $\| \cdot \|$ on $M$ and the norm $\| \cdot \|_0$ on $\mathbb{C}^{m \times 1}$ is defined by

$$
\kappa_0 (x) := \max_{u \neq 0} \frac{\| \sum_{j=1}^{m} u(j) x_j \|}{\| u \|_0} \max_{u \neq 0} \frac{\max_{u \neq 0} \| u \|_0}{\| \sum_{j=1}^{m} u(j) x_j \|}.
$$

(See, for instance, [19, 15].) It follows that

$$
\kappa_0 (x) = \frac{\max \{ \| \sum_{j=1}^{m} u(j) x_j \| : u \in \mathbb{C}^{m \times 1} \text{ and } \| u \|_0 = 1 \} }{\min \{ \| \sum_{j=1}^{m} u(j) x_j \| : u \in \mathbb{C}^{m \times 1} \text{ and } \| u \|_0 = 1 \} }.
$$

For example, let the norm $\| \cdot \|$ on $M$ be induced by an inner product and take the norm $\| \cdot \|_0$ on $\mathbb{C}^{m \times 1}$ to be the 2-norm $\| \cdot \|_2$. Then it can be seen that $\kappa_2 (x) = \| G^{1/2} \|_2 \| G^{-1/2} \|_2$, where $G$ is the Gram matrix of $x$. Thus $\text{cond} (x) \leq \kappa_2 (x)$. In general, a strict inequality may hold. For instance, let $M := \mathbb{C}^{2 \times 1}$ with the usual inner product, and let $x := [x_1, x_2]$, where $x_1 := [1, 0]^t$, $x_2 := [1, 1]^t$. Then $\text{cond} (x) = 2$, whereas $\kappa_2 (x) = (3 + \sqrt{5})/2$.

If, however, an ordered basis $x$ of $M$ is orthogonal, then $\| \sum_{j=1}^{m} u(j) x_j \| = \sum_{j=1}^{m} |u(j)|^2 \| x_j \|_2^2$ for all $u \in \mathbb{C}^{m \times 1}$, and so $\text{cond} (x) = \kappa_2 (x)$.

Given a normed space $M$, there exists an ordered basis $y := [y_1, \ldots, y_m]$ of $M$ such that $\| y_j \| = 1 = \| y_j^t \|$ for each $j = 1, \ldots, m$, and so $\text{cond} (y) = 1$. The proof of the existence of such a basis is not constructive since it involves locating a point in the $m$-fold product of the closed unit ball of $M$ at which a determinantal function attains its maximum. (See, for example, [8, Lemma 2.1].) If, however, the norm on $M$ is induced by an inner product, then starting with any ordered basis $x$ of $M$, we can actually construct an ordered orthonormal basis $y$ of $M$, for which $\text{cond} (y) = 1$, by employing one of the well-known algorithms such as the Gram-Schmidt orthonormalization procedure.

The celebrated Riesz Lemma yields an ordered basis $y := [y_1, \ldots, y_m]$ of $M$ such that $\| y_j \| = 1 = \text{dist} (y_j, \text{span} \{y_1, \ldots, y_{j-1} \})$ for each $j = 1, \ldots, m$, which in turn implies that $\text{dist} (y_j, \tilde{M}_j) \geq 1/2^{m-1}$ for each $j = 1, \ldots, m$. (See [1, Proposition 5.15].) Then $\text{cond} (y) \leq 2^{m-1}$. The estimate is good for small $m$, but even in this case, it is not clear how to implement the Riesz Lemma for a given norm on $M$. If, however, $M$ is a subspace of $\mathbb{C}^{n \times 1}$, then this implementation is equivalent to the computation of a minimum norm solution of an over-determined system of linear equations. Various algorithms are available for this purpose for specific norms on $\mathbb{C}^{n \times 1}$ such as the $p$-norms with $1 \leq p \leq \infty$. (See [20, Chapter 1], for $p = \infty$, [20, Chapter 4] for $1 < p < \infty$ and [20, Chapter 6] for $p = 1$. Also, see [13] and [1, Remark 5.16] for $p = 1$ and $p = \infty$.) One can, therefore, attempt to identify an $m$-dimensional subspace $\tilde{M}$ of a Banach space $X$ with an $m$-dimensional subspace $\tilde{M}$ of some $\mathbb{C}^{n \times 1}$, construct a well-conditioned ordered basis of $\tilde{M}$ with respect to a suitably chosen norm on $\mathbb{C}^{n \times 1}$, and transfer this basis back to $M$. In doing so, one has to ensure that the transferred basis is well-conditioned with respect to the given norm on $X$. 
We shall use notation and terminology of [14]. We outline some of the results in [14] here in an attempt to make our discussion self-contained. For more details and for proofs of the statements given below, the reader is referred to [14]. Let $X$ be a Banach space over $\mathbb{C}$ with a norm $\| \cdot \|$, and let $BL(X)$ denote Banach algebra of all bounded linear maps from $X$ to $X$. Let $S \in BL(X)$ be of finite rank; say $\text{rank}(S) \leq n$. Then there are elements $x_1, \ldots, x_n$ in $X$ and bounded linear functionals $f_1, \ldots, f_n$ on $X$ such that

$$S x = \sum_{j=1}^{n} f_j(x) x_j \quad \text{for all} \ x \in X.$$ 

We note that neither the elements $x_1, \ldots, x_n$ in $X$ nor the bounded linear functionals $f_1, \ldots, f_n$ on $X$ are required to be linearly independent. Define $Kx := [f_1(x), \ldots, f_n(x)]^t$ for $x \in X$ and $Lu := u(1)x_1 + \cdots + u(n)x_n$ for $u := [u(1), \ldots, u(n)]^t \in \mathbb{C}^{n \times 1}$, where the superscript $t$ denotes transpose. Then $S = LK$. Define $A : \mathbb{C}^{n \times 1} \to \mathbb{C}^{n \times 1}$ by $A := KL$. Then $AK = KS$.

We note that the $n \times n$ Gram matrix $[f_i(x_j)]$ represents the operator $A$ with respect to the standard basis of $\mathbb{C}^{n \times 1}$.

We specify a norm on $\mathbb{C}^{n \times 1}$, and let $K' : (\mathbb{C}^{n \times 1})^t \to X'$ denote the transpose of the continuous linear map $K : X \to \mathbb{C}^{n \times 1}$. Consider $U$ in $BL(X)$ such that the range of $K'$ is invariant under the transpose $U' : X' \to X'$ of $U$. Then $f_i \circ U$ belongs to the linear span of $f_1, \ldots, f_n$ for each $i = 1, \ldots, n$, and there are complex numbers $c_{i,j}$ such that $f_i \circ U = \sum_{j=1}^{n} c_{i,j} f_j$ for each $i = 1, \ldots, n$. Define a linear map $C : \mathbb{C}^{n \times 1} \to \mathbb{C}^{n \times 1}$ by $Cu := \left[ \sum_{j=1}^{n} c_{1,j} u(j), \ldots, \sum_{j=1}^{n} c_{n,j} u(j) \right]^t$ for $u := [u(1), \ldots, u(n)]^t$ in $\mathbb{C}^{n \times 1}$. Then $CK = KU$. We remark that if the functionals $f_1, \ldots, f_n$ are linearly independent, then the linear map $C$ is determined by the operator $U$. For several examples of the maps $K$ and $U$ which satisfy the invariance condition mentioned above, we refer to [14, Examples 2.1].

Let $T := S + U$ and $B := A + C$. It follows that $BK = KT$. Given a well-conditioned ordered basis of a spectral subspace of $B$, we would like to construct a well-conditioned basis for a spectral subspace of $T$.

For this purpose, we set up the following notation. Fix $m \in \mathbb{N}$, and let $\underline{X} := \{ x_1, \ldots, x_m : x_j \in X \text{ for } 1 \leq j \leq m \}$ with the norm given by $\| x \| := \max_{j=1, \ldots, m} \| x_j \|$. For $x := [x_1, \ldots, x_m] \in \underline{X}$, $F_1, \ldots, F_m$ in $BL(X)$ and $\Theta := [\theta_{i,j}] \in \mathbb{C}^{m \times m}$, define $F \in BL(\underline{X})$ by $F(x) := [F_1(x_1), \ldots, F_m(x_m)]$, that is, $F := \text{diag}(F_1, \ldots, F_m)$, and $x \Theta := \left[ \sum_{i=1}^{m} \theta_{i,1} x_i, \ldots, \sum_{i=1}^{m} \theta_{i,m} x_i \right] \in \underline{X}$. Then $\| F \| = \max_{j=1, \ldots, m} \| F_j \|$, and

$$\| x \Theta \| \leq \| x \| \| \Theta \|_1, \quad \text{where} \quad \| \Theta \|_1 := \max_{j=1, \ldots, m} \sum_{i=1}^{m} |\theta_{i,j}|.$$ 

Note that if $F_1 = \cdots = F_m = F$, then $\| F \| = \| F \|$ and $F(x) \Theta = (F(x)) \Theta$ for all $x \in \underline{X}$ and $\Theta \in \mathbb{C}^{m \times m}$. 
Observing that each spectral subspace of $B$ is invariant under $B$, we prove a preliminary result for invariant subspaces of $B$.

**Lemma 2.1.** Let $M$ be an invariant subspace of $B$ and $\underline{u} \in \mathbb{C}^{n \times m}$ be an ordered basis of $M$. Then there is a unique $\Theta \in \mathbb{C}^{m \times n}$ such that $B \underline{u} = \underline{u} \Theta$. Let a norm $\| \cdot \|_0$ on $\mathbb{C}^{n \times 1}$ be specified. Then
\[ \| \Theta \|_1 \leq m(\|K\| \|L\| + \|C\|) \text{cond}(\underline{y}). \]

**Proof.** Let $\underline{u} = [u_1, \ldots, u_m] \in \mathbb{C}^{n \times m}$. Since $M$ is invariant under $B$, there is a unique $\theta_{i,j} \in \mathbb{C}$ for $i, j = 1, \ldots, m$ such that $B u_j = \theta_{1,j} u_1 + \cdots + \theta_{m,j} u_m$ for each $j = 1, \ldots, m$. Letting $\Theta := [\theta_{i,j}] \in \mathbb{C}^{m \times m}$, we see that $B \underline{u} = \underline{u} \Theta$. Define $c_j := \|u_j\|_0$ and $d_j := \text{dist}(u_j, M_j)$ for $j = 1, \ldots, m$, where $M_j := \text{span}\{u_i : i = 1, \ldots, m, i \neq j\}$. Then for $i, j = 1, \ldots, m$,
\[ d_i |\theta_{i,j}| \leq \|Bu_j\|_0 \leq (\|A\| + \|C\|)\|u_j\|_0 \leq (\|K\| \|L\| + \|C\|)c_j, \]

since $B = A + C = KL + C$. If we let $c := \max\{c_j : j = 1, \ldots, m\}$ and $d := \min\{d_j : j = 1, \ldots, m\}$, then
\[ \|\Theta\|_1 = \max_{j=1,\ldots,m} \sum_{i=1}^m |\theta_{i,j}| \leq m(\|K\| \|L\| + \|C\|) \frac{c}{d} = m(\|K\| \|L\| + \|C\|) \text{cond}(\underline{y}), \]

as desired. \qed

Let us denote the spectrum of a bounded operator $F$ on a normed space by $\text{sp}(F)$. For ready reference, we state the following proposition which is a combination of Propositions 3.2 and 3.4 of [14].

**Proposition 2.2.** Let $E := \text{sp}(C) \cup \text{sp}(U)$ and $\Lambda \subseteq \text{sp}(T) \setminus E$. Then $\Lambda$ is a finite set. Further, each $\lambda \in \Lambda$ is an eigenvalue of $T$ and it is an isolated point of $\text{sp}(T)$. The linear map $K$ from the spectral subspace $M(T, \Lambda)$ associated with $T$ and $\Lambda$ to the spectral subspace $M(B, \Lambda)$ associated with $B$ and $\Lambda$ is bijective. Let the common dimension of $M(T, \Lambda)$ and $M(B, \Lambda)$ be $m$. Let $\underline{u} \in \mathbb{C}^{n \times m}$ form an ordered basis of $M(B, \Lambda)$, and let $\Theta \in \mathbb{C}^{m \times n}$ be such that $B \underline{u} = \underline{u} \Theta$. Then there is a unique $\varphi \in X$ such that $\varphi \Theta - \underline{U} \varphi = \underline{L} \underline{u}$, where $\underline{U} := \text{diag}(U, \ldots, U)$. In fact, $\varphi$ forms an ordered basis of $M(T, \Lambda)$ and satisfies $K \varphi = \underline{u}$.

It may be noted that $\text{sp}(\Theta) = \Lambda$, and so $\text{sp}(\Theta) \cap \text{sp}(U) = \emptyset$. Hence for every $\underline{y} \in X$, there is unique $\underline{x} \in X$ satisfying the Sylvester equation $\underline{x} \Theta - \underline{U} \underline{x} = \underline{y}$. (See [1, Proposition 1.50].) The block resolvent of $\underline{U}$ at $\Theta$ is the operator $R(\underline{U}, \Theta) \in BL(X)$ defined by $R(\underline{U}, \Theta) \underline{y} := \underline{x}$ for $\underline{y} \in X$.

**Proposition 2.3.** Consider a norm $\| \cdot \|_0$ on $\mathbb{C}^{n \times 1}$. Let $\underline{u} = [u_1, \ldots, u_m]$ form an ordered basis of the spectral subspace $M(B, \Lambda)$ associated with $B$ and $\Lambda$. Consider the ordered basis $\varphi := R(\underline{U}, \Theta) \underline{L} \underline{u}$ of the spectral subspace $M(T, \Lambda)$ associated with $T$ and $\Lambda$. If $\varphi := [\varphi_1, \ldots, \varphi_m]$, then
\[ \|\varphi\| \leq \|R(\underline{U}, \Theta)\| \|L\| \|\underline{u}\|_0 \]
and
\[ \text{dist}(u_j, \hat{M}_j(B, \Lambda)) \leq \|K\| \text{dist}(\varphi_j, \hat{M}_j(T, \Lambda)), \]
where \( \hat{M}_j(B, \Lambda) := \text{span}\{u_i : i = 1, \ldots, m, i \neq j\} \) and \( \hat{M}_j(T, \Lambda) := \text{span}\{\varphi_i : i = 1, \ldots, m, i \neq j\} \). As a consequence,
\[ \text{cond}(\varphi) \leq \|K\| \|L\| \|R(U, \Theta)\| \text{cond}(\underline{u}). \]

Proof. Since \( \|\varphi\| = \max_{j=1,\ldots,m} \|\varphi_j\| \) and \( \|L\| = \|L\| \)
\[ \|\varphi_j\| \leq \|\varphi\| = \|R(U, \Theta)L\underline{u}\| \leq \|R(U, \Theta)\| \|L\| \|\underline{u}\|_0 \]
for each \( j = 1, \ldots, m \), also, since \( K\varphi_i = u_i \) for all \( i = 1, \ldots, m \),
\[ \left\| u_j - \sum_{i \neq j} \alpha_i u_i \right\|_0 = \left\| K \left( \varphi_j - \sum_{i \neq j} \alpha_i \varphi_i \right) \right\|_0 \leq \|K\| \left\| \varphi_j - \sum_{i \neq j} \alpha_i \varphi_i \right\| \]
for each \( j = 1, \ldots, m \) and any \( \alpha_1, \ldots, \alpha_m \) in \( \mathbb{C} \). This shows that
\[ \text{dist}(u_j, \hat{M}_j(B, \Lambda)) \leq \|K\| \text{dist}(\varphi_j, \hat{M}_j(T, \Lambda)) \]
for each \( j = 1, \ldots, m \).
As a consequence,
\[ \text{cond}(\varphi) = \frac{\max\{\|\varphi_j\| : j = 1, \ldots, m\}}{\min\{\text{dist}(\varphi_j, \hat{M}_j(T, \Lambda)) : j = 1, \ldots, m\}} \]
\[ \leq \frac{\|R(U, \Theta)\| \|L\| \max\{\|u_j\| : j = 1, \ldots, m\}\|K\|}{\min\{\text{dist}(u_j, \hat{M}_j(B, \Lambda)) : j = 1, \ldots, m\}} \]
\[ = \|K\| \|L\| \|R(U, \Theta)\| \text{cond}(\underline{u}), \]
as desired. \( \square \)

It is clear from the above proposition that the ordered basis \( \varphi \) of the spectral subspace \( M(T, \Lambda) \) would be well-conditioned, provided the ordered basis \( \underline{u} \) of \( M(B, \Lambda) \) is well-conditioned and the norms of \( K, L \) and \( R(U, \Theta) \) are of moderate size. In general, it is not easy to find a bound for the norm of the block resolvent \( R(U, \Theta) \). Let us consider a special situation where each component of the ordered basis \( \underline{u} \in \mathbb{C}^{n \times m} \) of the spectral subspace \( M(B, \Lambda) \) is in fact an eigenvector of \( B \). Then there is \( \lambda_j \in \Lambda \) such that \( Bu_j = \lambda_j u_j \) for each \( j = 1, \ldots, m \) and \( \Theta = \text{diag}(\lambda_1, \ldots, \lambda_m) \). The equation \( x \Theta - U_\underline{x} = \underline{y} \) is reduced to the \( m \) equations \( \lambda_j x_j - U x_j = y_j, j = 1, \ldots, m \). Since \( \text{sp}(U) \) and \( \Lambda \) are disjoint, we see that \( \lambda_j I - U \) is invertible in \( BL(X) \) and \( x_j = (\lambda_j I - U)^{-1} y_j \) for each \( j = 1, \ldots, m \), that is, \( R(U, \Theta) = \text{diag}((\lambda_1 I - U)^{-1}, \ldots, (\lambda_m I - U)^{-1}) \). It follows that
\[ \|R(U, \Theta)\| = \max_{j=1,\ldots,m} \| (\lambda_j I - U)^{-1} \|. \]
In this case, \( \varphi = R(U, \Theta)L\underline{u} = [(\lambda_1 I - U)^{-1} L_{u_1}, \ldots, (\lambda_m I - U)^{-1} L_{u_m}] \). As a word of caution, we remark that this special situation is of practical relevance only when \( \Lambda \) consists of a single semisimple eigenvalue \( \lambda \) of \( T \), so that \( \varphi = (\lambda I - U)^{-1} L\underline{u} \). This is because even if all distinct eigenvalues of \( B \) in \( \Lambda \) are semisimple, most algorithms for calculation of a basis of \( M(B, \Lambda) \) yield an ordered basis \( \underline{u} = [u_1, \ldots, u_m] \), where each \( u_j \) is a linear combination of eigenvectors of \( B \).
corresponding to different eigenvalues. In the next section, we shall consider a special type of operator, namely a multiplication operator $U$, and find meaningful bounds for $\|R(\underline{U}, \Theta)\|$. 

Let us now turn to the bounds for the norms of $K$ and $L$. They depend on the given norm $\|\cdot\|$ on the Banach space $X$, on the specified norm $\|\cdot\|_0$ on $\mathbb{C}^{n \times 1}$ and on the presentation $Sx = \sum_{j=1}^{n} f_j(x)x_j$, $x \in X$, of the finite-rank operator $S$. We observe that $\max\{\|S\|, \|A\|\} \leq \|K\| \|L\|$ since $S = KL$ and $A = KL$. Hence there is a sort of balance between the norms of $K$ and $L$. It is often possible to specify a suitable norm on $\mathbb{C}^{n \times 1}$ such as the 1-norm, or the 2-norm, or the $\infty$-norm, so that $\|K\|$ and $\|L\|$ are less than a moderate constant, independent of the rank of the operator $S$. (See Examples 4.2 and 4.3 of [6], or Examples 5.11, 5.12 and 5.13 of [1].) We further illustrate this phenomenon by considering two finite-rank operators not treated earlier.

Our first example is modelled on a modified projection method that was introduced recently by Kulkarni in [12], and that has proved to be efficient in operator approximation theory.

**Proposition 2.4.** Let $F \in BL(X)$, and let $\pi \in BL(X)$ be a finite rank projection operator given by $\pi(x) := \sum_{i=1}^{n} e'_i(x)e_i$, where $e_1, \ldots, e_n$ are in $X$ and $e'_1, \ldots, e'_n$ are in $X'$ such that $e'_i(e_j) = \delta_{i,j}$ for $i, j = 1, \ldots, n$. Define

$$S := \pi F + F \pi - \pi F \pi.$$ 

Then $\text{rank}(S) \leq 2n$ and $\|S\| \leq \|\pi\| \|F\|(2 + \|\pi\|)$. Define $K : X \to \mathbb{C}^{2n \times 1}$ by

$$(Kx)(i) = \begin{cases} e'_i(Fx) - \sum_{j=1}^{n} e'_j(x)e'_i(Fe_j) & \text{if } 1 \leq i \leq n, \\ e'_{i-n}(x) & \text{if } n+1 \leq i \leq 2n \end{cases} \quad \text{for } x \in X,$$

and $L : \mathbb{C}^{2n \times 1} \to X$ by

$$Lu = \sum_{i=1}^{n} u(i)e_i + \sum_{i=1}^{n} u(n+i)Fe_i \quad \text{for } u \in \mathbb{C}^{2n \times 1}.$$ 

Then $S = KL$.

(i) Let $X$ be an inner product space, $\{e_1, \ldots, e_n\}$ be an orthonormal set in $X$. For $i = 1, \ldots, n$, let $e'_i(x) := \langle x, e_i \rangle$, $x \in X$. Then $\|\pi\| = 1$. Also, with respect to the 2-norm on $\mathbb{C}^{2n \times 1}$,

$$\|K\| \leq \sqrt{4\|F\|^2 + 1} \quad \text{and} \quad \|L\| \leq 1 + \|F\|.$$ 

(ii) Let $J$ be a compact Hausdorff space, and let $X := C(J)$, the set of all complex-valued continuous functions on $J$ along with the sup norm $\|\cdot\|_{\sup}$. Consider distinct points $t_1, \ldots, t_n$ in $J$, and let $e_1, \ldots, e_n$ be in $X$ such that $e_j(t_i) = \delta_{i,j}$ for $i, j = 1, \ldots, n$. For $i = 1, \ldots, n$, let $e'_i(x) := x(t_i)$, $x \in X$. Then $\|\pi\| = \|\sum_{i=1}^{n} e_i\|_{\sup}$. Also, with respect to the $\infty$-norm on $\mathbb{C}^{2n \times 1}$,

$$\|K\| \leq \max \{\|F\|(1 + \|\pi\|), 1\} \quad \text{and} \quad \|L\| \leq (1 + \|F\|\|\pi\|).$$
Proof. It is easy to see that $\|S\| \leq \|\pi\| \|F\|(2 + \|\pi\|)$ and

$$Sx = \sum_{i=1}^{n} \left( e'_i(Fx) - \sum_{j=1}^{n} e'_j(x) e'_i(Fe_j) \right) e_i + \sum_{i=1}^{n} e'_i(x) Fe_i \quad \text{for } x \in X.$$  

Hence rank$(S) \leq 2n$ and $S = LK$, where $K$ and $L$ are as stated.

(i) Since $\pi$ is an orthogonal projection, its norm is equal to 1. For $x \in X$,

$$\|Kx\|_2^2 = \sum_{i=1}^{n} \left| \langle Fx, e_i \rangle - \sum_{j=1}^{n} \langle x, e_j \rangle \langle Fe_j, e_i \rangle \right|^2 + \sum_{i=1}^{n} \left| \langle x, e_i \rangle \right|^2$$

$$\leq 2 \sum_{i=1}^{n} |\langle Fx, e_i \rangle|^2 + 2 \sum_{i=1}^{n} \left| \sum_{j=1}^{n} \langle x, e_j \rangle \langle Fe_j, e_i \rangle \right|^2 + \sum_{i=1}^{n} \left| \langle x, e_i \rangle \right|^2$$

$$\leq 2 \|Fx\|^2 + 2 \|F\|^2 \|x\|^2 + \|x\|^2$$

by Bessel’s inequality. Hence $\|K\| \leq \sqrt{4\|F\|^2 + 1}$. Also, for $u \in \mathbb{C}^{2n \times 1}$,

$$\|Lu\| \leq \left\| \sum_{i=1}^{n} u(i) e_i \right\| + \left\| F \left( \sum_{i=1}^{n} u(n+i) e_i \right) \right\|$$

$$\leq \left( \sum_{i=1}^{n} |u(i)|^2 \right)^{1/2} + \|F\| \left( \sum_{i=1}^{n} |u(n+i)|^2 \right)^{1/2}$$

$$\leq (1 + \|F\|) \|u\|_2$$

by Pythagoras’ theorem. Hence $\|L\| \leq 1 + \|F\|$.

(ii) For each $t_0 \in J$, there is $x_0 \in C(J)$ satisfying $\|x_0\|_{\sup} \leq 1$ and $x_0(t_i) := \text{sgn} e_i(t_0)$ for $i = 1, \ldots, n$ by Tietze’s extension theorem. Hence the equality $\|\pi\| = \|\sum_{i=1}^{n} e_i\|_{\sup}$ follows. For $x \in C(J)$ and $i = 1, \ldots, n$,

$$|(Kx)(t_i)| = \left| \langle Fx, t_i \rangle - \left( F \sum_{j=1}^{n} x(t_j) e_j \right)(t_i) \right|$$

$$\leq \|F\| \left( \|x\|_{\sup} + \left\| \sum_{j=1}^{n} x(t_j) e_j \right\|_{\sup} \right)$$

$$\leq \|F\| \left( \|x\|_{\sup} + \left\| \sum_{j=1}^{n} e_j \right\|_{\sup} \|x\|_{\sup} \right)$$

$$= \|F\| (1 + \|\pi\|) \|x\|_{\sup}.$$
Next, for \( x \in C(J) \) and \( i = n + 1, \ldots, 2n, \) \( \|(Kx)(i)\| = |x(t_{i-n})| \leq \|x\|_{\sup}. \) Hence \( \|K\| \leq \max \{ \|F\|(1 + \|\pi\|), 1 \}. \) Also, for \( u \in \mathbb{C}^{2n \times 1}, \)

\[
\|Lu\|_{\sup} \leq \left\| \sum_{i=1}^{n} u(i)e_i \right\|_{\sup} + \left\| F \left( \sum_{i=1}^{n} u(n+i)e_i \right) \right\|_{\sup} \\
\leq \left\| \sum_{j=1}^{n} e_j \right\|_{\sup} \|u\|_{\infty} + \left\| F \right\| \left\| \sum_{j=1}^{n} e_j \right\|_{\sup} \|u\|_{\infty} \\
\leq (1 + \|F\|)\|\pi\|\|u\|_{\infty}.
\]

Hence \( \|L\| \leq (1 + \|F\|)\|\pi\|. \) \( \square \)

It may be noted that in part (i) of the above proposition, the upper bounds for \( \|K\| \) and \( \|L\| \) do not depend on the rank of \( S \). The same would hold in part (ii) as well, if \( \|\pi\| \) is bounded by a constant, independent of the rank of \( \pi \). This happens, for example, if \( \pi \) is a term of a sequence \( (\pi_n) \) of projections in \( BL(X) \) converging pointwise to the identity operator on \( X \).

Our second example is modelled on a degenerate kernel method based on interpolation in both variables of a kernel and is often used for approximating integral operators. It will show how a judicious choice of the norm on \( \mathbb{C}^{n \times 1} \) must be made to achieve a moderate size of \( \|K\| \) as well as of \( \|L\| \).

**Proposition 2.5.** Let \( (J, \mu) \) be a finite measure space, and \( X := L^\infty(J, \mu) \). Let \( e_1, \ldots, e_n \) be in \( X \) such that \( e_j \geq 0 \) for all \( j = 1, \ldots, n \) and \( \sum_{j=1}^{n} e_j = 1 \). Given \( \alpha_{i,j} \in \mathbb{C} \), let \( S \) denote the Fredholm integral operator having a degenerate kernel given by \( k(s, t) := \sum_{i,j=1}^{n} \alpha_{i,j} e_i(s)e_j(t) \) for \( s, t \in J \), that is,

\[
(Sx)(s) := \int_{J} k(s,t)x(t)d\mu(t) \quad \text{for } x \in X \text{ and } s \in J.
\]

Then \( \text{rank}(S) \leq n \) and \( \|S\| \leq \alpha \mu(J) \), where \( \alpha := \max \{ |\alpha_{i,j}| : 1 \leq i, j \leq n \} \).

(i) Define \( K : X \to \mathbb{C}^{n \times 1} \) by

\[
(Kx)(i) := \sum_{j=1}^{n} \alpha_{i,j} \int_{J} xe_j d\mu \quad \text{for } i = 1, \ldots, n \text{ and } x \in X,
\]

and \( L : \mathbb{C}^{n \times 1} \to X \) by \( Lu := \sum_{i=1}^{n} u(i)e_i \) for \( u \in \mathbb{C}^{n \times 1}. \) Then \( S = LK. \) With respect to the \( \infty \)-norm on \( \mathbb{C}^{n \times 1}, \)

\[
\|K\| \leq \alpha \mu(J) \quad \text{and} \quad \|L\| \leq 1.
\]

(ii) Define \( K : X \to \mathbb{C}^{n \times 1} \) by

\[
(Kx)(j) := \int_{J} xe_j d\mu \quad \text{for } j = 1, \ldots, n \text{ and } x \in X,
\]

and \( L : \mathbb{C}^{n \times 1} \to X \) by \( Lu := \sum_{j=1}^{n} u(j) \left( \sum_{i=1}^{n} \alpha_{i,j} e_i \right) \) for \( u \in \mathbb{C}^{n \times 1}. \) Then \( S = LK. \) With respect to the \( 1 \)-norm on \( \mathbb{C}^{n \times 1}, \)

\[
\|K\| \leq \mu(J) \quad \text{and} \quad \|L\| \leq \alpha.
\]
Proof. (i) It is clear that $S = LK$. For $x \in X$,
\[
\|Kx\|_{\infty} \leq \|x\|_{\text{esssup}} \max \left\{ \sum_{j=1}^{n} |\alpha_{i,j}| \int_{J} e_{j}d\mu : i = 1, \ldots, n \right\} \leq \alpha \mu(J)\|x\|_{\text{esssup}}.
\]
Hence $\|K\| \leq \alpha \mu(J)$. Since $|(Lu)(s)| \leq \|u\|_{\infty}$ for all $u \in \mathbb{C}^{n \times 1}$ and $s \in J$, we see that $\|L\| \leq 1$.

(ii) Again it is clear that $S = LK$. For $x \in X$,
\[
\|Kx\|_{1} \leq \|x\|_{\text{esssup}} \sum_{j=1}^{n} \int_{J} e_{j}d\mu = \mu(J)\|x\|_{\text{esssup}}.
\]
Hence $\|K\| \leq \mu(J)$. Since $|(Lu)(s)| \leq \sum_{j=1}^{n} |u(j)|\alpha(\sum_{i=1}^{n} e_{i}(s)) = \alpha\|u\|_{1}$ for all $u \in \mathbb{C}^{n \times 1}$ and $s \in J$, we see that $\|L\| \leq \alpha$. \hfill \Box

In the above proposition if we consider the 1-norm on $\mathbb{C}^{n \times 1}$ in part (i), then we would have $\|K\| \leq \sum_{i=1}^{n} \beta_{i}$, where $\beta_{i} := \max\{|\alpha_{i,j}| : 1 \leq j \leq n\}$ and $\|L\| \leq \max\{|e_{i}|_{\text{esssup}} : 1 \leq i \leq n\} \leq 1$. Similarly, if we consider the $\infty$-norm on $\mathbb{C}^{n \times 1}$ in part (ii), then we would have $\|K\| \leq \max\{ \int_{J} e_{j}d\mu : 1 \leq j \leq n\} \leq 1$ and $\|L\| \leq \sum_{i=1}^{n} \gamma_{i}$, where $\gamma_{j} := \max\{|\alpha_{i,j}| : 1 \leq i \leq n\}$. The upper bound of $\|K\|$ in part (i) and of $\|L\|$ in part (ii) would in general increase with the rank of $S$, even when $\alpha$ does not. This would be undesirable.

3. Block resolvent of a multiplication operator

In this section, we consider a right multiplication operator $U$ on a (not necessarily commutative) Banach algebra $X$ over $\mathbb{C}$ with the identity element $1_{X}$. Let $\Theta \in \mathbb{C}^{m \times m}$ be such that $\text{sp}(U) \cap \text{sp}(\Theta) = \emptyset$. Our goal is to find estimates for the norm of the block resolvent $R(U, \Theta)$ of $U$ at $\Theta$.

For $\ell, m \in \mathbb{N}$, let $X^{\ell \times m}$ denote the set of all $\ell \times m$ matrices with entries in $X$. As in Section 2, let $X := X^{1 \times m}$. For $Z := [\zeta_{i,j}], W := [\omega_{i,j}] \in X^{\ell \times m}$ and $\alpha \in \mathbb{C}$, let $Z + W := [\zeta_{i,j} + \omega_{i,j}]$ and $\alpha Z := [\alpha \zeta_{i,j}]$. Then $X^{\ell \times m}$ is a linear space over $\mathbb{C}$. In fact, if for $x \in X$ and $Z := [\zeta_{i,j}] \in X^{\ell \times m}$, we define $xZ := [x\zeta_{i,j}] \in X^{\ell \times m}$, then $X^{\ell \times m}$ is a left $X$-module, that is, $x(W + Z) = xW + xZ, (x + y)Z = xZ + yZ, (xy)Z = x(yZ)$ and $1_{X}Z = Z$ for all $x, y \in X$ and $W, Z \in X^{\ell \times m}$.

For $\ell, m, p \in \mathbb{N}$, $W := [\omega_{i,j}] \in X^{\ell \times m}$ and $Z := [\zeta_{i,j}] \in X^{m \times p}$, define $WZ := [\sum_{k=1}^{m} \omega_{i,k} \zeta_{k,j}] \in X^{\ell \times p}$. Letting $\ell = p = m$, we see that $X^{m \times m}$ is an algebra over $\mathbb{C}$ with the identity element $1_{m} := \text{diag}(1, \ldots, 1)$. Also, letting $\ell = 1$ and $p = m$, we see that $xZ = [\sum_{i=1}^{m} x_{i}\zeta_{i,1}, \ldots, \sum_{i=1}^{m} x_{i}\zeta_{i,m}]$ for $x := [x_{1}, \ldots, x_{m}] \in X$ and $Z := [\zeta_{i,j}] \in X^{m \times m}$. If $\Theta := [\theta_{i,j}] \in \mathbb{C}^{m \times m}$ and we let $Z := [\theta_{i,j}1_{X}] \in X^{m \times m}$, then it is clear that $xZ = x\Theta$.

For $Z \in X^{m \times m}$, define $\Phi(Z) : X \rightarrow X$ by
\[
\Phi(Z)x := xZ.
\]
Then the map $\Phi(Z)$ is $X$-linear, that is, $\Phi(Z)(x + y) = \Phi(Z)x + \Phi(Z)y$, and $\Phi(Z)(ax) = a\Phi(Z)x$ for all $x, y \in X$ and $a \in X$. 
We have already defined the norm \( \| x \| := \max_{j=1,...,m} \| x_j \| \) of \( x := [x_1,\ldots,x_m] \in X \). For \( Z := [\zeta_{i,j}] \in X^{m\times m} \), define

\[
\| Z \|_1 := \max_{j=1,...,m} \sum_{i=1}^m \| \zeta_{i,j} \|.
\]

It is easy to check that \( \| \cdot \|_1 \) is a complete norm on \( X^{m\times m} \), \( \| WZ \|_1 \leq \| W \|_1 \| Z \|_1 \) for all \( W,Z \in X^{m\times m} \) and \( \| I_m \| = 1 \). Thus \( X^{m\times m} \) is a Banach algebra over \( \mathbb{C} \) with the identity element \( I_m \). Also, for each \( Z \in X^{m\times m} \), \( \| \varphi Z \| \leq \| \varphi \| \| Z \|_1 \) for all \( \varphi \in \mathcal{X} \), so that \( \Phi(Z) \in BL(\mathcal{X}) \) and \( \| \Phi(Z) \| \leq \| Z \|_1 \).

Strict inequality may hold here as the following simple example shows. Let \( X \) denote the Banach algebra of all complex-valued functions defined on a set \( \{t_1,t_2\} \), and define \( \| x \| := \max\{|x(t_1)|,|x(t_2)|\} \) for \( x \in X \). Consider \( \zeta_1,\zeta_2 \in X \) defined by \( \zeta_1(t_1) = 1 = \zeta_2(t_2) \) and \( \zeta_1(t_2) = 0 = \zeta_2(t_1) \). Let \( m = 2 \) and \( Z := \begin{bmatrix} \zeta_1 & 0 \\ \zeta_2 & 0 \end{bmatrix} \in X^{2\times2} \). Since \( \| \zeta_1 \| = 1 = \| \zeta_2 \| \), we obtain \( \| Z \|_1 = 2 \).

On the other hand, for \( z := [x_1,x_2] \in X \),

\[
\| \Phi(Z) z \| = \| [x_1\zeta_1 + x_2\zeta_2, 0] \| = \| x_1\zeta_1 + x_2\zeta_2 \| = \max\{|x_1(t_1)|,|x_2(t_2)|\},
\]

so that \( \| \Phi(Z) \| = 1 \). Thus \( \| \Phi(Z) \| < \| Z \|_1 \).

The map \( \Phi : X^{m\times m} \to BL(\mathcal{X}) \) satisfies \( \Phi(W + Z) = \Phi(W) + \Phi(Z) \), \( \Phi(\alpha Z) = \alpha \Phi(Z) \) and \( \Phi(WZ) = \Phi(W)\Phi(Z) \) for all \( W,Z \in X^{m\times m} \) and \( \alpha \in \mathbb{C} \), and further, \( \Phi(I_m) = I \). Also, \( \Phi \) is injective. To see this, consider \( e_i := [0,\ldots,0,1,0,\ldots,0] \in \mathcal{X} \), where \( 1 \) appears in the \( i \)th component, for \( i = 1,\ldots,m \), and note that \( \Phi(Z) e_i \) is the \( i \)th row of \( Z \). Hence \( Z = 0 \) if and only if \( \Phi(Z) = 0 \). Thus \( \Phi \) is an injective anti-homomorphism, that is, an anti-monomorphism, from the Banach algebra \( X^{m\times m} \) to the Banach algebra \( BL(\mathcal{X}) \). Also, \( \| \Phi \| = 1 \), but \( \Phi \) is not an isometry.

For a fixed \( \xi \in X \), consider the right multiplication operator \( U \in BL(X) \) defined by \( Ux := x\xi \). If \( \xi \) is invertible in \( X \) and \( \eta := \xi^{-1} \), then \( U \) is invertible in \( BL(X) \) and \( U^{-1}y = y\eta \) for all \( y \in X \). Conversely, let \( U \) be invertible in \( BL(X) \), and let \( V := U^{-1} \). Since \( U \) is \( X \)-linear, so is \( V \) and hence \( Vy = y(V1_X) \) for all \( y \in X \). This shows that \( \xi \) is invertible in \( X \) and \( \xi^{-1} = V1_X \). As a consequence, \( \text{sp}(U) = \text{sp}(\xi) \). The following proposition is a generalization of this result.

**Proposition 3.1.** Let \( X \) be a Banach algebra with the identity element \( 1_X \), \( m \in \mathbb{N} \) and \( Z \in X^{m\times m} \). Then \( \text{sp}(\Phi(Z)) = \text{sp}(Z) \).

**Proof.** First we show that \( Z \) is invertible in \( X^{m\times m} \) if and only if \( \Phi(Z) \) is invertible in \( BL(\mathcal{X}) \). If \( Z \) is invertible in \( X^{m\times m} \), then since \( \Phi \) is an anti-homomorphism, it follows that \( \Phi(Z) \) is invertible and \( \Phi(Z^{-1}) \) is its inverse. Conversely, let \( \Phi(Z) \) be invertible in \( BL(\mathcal{X}) \) and let \( \Phi := \Phi(Z)^{-1} \). Consider \( W \in X^{m\times m} \) whose \( i \)th row of is \( F_i \), that is, \( F_i = F_{i,h} \), for \( i = 1,\ldots,m \).
Since the map $\Phi(Z)$ is $X$-linear, so is its inverse $F$. Hence

$$Fy = F\left(\sum_{i=1}^{m} y_i \xi_i\right) = \sum_{i=1}^{m} y_i F\xi_i = yW \quad \text{for all } y := [y_1, \ldots, y_m] \in X.$$ 

Thus $\Phi(Z)^{-1} = F = \Phi(W)$. Again, since $\Phi$ is an anti-monomorphism, we see that $\Phi(WZ) = \Phi(Z)\Phi(W) = I = \Phi(W)\Phi(Z) = \Phi(ZW)$, and so $WZ = I_m = ZW$, that is, $Z$ is invertible and $Z^{-1} = W$. It follows that for $\alpha \in \mathbb{C}$, $\alpha Z - Z$ is invertible in $X^{m \times m}$ if and only if $\alpha I - \Phi(Z)$ is invertible in $BL(X)$, that is, $\text{sp}(\Phi(Z)) = \text{sp}(Z)$.

\textbf{Corollary 3.2.} Let $X$ be a Banach algebra with the identity element $1_X$, $\xi \in X$, and let $Ux := x \xi$ for $x \in X$. For $m \in \mathbb{N}$ and $\Theta := [\theta_{i,j}] \in \mathbb{C}^{m \times m}$, define 

$$Z := [\theta_{i,j}1_X] - \text{diag}(\xi, \ldots, \xi).$$

Then $Z$ is invertible in $X^{m \times m}$ if and only if $\text{sp}(\xi) \cap \text{sp}(\Theta) = \emptyset$. In that event, $R(U, \Theta) = \Phi(Z^{-1})$, and consequently, $||R(U, \Theta)|| \leq ||Z^{-1}||_1$.

\textbf{Proof.} As we have seen in Proposition 3.1, $Z$ is invertible in $X^{m \times m}$ if and only if $\Phi(Z)$ is invertible in $BL(X)$. But $\Phi(Z) \xi = y$ means $\xi - U \xi = y$ for all $\xi, y$ in $X$. This Sylvester equation has a unique solution $\xi$ for each $y \in X$ if and only if $\text{sp}(U) \cap \text{sp}(\xi) = \emptyset$, that is, $\text{sp}(\xi) \cap \text{sp}(\Theta) = \emptyset$. (See [1, Proposition 1.50].) In that event, $R(U, \Theta)y = x = \Phi(Z)^{-1}y = \Phi(Z^{-1})y$ for all $x, y$ in $X$, that is, $R(U, \Theta) = \Phi(Z^{-1})$. The final inequality is obvious. □

As we have indicated earlier, the upper bound $||Z^{-1}||_1$ of $||R(U, \Theta)||$ may not be sharp. For example, consider the Banach algebra $X$ of all complex-valued functions defined on $[0, 1]$, and define $||x|| := \max\{|x(0)|, |x(1)|\}$ for $x \in X$. Let $m = 2$ and $\Theta := \begin{bmatrix} 3/2 & 1 \\ 0 & 1/3 \end{bmatrix} \in \mathbb{C}^{2 \times 2}$. Consider $\xi \in X$ defined by $\xi(0) := 0$ and $\xi(1) := 1$. Then $\text{sp}(\Theta) = \{3/2, 1/3\}$ and $\text{sp}(\xi) = \{0, 1\}$ are disjoint. Hence $Z := \begin{bmatrix} (3 - 2\xi)/2 & 1 \\ 0 & (1 - 3\xi)/3 \end{bmatrix}$ is invertible in $X^{2 \times 2}$, and 

$$Z^{-1} = \begin{bmatrix} 2/(3 - 2\xi) & -6/(3 - 2\xi)(1 - 3\xi) \\ 0 & 3/(1 - 3\xi) \end{bmatrix}.$$

Let $\omega_1 := 2/(3 - 2\xi)$, $\omega_2 := -6/(3 - 2\xi)(1 - 3\xi)$ and $\omega_3 := 3/(1 - 3\xi)$. Then $||Z^{-1}||_1 = \max\{||\omega_1||, ||\omega_2||, ||\omega_3||\} = \max\{2, 3 + 3\} = 6$. On the other hand, for $y := [y_1, y_2] \in X$,

$$||R(U, \Theta)y|| = ||yZ^{-1}|| = \max\{||y_1\omega_1||, ||y_1\omega_2 + y_2\omega_3||\},$$

where $||y_1\omega_1|| = 2\max\{|y_1(0)|/3, |y_1(1)|\} \leq 2||y||$ and $||y_1\omega_2 + y_2\omega_3|| = \max\{|-2y_1(0) + 3y_2(0)/2, |3y_1(1) - 3y_2(1)/2|\} \leq 5||y||$. Hence $||R(U, \Theta)|| = \leq 5$, and so $||R(U, \Theta)|| < ||Z^{-1}||_1$.

In view of the above remark, one would like to find a sharper upper bound for $||R(U, \Theta)||$. With this in mind, we consider the case of a Banach algebra $X$ over $\mathbb{C}$ with an identity element $1_X$ such that $||x^2|| = ||x||^2$ for all $x \in X$. Then $X$ is commutative by a result of Hirschfeld and Zelazko [10], and the
Gelfand transform of $X$ is an isometry [9, Corollary 5.3]. Hence we may assume that $X$ is a function algebra on a compact Hausdorff space $J$, that is, $X$ is a closed subalgebra of the Banach algebra $C(J)$ of all complex-valued continuous functions on $J$ (with the sup norm $\| \cdot \|_{\text{sup}}$), $X$ contains the constant function 1, and $X$ separates the points of $J$.

**Proposition 3.3.** Let $X$ be a function algebra on a compact Hausdorff space $J$, $m \in \mathbb{N}$ and $Z \in X^{m \times m}$. If $Z := [z_{i,j}]$ and for $s \in J$, we let $Z(s) := [z_{i,j}(s)] \in \mathbb{C}^{m \times m}$, then $\|\Phi(Z)\| = \sup_{s \in J} \|Z(s)\|_1$.

**Proof.** Let $x := [x_1, \ldots, x_m] \in X$. Then $|x_i(s)| \leq \|x_i\|_{\text{sup}} \leq \|x\|$ for each $i = 1, \ldots, m$ and $s \in J$. Hence

$$\|xZ\| = \max_{j=1,\ldots,m} \sum_{i=1}^m \|x_i z_{i,j}\|_{\text{sup}} = \max_{j=1,\ldots,m} \sup_{s \in J} \left| \sum_{i=1}^m x_i(s) z_{i,j}(s) \right|$$

$$\leq \|x\| \max_{j=1,\ldots,m} \sup_{s \in J} \left| \sum_{i=1}^m z_{i,j}(s) \right| = \|x\| \sup_{s \in J} \max_{j=1,\ldots,m} \sum_{i=1}^m \left| z_{i,j}(s) \right|$$

$$= \|x\| \sup_{s \in J} \|Z(s)\|_1.$$  

This shows that $\|\Phi(Z)\| \leq \sup_{s \in J} \|Z(s)\|_1$. On the other hand, fix $s \in J$ and $j \in \{1, \ldots, m\}$. For $i = 1, \ldots, m$, let $x_i \in X$ be the constant function equal to $\text{sgn} z_{i,j}(s)$ on $J$, where $\text{sgn}$ denotes the signum of a complex number. Then $\|x_i\|_{\text{sup}} = 1$ and

$$\sum_{i=1}^m \left| z_{i,j}(s) \right| = \sum_{i=1}^m x_i(s) z_{i,j}(s) \leq \left\| \sum_{i=1}^m x_i z_{i,j} \right\|_{\text{sup}} \leq \|xZ\| \leq \|\Phi(Z)\|.$$  

Taking maximum over $j \in \{1, \ldots, m\}$ and then taking supremum over $s \in J$, we see that $\sup_{s \in J} \|Z(s)\|_1 \leq \|\Phi(Z)\|$. This completes the proof. 

We are now in a position to give an estimate for $\|R(U, \Theta)\|$, where $U$ is the multiplication operator by an element $\xi$ of a function algebra, in terms of $\|\Theta\|_1, \|\xi\|_{\text{sup}}$ and the distance between range $(\xi)$ and $\text{sp}(\Theta)$. We shall use the following result of Kato.

**Lemma 3.4.** If $\Theta \in \mathbb{C}^{m \times m}$ is nonsingular and $\epsilon := \min\{|\lambda| : \lambda \in \text{sp}(\Theta)\}$,

then

$$\|\Theta^{-1}\|_1 \leq \frac{m^{m/2}}{\epsilon^m} \|\Theta\|^{m-1}.$$  

**Proof.** See [11], or [1, Lemma 2.20].

**Proposition 3.5.** Let $X$ be a function algebra on a compact Hausdorff space $J$. Let $\xi \in X$, and $Ux := x \xi$ for $x \in X$. Let $m \in \mathbb{N}$ and $\Theta \in \mathbb{C}^{m \times m}$ satisfy $\text{sp}(\xi) \cap \text{sp}(\Theta) = \emptyset$. Then

$$\|R(U, \Theta)\| \leq \frac{m^{m/2}}{\epsilon^m} (\|\Theta\|_1 + \|\xi\|_{\text{sup}})^{m-1},$$  

where $\epsilon := \text{dist}(\text{range}(\xi), \text{sp}(\Theta))$.  

Proof. Let $\Theta := [\theta_{i,j}]$ and $Z := [\theta_{i,j} - \text{diag}(\xi, \ldots, \xi)]$. By Corollary 3.2, $Z$ is invertible in $X^{m \times m}$ and $R(U, \Theta)y = yZ$ for all $y \in X$. Also, by Proposition 3.3, $\|R(U, \Theta)\| = \sup_{s \in J} \|Z^{-1}(s)\|_1$.

Since $Z$ is invertible in $X^{m \times m}$, we see that $\det Z$ is invertible in $X$ and $Z^{-1} = (\det Z)^{-1} \text{adj} Z$, where $\text{adj}$ denotes the adjugate or the classical adjoint. Also, for each $s \in J$, $\det Z(s) \neq 0$ and $Z(s)^{-1} = (\det Z(s))^{-1} \text{adj} Z(s)$, so that $Z^{-1}(s) = Z(s)^{-1}$. By Lemma 3.4, we see that for each $s \in J$,

$$\|Z(s)^{-1}\|_1 \leq \frac{m^{m/2}}{\epsilon(s)^m} \|Z(s)\|_1^{m-1},$$

where $\epsilon(s) := \min\{|\alpha| : \alpha \in \text{sp}(Z(s))\}$.

Now $Z(s) = \Theta - \text{diag}(\xi(s), \ldots, \xi(s))$ for each $s \in J$, and so

$$\epsilon(s) = \text{dist}(\xi(s), \text{sp}(\Theta)) \geq \text{dist}(\text{range}(\xi), \text{sp}(\Theta)) > 0,$$

since the range of $\xi$ is a closed subset of $\text{sp}(\xi)$. Also, $\|Z(s)\|_1 \leq \|\Theta\|_1 + \|\xi(s)\|_1 \leq \|\Theta\|_1 + \|\xi\|_{\text{sup}}$ for each $s \in J$. Hence the desired result follows. \hfill $\Box$

The above result shows that for the multiplication operator $U$ on a function algebra, $\|R(U, \Theta)\|$ is small if $\|\Theta\|_1$ and $\|\xi\|_{\text{sup}}$ are small, and if $\epsilon$ is not too small, that is, if range$(\xi)$ and $\text{sp}(\Theta)$ are well-separated. In case $\Theta$ represents a finite dimensional operator $B$ restricted to a spectral subspace, we have given an estimate for $\|\Theta\|_1$ in Lemma 2.1. When $m$ is small, the upper bound for $\|R(U, \Theta)\|$ given in Proposition 3.5 is of moderate size. For example, if $m = 2$, we obtain $\|R(U, \Theta)\| \leq 2(\|\Theta\|_1 + \|\xi\|_{\text{sup}})/\epsilon^2$.

As a variant of the above treatment, consider a finite measure space $(J, \mu)$, and let $X$ denote the commutative Banach algebra $L^\infty(J, \mu)$ along with the essential-sup norm $\|\cdot\|_{\text{esssup}}$. The equivalence class of the constant function 1 is the identity element of $X$. Fix $\xi \in X$, and as before, let $U$ denote the multiplication operator by $\xi$. Then $\|U\| = \|\xi\|_{\text{esssup}}$ and $\text{sp}(U) = \text{sp}(\xi)$, which coincides with the essential range of $\xi$ defined by

$$\text{essrange}(\xi) := \{\alpha \in \mathbb{C} : \mu\{t \in J : |\xi(t) - \alpha| < \epsilon\} > 0 \text{ for every } \epsilon > 0\}.$$ 

Let $m \in \mathbb{N}$, $x := [x_1, \ldots, x_m] \in X$ and $Z := [\xi_{i,j}] \in X^{m \times m}$. Then

$$\|xZ\| \leq \|x\| \|Z\|_1, \quad \text{where } \|Z\|_1 = \max_{j = 1, \ldots, m} \sum_{i = 1}^m \|\xi_{i,j}\|_{\text{esssup}}.$$ 

In fact, since $|x_i(s)| \leq \|x_i\|_{\text{esssup}} \leq \|x\|$ for each $i = 1, \ldots, m$ and almost all $s \in J$, we may prove, as in the proof of Corollary 3.2, that

$$\|xZ\| \leq \|x\|_{\text{esssup}} \mathcal{R}_Z, \quad \text{where } \mathcal{R}_Z(s) := \|Z(s)\|_1 \text{ for } s \in J.$$ 

We then have the following analogue of Proposition 3.5.

**Proposition 3.6.** Let $(J, \mu)$ be a finite measure space and $X := L^\infty(J, \mu)$. Consider a bounded measurable function $\xi$ on $J$ such that $\xi(s) \in \text{essrange}(\xi)$.
for almost all \( s \in J \), and let \( U x := x \xi \) for \( x \in X \). Let \( m \in \mathbb{N} \) and \( \Theta \in \mathbb{C}^{m \times m} \) satisfy essrange (\( \xi \)) \( \cap \text{sp}(\Theta) = \emptyset \). Then
\[
\| R(L, \Theta) \| \leq \frac{m^{m/2}}{e^m} (\| \Theta \|_1 + \| \xi \|_{\text{essup}})^{m-1}, \text{ where } \epsilon := \text{dist}(\text{essrange}(\xi), \text{sp}(\Theta)).
\]

In particular, if \( J \) is a bounded Lebesgue measurable subset of a Euclidean space and \( \mu \) is the Lebesgue measure, then the above inequality holds for every bounded measurable function \( \xi \) on \( J \).

**Proof.** Our proof is a modification of the proof of Proposition 3.5. Let \( \Theta := [\theta_{i,j}] \) and \( Z := [\theta_{i,j}] - \text{diag}(\xi, \ldots, \xi) \). Since \( \text{sp}(\xi) = \text{essrange}(\xi) \), Corollary 3.2 shows that \( Z \) is invertible in \( X^{m \times m} \) and \( R(L, \Theta)y = y Z^{-1} \) for all \( y \in X \). Hence \( \| R(L, \Theta) \| \leq \text{essup} \mathfrak{N}_{Z^{-1}} \), where \( \mathfrak{N}_{Z^{-1}}(s) := \| Z^{-1}(s) \|_1 \) for \( s \in J \), as we have noted just before stating this proposition.

Since \( \det Z \) is invertible in \( L^\infty(J, \mu) \), we see that \( \det Z(s) \neq 0 \), \( Z(s) \) is invertible in \( C^{m \times m} \) and \( Z^{-1}(s) = Z(s)^{-1} \) for almost all \( s \in J \). Lemma 3.4 shows that
\[
\| Z^{-1}(s) \|_1 \leq \frac{m^{m/2}}{\epsilon(s)^m} \| Z(s) \|_1^{m-1}, \text{ where } \epsilon(s) := \text{dist}(\xi(s), \text{sp}(\Theta)),
\]
for almost all \( s \in J \). By our assumption, \( \xi(s) \) belongs to essrange (\( \xi \)), and so \( \text{dist}(\xi(s), \text{sp}(\Theta)) \geq \text{dist}(\text{essrange}(\xi), \text{sp}(\Theta)) > 0 \) for almost all \( s \in J \). Also, \( \| Z(s) \|_1 \leq \| \Theta \| + |\xi(s)| \leq \| \Theta \|_1 + \| \xi \|_{\text{essup}} \) for almost all \( s \in J \). Consequently,
\[
\text{essup} \mathfrak{N}_{Z^{-1}} \leq \frac{m^{m/2}}{e^m} (\| \Theta \| + \| \xi \|_{\text{essup}})^{m-1}, \text{ where } \epsilon := \text{dist}(\text{essrange}(\xi), \text{sp}(\Theta)).
\]

Suppose now that \( J \) is a bounded Lebesgue measurable subset of a Euclidean space and \( \mu \) is the Lebesgue measure on \( J \). Suppose \( s \in J \) and \( \xi(s) \notin \text{essrange}(\xi) \). Then there is \( \epsilon > 0 \) such that \( |\xi(t) - \xi(s)| \geq \epsilon \) for almost all \( t \in J \). For \( r > 0 \), if we let \( \Delta(s, r) := \{ t \in J : |t - s| < r \} \), where \( |t - s| \) denotes the Euclidean distance between \( t, s \in J \), then
\[
\frac{1}{\mu(\Delta(s, r))} \int_{\Delta(s, r)} |\xi(t) - \xi(s)| d\mu(t) \geq \epsilon.
\]
Therefore \( \lim_{r \to 0^+} \frac{1}{\mu(\Delta(s, r))} \int_{\Delta(s, r)} |\xi(t) - \xi(s)| d\mu(t) \neq 0 \), that is, \( s \) is not a Lebesgue point of \( \xi \). Thus \( \xi(s) \in \text{essrange}(\xi) \) for every Lebesgue point \( s \) of \( \xi \). But since \( \xi \) is Lebesgue integrable on \( J \), almost every point of \( J \) is a Lebesgue point of \( \xi \). (See, for example, [17, Theorem 7.7].) Hence \( \xi(s) \in \text{essrange}(\xi) \) for almost all \( s \in J \). Thus the stated upper bound for \( \| R(L, \Theta) \| \) holds for every bounded measurable function \( \xi \) on \( J \). \( \square \)

4. Uniform conditioning

Let \((M_n)\) be a sequence of finite dimensional normed spaces over \( \mathbb{C} \). For notational ease, we shall denote the norm on each \( M_n \) by the same symbol \( \| \cdot \| \). This would suit our purpose later when we assume that each \( M_n \) is a subspace of a Banach space \( X \) over \( \mathbb{C} \). For \( n \in \mathbb{N} \), let \( m_n \) be the
dimension of \( M_n \), \( x_n := [x_{n,1}, \ldots, x_{n,m_n}] \) an ordered basis of \( M_n \), and \( x'_n := [x'_{n,1}, \ldots, x'_{n,m_n}] \) the ordered dual basis of \( M'_n \). Let \( \gamma_n := \max\{\|x_{n,j}\| : j = 1, \ldots, m_n\} \) and \( \gamma'_n := \max\{\|x'_{n,j}\| : j = 1, \ldots, m_n\} \) for \( n \in \mathbb{N} \). We shall say that the sequence \( \mathbf{x} := (x_n) \) of ordered bases is uniformly conditioned if the sequences \( (\gamma_n) \) and \( (\gamma'_n) \) are bounded. In this case, we define the condition number of the sequence \( \mathbf{x} := (x_n) \) of ordered bases of \( (M_n) \) by

\[
\text{cond} (\mathbf{x}) := \sup\{\gamma_n : n \in \mathbb{N}\} \sup\{\gamma'_n : n \in \mathbb{N}\}.
\]

Since \( \gamma'_n = 1/\delta_n \), where \( \delta_n := \min\{\text{dist}(x_{n,j}, \hat{M}_{n,j}) : j = 1, \ldots, m_n\} \), it follows that the sequence \( (\gamma_n) \) is bounded if and only if the sequence \( (\delta_n) \) is bounded away from 0, and in that event

\[
\text{cond} (\mathbf{x}) := \frac{\sup\{\gamma_n : n \in \mathbb{N}\}}{\inf\{\delta_n : n \in \mathbb{N}\}}.
\]

Since \( \text{cond} (x_n) = \gamma_n/\delta_n \) for each \( n \in \mathbb{N} \), we obtain \( \text{cond} (\mathbf{x}) \geq \sup\{\text{cond} (x_n) : n \in \mathbb{N}\} \). If, in particular, either the sequence \( (\gamma_n) \) or the sequence \( (\delta_n) \) is a constant sequence, then it follows that \( \text{cond} (\mathbf{x}) = \sup\{\text{cond} (x_n) : n \in \mathbb{N}\} \).

But a strict inequality may hold in general. For example, if \( m_n = 1 \) and \( x_n \) is a nonzero element of \( M_n \) for each \( n \in \mathbb{N} \), then \( \text{cond} (x_n) = 1 \) for each \( n \in \mathbb{N} \), but the sequence \( \mathbf{x} := (x_n) \) is uniformly conditioned if and only if the sequence \( (\|x_n\|) \) is bounded as well as bounded away from 0, and in that event, \( \text{cond} (\mathbf{x}) = \sup\{\|x_n\| : n \in \mathbb{N}\}/\inf\{\|x_n\| : n \in \mathbb{N}\} \). If a sequence \( \mathbf{x} \) of ordered bases of finite dimensional normed spaces is not uniformly conditioned, that is, if at least one of the sequences \( (\gamma_n) \) and \( (\gamma'_n) \) is unbounded, then we define \( \text{cond} (\mathbf{x}) := \infty \).

We observe that a sequence \( \mathbf{x} := (x_n) \) of ordered bases is uniformly conditioned if and only if there are positive real numbers \( \gamma \) and \( \delta \) such that \( \|x_{n,j}\| \leq \gamma \) and \( \text{dist}(x_{n,j}, \hat{M}_{n,j}) \geq \delta \), for all \( n \in \mathbb{N} \) and \( j = 1, \ldots, m \), where \( \hat{M}_{n,j} = \text{span}\{x_{n,i} : i = 1, \ldots, m, i \neq j\} \), and in that event,

\[
\text{cond} (\mathbf{x}) = \frac{\sup\{\|x_{n,j}\| : n \in \mathbb{N}, j = 1, \ldots, m_n\}}{\inf\{\text{dist}(x_{n,j}, \hat{M}_{n,j}) : n \in \mathbb{N}, j = 1, \ldots, m_n\}} \leq \gamma/\delta.
\]

In [1, §5.1.2], such a sequence of bases was called ‘uniformly well-conditioned’. However, it seems appropriate to use the nomenclature ‘uniformly conditioned’, since the adverb ‘well’ indicates a judgment on the part of the user. We shall say that a sequence \( \mathbf{x} := (x_n) \) is uniformly well-conditioned if its condition number is of moderate size, relative to the problem at hand depending on the desired accuracy and on the availability of computing precision. If the ordered basis \( x_n \) is orthogonal for each \( n \in \mathbb{N} \), then

\[
\text{cond} (\mathbf{x}) = \frac{\sup\{\|x_{n,j}\| : n \in \mathbb{N}, j = 1, \ldots, m_n\}}{\inf\{\|x_{n,j}\| : n \in \mathbb{N}, j = 1, \ldots, m_n\}}.
\]

If, in addition, the norms of all \( x_{n,j} \)'s are equal, then \( \text{cond} (\mathbf{x}) = 1 \), and the sequence \( \mathbf{x} = (x_n) \) of ordered bases is best-conditioned.
We shall now use the results of Sections 2 and 3 to obtain a sequence of uniformly conditioned ordered bases of spectral subspaces of a sequence of operators. Let $X$ be a Banach space, and let $c_{00}$ denote the linear space of all complex sequences having all but finitely many terms equal to 0 along with a specified norm $\| \cdot \|_0$. For each $n \in \mathbb{N}$, we may consider $\mathbb{C}^{n \times 1}$ to be a subspace of $c_{00}$ via the map $[\alpha_1, \ldots, \alpha_n] \in \mathbb{C}^{n \times 1} \rightarrow (\alpha_1, \ldots, \alpha_n, 0, 0, \ldots) \in c_{00}$. For $n \in \mathbb{N}$, let $S_n \in BL(X)$ be of rank $\leq n$ and, as shown in Section 2, let $S_n = K_n L_n$, where $K_n : X \rightarrow \mathbb{C}^{n \times 1}$ and $L_n : \mathbb{C}^{n \times 1} \rightarrow X$ are continuous linear maps. If for $n \in \mathbb{N}$, we define $A_n : \mathbb{C}^{n \times 1} \rightarrow \mathbb{C}^{n \times 1}$ by $A_n := K_n L_n$, then $A_n K_n = K_n S_n$. For $n \in \mathbb{N}$, consider $U_n$ in $BL(X)$ such that the range of $K_n'$ is invariant under $U_n$, so that there is a (continuous) linear map $C_n : \mathbb{C}^{n \times 1} \rightarrow \mathbb{C}^{n \times 1}$ such that $C_n K_n = K_n U_n$. Finally, for $n \in \mathbb{N}$, let $T_n := S_n + U_n$ and $B_n := A_n + C_n$, so that $B_n K_n = K_n T_n$. We shall use this notation in the sequel.

Let $\Lambda_n$ be a subset of $\text{sp}(B_n)$ which does not intersect $\text{sp}(C_n) \cup \text{sp}(U_n)$. Suppose the spectral subspace $M(B_n, \Lambda_n)$ associated with $B_n$ and $\Lambda_n$ is of dimension $m_n$. (Obviously, $m_n \leq n$ for each $n \in \mathbb{N}$.) Then for each $n \in \mathbb{N}$, the spectral subspace $M(T_n, \Lambda_n)$ associated with $T_n$ and $\Lambda_n$ is also of dimension $m_n$; in fact, if $u_n$ forms an ordered basis of $M(B_n, \Lambda_n)$, then there is a unique $\Theta_n \in \mathbb{C}^{m_n \times m_n}$ such that $B_n u_n = u_n \Theta_n$ and there is a unique ordered basis $\varphi_n$ of $M(T_n, \Lambda_n)$ such that $\varphi_n \Theta_n - U_n \varphi_n = L_n u_n$. The following result shows that if the sequence $u := (u_n)$ of ordered bases is uniformly conditioned, then so is the sequence of ordered bases $\varphi := (\varphi_n)$ of ordered bases, provided the norms of $K_n$, $L_n$ and $R(U_n, \Theta_n)$ are bounded.

**Theorem 4.1.** Consider a norm $\| \cdot \|_0$ on $c_{00}$. For each $n \in \mathbb{N}$, let $\Lambda_n$ be a subset of $\text{sp}(B_n)$ which is disjoint from $\text{sp}(C_n) \cup \text{sp}(U_n)$, and let $m_n$ be the sum of the algebraic multiplicities of eigenvalues in $\Lambda_n$. Let $(u_n)$ be a sequence of ordered bases of the spectral subspaces $M(B_n, \Lambda_n)$ associated with $B_n$ and $\Lambda_n$, and define $\varphi_n := R(U_n, \Theta_n) L_n u_n$ for $n \in \mathbb{N}$. If the sequence $(u_n)$ is uniformly conditioned, and if there are positive numbers $\alpha, \beta, \eta$ such that $\|K_n\| \leq \alpha$, $\|L_n\| \leq \beta$ and $\|R(U_n, \Theta_n)\| \leq \eta$ for all $n \in \mathbb{N}$, then the sequence $(\varphi_n)$ of ordered bases of the spectral subspaces $M(T_n, \Lambda_n)$ associated with $T_n$ and $\Lambda_n$ is uniformly conditioned. In fact,

$$\text{cond } (\varphi) \leq \alpha \beta \eta \text{cond } (u).$$

**Proof.** For $n \in \mathbb{N}$, let $u_n := [u_{n,1}, \ldots, u_{n,m_n}]$, $c_n := \max\{\|u_{n,j}\|_0 : j = 1, \ldots, m_n\}$ and $d_n := \min\{\text{dist}(u_{n,j}, \tilde{M}_{n,j}(B_n, \Lambda_n)) : j = 1, \ldots, m_n\}$. Since the sequence $(u_n)$ is uniformly conditioned, $c := \sup\{c_n : n \in \mathbb{N}\} < \infty$ and $d := \inf\{d_n : n \in \mathbb{N}\} > 0$. Now let $\varphi_n := [\varphi_{n,1}, \ldots, \varphi_{n,m_n}]$, $\gamma_n := \max\{\|\varphi_{n,j}\| : j = 1, \ldots, m_n\}$ and $\delta_n := \min\{\text{dist}(\varphi_{n,j}, \tilde{M}_{n,j}(T_n, \Lambda_n)) : j = 1, \ldots, m_n\}$. By Proposition 2.3,

$$\gamma_n \leq \|R(U_n, \Theta_n)\| \|L_n\| c_n \leq \eta \beta c$$

and

$$d \leq d_n \leq \|K_n\| \delta_n \leq \alpha \delta_n.$$
for all $n \in \mathbb{N}$. Thus the sequence $(\gamma_n)$ is bounded and the sequence $(\delta_n)$ is bounded away from 0, that is, the sequence $(\varphi_n)$ of ordered bases of $M(T_n, \Lambda_n)$ is uniformly conditioned, and since $\text{cond}(\mathbf{u}) = c/d$, we obtain

$$\text{cond}(\varphi) = \frac{\sup \{\gamma_n : n \in \mathbb{N}\}}{\inf \{\delta_n : n \in \mathbb{N}\}} \leq \frac{\beta \eta c}{d/\alpha} = \alpha \beta \eta \text{cond}(\mathbf{u}).$$

as desired. \qed

**Remark 4.2.** As we have mentioned in Section 2, it is often possible choose a norm $\| \cdot \|_0$ on $c_00$ in consonance with the given norm on the Banach space $X$, so that the sequences $(\|K_n\|)$ and $(\|L_n\|)$ remain bounded. This was illustrated in Propositions 2.4 and 2.5. In general, the question of establishing a bound for the sequence $(\|R(U_n, \Theta_n)\|)$ is tricky. In the special situation where each $\Lambda_n$ consists of a single semisimple eigenvalue $\lambda_n$ of $B_n$, the spectral subspace $M(B_n, \Lambda_n)$ is in fact the eigenspace $N(\lambda_n I - B_n)$ of dimension $m_n$. The same holds for $M(T_n, \Lambda_n)$ since the ascent of $\lambda_n$ as an eigenvalue of $T_n$ is the same as its ascent as an eigenvalue of $B_n$. (See Proposition 2.4 of [14].) In this case, $\|R(U_n, \Theta_n)\| = \|(\lambda_n I - U_n)^{-1}\| = \|(\lambda_n I - U_n)^{-1}\|$ for each $n \in \mathbb{N}$, so that any bound of $(\|\lambda_n I - U_n\|)$ would work as the positive number $\eta$ of Theorem 4.1. If, in addition, each $U_n = 0$, then the sequence $(\|R(U_n, \Theta_n)\|)$ is bounded if and only if the sequence $(\lambda_n)$ of eigenvalues is bounded away from 0.

We now consider the case when the Banach space $X$ has a multiplicative structure and each $U_n$ is a multiplication operator on $X$.

**Corollary 4.3.** Let $X$ be a Banach algebra with the identity element $1_X$. For $n \in \mathbb{N}$, let $\xi_n \in X$ and define $U_n x := x \xi_n$ for $x \in X$. Consider a norm $\| \cdot \|_0$ on $c_00$. For each $n \in \mathbb{N}$, let $\Lambda_n$ be a subset of $\text{sp}(B_n)$, which is disjoint from $\text{sp}(C_n) \cup \text{sp}(\xi_n)$, and let $m_n$ be the sum of the algebraic multiplicities of eigenvalues in $\Lambda_n$. Let $(\mathbf{u}_n)$ be a sequence of ordered bases of the spectral subspaces $M(B_n, \Lambda_n)$ associated with $B_n$ and $\Lambda_n$, and define $\varphi_n := L_n u_n Z_n^{-1}$, where $Z_n := \Theta_n - \text{diag}(\xi_n, \ldots, \xi_n)$ for $n \in \mathbb{N}$. If the sequence $(\mathbf{u}_n)$ is uniformly conditioned, and if there are positive numbers $\alpha, \beta, \eta$ such that $\|K_n\| \leq \alpha$, $\|L_n\| \leq \beta$ and $\|Z_n^{-1}\|_1 \leq \eta$ for all $n \in \mathbb{N}$, then the sequence $(\varphi_n)$ of ordered bases of the spectral subspaces $M(T_n, \Lambda_n)$ associated with $T_n$ and $\Lambda_n$ is uniformly conditioned. In fact, $\text{cond}(\varphi) \leq \alpha \beta \eta \text{cond}(\mathbf{u})$.

**Proof.** By Corollary 3.2, $Z_n$ is invertible in $X^{m_n \times m_n}$, and $R(U_n, \Theta_n)y = y Z_n^{-1}$, $y \in X^{1 \times m_n}$ for each $n \in \mathbb{N}$. Since $\|R(U_n, \Theta_n)\| \leq \|Z_n^{-1}\|_1$ for each $n \in \mathbb{N}$, the result follows from Theorem 4.1. \qed
Remark 4.4. As we have noted in Section 3, the estimate \( \| R(U_n, \Theta_n) \| \leq \| Z_n^{-1} \|_1 \) given in the above corollary may not be sharp in general. In the special case where the Banach algebra \( X \) is in fact a function algebra, Proposition 3.6 gives the following estimate:

\[
\| R(U_n, \Theta_n) \| \leq \frac{m_n^{n/2}}{\epsilon_n^{m_n/2}} (\| \Theta_n \|_1 + \| \xi_n \|_{\text{sup}})^{m_n - 1},
\]

where \( \epsilon_n := \text{dist}(\text{range}(\xi_n), \text{sp}(\Theta_n)) \), and by Lemma 2.1,

\[
\| \Theta_n \|_1 \leq m_n (\| K_n \| \| L_n \| + \| C_n \|) \text{cond}(u_n) \leq m_n (\alpha \beta + \| C_n \|) \text{cond}(u)
\]

for each \( n \in \mathbb{N} \). Thus, if \( X \) is a function algebra, then instead of assuming that the sequence \( (\| Z_n^{-1} \|_1) \) is bounded, we may assume that the sequences \( (m_n) \), \( (\| C_n \|) \) and \( (\| \xi \|_{\text{sup}}) \) are bounded, and the sequence \( \epsilon_n \) is bounded away from 0 in Corollary 4.3, and then the sequence \( \varphi = (\varphi_n) \) of ordered bases would be uniformly conditioned. In view of Proposition 3.6, a similar result holds for \( X := L^\infty(J, \mu) \) if we replace \( \| \xi_n \|_{\text{sup}} \) and \( \text{range}(\xi_n) \) by \( \| \xi_n \|_{\text{ess} \sup} \) and \( \text{essrange}(\xi_n) \) respectively.

If each \( U_n := 0 \), then we do not need any multiplicative structure on the Banach space \( X \). In this case, \( \xi_n = 0 \) and we may take \( C_n = 0 \), so that \( B_n = A_n, Z_n = \Theta_n \) and \( \epsilon_n = \text{dist}(0, \Theta_n) \) for each \( n \in \mathbb{N} \). This special case (with \( m_n = m \) for all \( n \in \mathbb{N} \)) was treated in [6, Theorem 4.1] albeit with a different estimate; this estimate was later improved in [1, Theorem 5.9].

5. Spectral Approximation

Let \( X \) be a Banach space over \( \mathbb{C} \), let \( F \in BL(X) \) and \( \Lambda \subseteq \text{sp}(F) \) be a spectral set of finite type, that is, \( \Lambda \) as well as \( \text{sp}(F) \setminus \Lambda \) are closed sets and the spectral subspace \( M(F, \Lambda) \) associated with \( F \) and \( \Lambda \) is finite-dimensional. One of the central problems in spectral approximation is to construct approximations of an ordered basis of \( M(F, \Lambda) \). This is usually accomplished by considering a sequence \((T_n)\) in \( BL(X) \) which approximates \( F \) in such a manner that a sequence \((\Lambda_n)\) of spectral sets of \((T_n)\) would approximate \( \Lambda \), and a sequence \((\varphi_n)\) of ordered bases of the spectral subspaces \( M(F_n, \Lambda_n) \) would turn out to be approximate ordered bases of \( M(F, \Lambda) \). Classically, the sequence \((T_n)\) is required to converge to \( F \) in the operator norm or in a collectively compact manner. More recently, it has been shown that the following conditions are sufficient for this purpose: (i) the sequence \((T_n)\) is bounded, (ii) \( \| (T_n - F)F \| \to 0 \), and (iii) \( \| (T_n - F)T_n \| \to 0 \). If these conditions are satisfied, the sequence \((T_n)\) is said to be \( \nu \)-convergent to \( F \). (See [1, §2.1].) In this case, for all large \( n \in \mathbb{N} \), the dimension of \( M(T_n, \Lambda_n) \) is equal to the dimension of \( M(F, \Lambda) \), provided \( 0 \notin \Lambda \).

Since one is interested in constructing an ordered basis \( \varphi_n \) of the spectral space \( M(T_n, \Lambda_n) \), one usually chooses the approximating operator \( T_n \) to be of finite rank. Some of the well-known finite rank approximations of \( F \) are as follows: projection approximation \( \pi_n F \), Sloan approximation \( F \pi_n \), Galerkin approximation \( \pi_n F \pi_n \) and a modified projection approximation.
\( \pi_n F + F \pi_n - \pi_n F \pi_n \) (considered in Proposition 2.4), where \( \pi_n \) is a finite-rank projection in \( BL(X) \) such that the sequence \( (\pi_n) \) converges to the identity operator on \( X \) pointwise. For an integral operator \( F \in BL(X) \), the most commonly used finite-rank approximations are the Nyström approximation and the degenerate kernel approximation. In some situations, however, we need to use an approximation of the kind \( T_n = S_n + U_n \), where \( S_n \) is of finite rank and \( U_n \) is, in general, of infinite rank. We shall now describe two such situations to illustrate the applicability of our results proved in Section 4. Both involve the multiplication operator treated at length in Section 3.

### 5.1. Integral Operator with a Weakly Singular Kernel

Let \( X := C([0, 1]) \) with the sup norm \( \| \cdot \|_{\text{sup}} \), and let \( F \) denote a Fredholm integral operator on \( X \) having a weakly singular kernel given by

\[
(Fx)(s) = \int_0^1 \kappa(|s-t|)x(t)dt \quad \text{for } x \in X \text{ and } s \in [0, 1],
\]

where \( \kappa : (0, 1) \to \mathbb{R} \) is a nonnegative decreasing function such that \( \kappa(t) \to \infty \) as \( t \to 0^+ \) and \( \int_0^1 \kappa(t)dt < \infty \). For example, let \( \kappa(t) := -\ln t \) or \( \kappa(t) := t^{-\alpha} \), where \( 0 < \alpha < 1 \), for \( t \in (0, 1] \). For \( n \in \mathbb{N} \), let \( 0 < h_n < 1 \), and define

\[
\kappa_n(t) := \begin{cases} 
\kappa(h_n) & \text{if } 0 \leq t \leq h_n, \\
\kappa(t) & \text{if } h_n < t \leq 1.
\end{cases}
\]

If \( F_n \) denotes the Fredholm integral operator on \( X \) having the continuous kernel \( \kappa_n \), then \( \|F - F_n\| \to 0 \) as \( h_n \to 0 \). (See [3, Theorem 1].) Hence the operator \( F \) is a compact, and so every spectral set of \( F \) not containing 0 is of finite type. Let \( \Lambda \) be a such a spectral set of \( F \). We address the problem of finding a sequence of approximate uniformly conditioned ordered bases of the spectral subspace \( M(F, \Lambda) \). The singularity subtraction technique proposed by Kantorovich and Krylov involves writing

\[
(Fx)(s) = \int_0^1 \kappa(|s-t|)(x(t) - x(s))dt + x(s) \int_0^1 \kappa(|s-t|)dt
\]

for \( x \in X \) and \( s \in [0, 1] \). For \( n \in \mathbb{N} \), let \( Q_n \) be a quadrature formula with distinct nodes \( t_{n,1}, \ldots, t_{n,n} \) in \([0, 1]\), where \( t_{n,1} = 0, t_{n,1} := h_n \), and with nonnegative weights \( w_{n,1}, \ldots, w_{n,n} \). We assume that \( Q_n(x) \to \int_0^1 x(t)dt \) for every \( x \in X \). Fix \( n \in \mathbb{N} \). The \( n \)th Kantorovich-Krylov approximation of \( F \) is given by \( T_n := S_n + U_n \), where

\[
(S_n x)(s) := \sum_{j=1}^n w_{n,j} \kappa_n(|s-t_{n,j}|)x(t_{n,j}) \quad \text{and}
\]

\[
(U_n x)(s) := x(s) \left( \int_0^1 \kappa(|s-t|)dt - \sum_{j=1}^n w_{n,j} \kappa_n(|s-t|) \right)
\]

for \( x \in X \) and \( s \in [0, 1] \). Here \( S_n \) is a variant of the Nyström approximation, and \( U_n \) is a multiplication operator. For for \( j = 1, \ldots, n \), define \( x_{n,j} \in X \) by
Let \( \xi_n \in X \) by \( \xi_n(s) := \int_0^1 \kappa(|s-t|) dt - \sum_{j=1}^n w_{n,j} \kappa_n(|s-t|) \) for \( s \in [0,1] \). Then \( U_n x = x \xi_n \) for \( x \in X \). Also, \( f_{n,j}(U_n x) = f_{n,j}(x \xi_n) = x(t_{n,j}) \xi_n(t_{n,j}) = \xi_n(t_{n,j}) f_{n,j}(x) \) for all \( x \in X \), that is, \( f_{n,j} \circ U_n = U_n f_{n,j} \) for \( j = 1, \ldots, n \). Hence the range of \( K'_n \) is invariant under \( U_n \). Define \( C_n : \mathbb{C}^{n \times 1} \to \mathbb{C}^{n \times 1} \) by \( C_n u := [\xi_n(t_{n,1}) u(1), \ldots, \xi_n(t_{n,n}) u(n)]^t \). Note that \( \text{sp}(C_n) = \{ \xi_n(t_{n,j}) : j = 1, \ldots, n \} \subseteq \text{range}(\xi_n) = \text{sp}(U_n) \).

Let us consider the \( \infty \)-norm on \( \mathbb{C}^{n \times 1} \). It follows that \( \| K_n \| \leq 1 \) and \( \| L_n \| \leq \| \sum_{j=1}^n |x_{n,j}| \|_{\text{sup}} \). To ensure that the sequence \( (\| \sum_{j=1}^n |x_{n,j}| \|_{\text{sup}}) \) is bounded, we assume that the sequence \( (Q_n) \) of quadrature formulae satisfies an additional hypothesis, namely the sum of the weights corresponding to the nodes lying in any half-open subinterval of \( [0,1] \) is less than or equal to a constant times the length of that subinterval, the constant being independent of \( n \). (This hypothesis was first introduced in [7].) Most quadrature formulae induced by piecewise linear interpolatory projections, such as the compound mid-point rule, the compound trapezoidal rule, the compound Gauss two-point rule, satisfy this requirement.) Then Lemma 4.17 of [1] can be used to prove that the sequence \( (\| L_n \|) \) is bounded. Under the stated hypothesis on the sequence \( (Q_n) \), it is proved in [1, Proposition 4.18, Theorem 4.19] that \( \| F x - S_n x \|_{\text{sup}} \to 0 \) for each \( x \in X \), and \( (T_n) \) is \( \nu \)-convergent to \( F \). Since \( U_n(1) = \xi_n = F(1) - S_n(1) \) for \( n \in \mathbb{N} \), we see that \( \| \xi_n \|_{\text{sup}} \to 0 \), and since \( \| C_n \| = \max\{|\xi_n(t_{n,j})| : j = 1, \ldots, n\} \leq \| \xi_n \|_{\text{sup}} \), we obtain \( \| C_n \| \to 0 \).

Now since the spectral set \( \Lambda \) of \( F \) is of finite type and since it does not contain 0, it consists of a finite number of nonzero eigenvalues of \( F \) each having a finite algebraic multiplicity. Let the sum of these algebraic multiplicities be equal to \( m \). Then there are finitely many closed disks in \( \mathbb{C} \) whose union \( D \) contains \( \Lambda \) but does not contain either 0 or any other eigenvalue of \( F \). For \( n \in \mathbb{N} \), let \( \Lambda_n := \text{sp}(T_n) \cap D \). Since \( (T_n) \) is \( \nu \)-convergent to \( F \), \( \Lambda_n \) is a spectral set of \( T_n \) and if \( m_n \) denotes the dimension of the associated spectral space \( M(T_n, \Lambda_n) \), then \( m_n = m \) for all large \( n \in \mathbb{N} \). (See [1, Theorem 2.12].) Also, since \( 0 \notin D \) and \( \| \xi_n \|_{\text{sup}} \to 0 \), there is \( \epsilon > 0 \) such that \( \epsilon_n := \text{dist} (\text{range} \( (\xi_n) \), \Lambda_n) \geq \epsilon \) for all large \( n \in \mathbb{N} \).

5.2. Sum of an integral operator with a continuous kernel and a multiplication operator. Let \( X := L^\infty([a,b], \mu) \), with the essential sup norm \( \| \cdot \|_{\text{ess sup}} \), where \( \mu \) is the Lebesgue measure. Let \( k(\cdot, \cdot) \in C([a,b] \times [a,b]) \), \( \xi \in C([a,b]) \), and let \( F \in BL(X) \) be given by

\[
(Fx)(s) = \int_a^b k(s,t)x(t)d\mu(t) + x(s)\xi(s) \quad \text{for } x \in X \text{ and } s \in [a,b].
\]
Such operators arise in the study of electromagnetic waveguide problems (for instance, in evaluation of propagation constants of a dielectric guide) as well as in population genetics (for instance, in continuum-of-alleles models). (See [16] and the references mentioned therein.) The operator $F$ is not compact unless $\xi = 0$. Let us assume, however, that $\Lambda$ is a spectral set of $F$ of finite type which is disjoint from the range of $\xi$. We address the problem of finding a sequence of uniformly conditioned approximate ordered bases of the spectral subspace $M(F, \Lambda)$.

For $n \in \mathbb{N}$, consider $a := t_{n,0} < t_{n,1} < \cdots < t_{n,n} = b$ such that the mesh $h_n$ of this partition tends to zero. Fix $n \in \mathbb{N}$. Let $E_{n,1} := [t_{n,0}, t_{n,1}], E_{n,2} := (t_{n,1}, t_{n,2}], \ldots, E_{n,n} := (t_{n,n-1}, t_{n,n}],$ and for $j = 1, \ldots, n$, let $\chi_{n,j}$ denote the characteristic function of $E_{n,j}$. Define $T_n := S_n + U_n$, where

$$
(S_n x)(s) := \sum_{j=1}^{n} k(s, t_{n,j}) \int_{E_{n,j}} x(t) d\mu(t) \quad \text{and} \quad (U_n x)(s) := x(s) \sum_{j=1}^{n} \xi(t_{n,j}) \chi_{n,j}(s)
$$

for $x \in X$ and $s \in [a, b]$. Here $S_n$ is a degenerate kernel approximation based on piecewise constant interpolation in the second variable, and $U_n$ is a multiplication operator. For $j = 1, \ldots, n$, define $x_{n,j} \in X$ by $x_{n,j}(s) := k(s, t_{n,j})$ for $s \in [a, b]$, and define $f_{n,j} \in X'$ by $f_{n,j}(x) := \int_{E_{n,j}} x(t) d\mu(t)$ for $x \in X$. Then $S_n x = \sum_{j=1}^{n} f_{n,j}(x) x_{n,j}$ for $x \in X$. If we define $K_n : X \to \mathbb{C}^{n \times 1}$ by $K_n x := [f_{n,1}(x), \ldots, f_{n,n}(x)]^t$ and $L_n : \mathbb{C}^{n \times 1} \to X$ by $L_n u := u(1)x_{n,1} + \cdots + u(n)x_{n,n}$ as usual, then $S_n = L_n K_n$. Define $\xi_n \in X$ by $\xi_n(s) := \sum_{j=1}^{n} \xi(t_{n,j}) \chi_{n,j}(s)$ for $s \in [a, b]$. Then $U_n x = x \xi_n$ for $x \in X$. Also, $f_{n,j}(U_n x) = f_{n,j}(x \xi_n) = \int_{E_{n,j}} x(t) \xi_n(t) d\mu(t) = \xi(t_{n,j}) f_{n,j}(x)$ for all $x \in X$, that is, $f_{n,j} \circ U_n = \xi(t_{n,j}) f_{n,j}$ for $j = 1, \ldots, n$. Hence the range of $K_n$ is invariant under $U_n$. Define $C_n : \mathbb{C}^{n \times 1} \to \mathbb{C}^{n \times 1}$ by

$$
C_n u := [\xi(t_{n,1}) u(1), \ldots, \xi(t_{n,n}) u(n)]^t.
$$

Note that essrange ($\xi$) = range ($\xi$) since $\xi$ is continuous, and clearly, essrange ($\xi_n$) = $\{\xi(t_{n,j}) : j = 1, \ldots, n\}$. Thus $\text{sp}(C_n) = \text{range} (\xi_n) = \text{sp}(U_n)$.

Let us consider the 1-norm on $\mathbb{C}^{n \times 1}$. It follows that $\|K_n\| \leq b - a$ and $\|L_n\| \leq \sup\{ |k(s, t)| : s, t \in [a, b] \}$. Also, $\|C_n\| = \max\{ |\xi(t_{n,j})| : j = 1, \ldots, n\} = \|\xi_n\|_{\text{esssup}} \leq \|\xi\|_{\text{esssup}}$ for $n \in \mathbb{N}$. (Note: The natural choice for a norm on $\mathbb{C}^{n \times 1}$ would have been the $\infty$-norm. With this choice, $\|K_n\| = h_n$ and $\|L_n\| = \| \sum_{j=1}^{n} |k(\cdot, t_{n,j})| \|_{\text{esssup}}$ for $n \in \mathbb{N}$. Then $\|K_n\| \to 0$, while $(\|L_n\|)$ would be unbounded. To rectify this situation, we may resort to the following scaling operation: Multiply $x_{n,j}$ and divide $f_{n,j}$ by the length of the subinterval $E_{n,j}$ for all $n \in \mathbb{N}$ and $j = 1, \ldots, n$. In this scaled version, we still have $S_n = L_n K_n$ for all $n \in \mathbb{N}$, and it is easy to see that $\|K_n\| \leq 1$ and $\|L_n\| \leq (b - a) \sup\{ |k(s, t)| : s, t \in [a, b] \}$ for $n \in \mathbb{N}$ with respect to the $\infty$-norm on $\mathbb{C}^{n \times 1}$. However, for large $n \in \mathbb{N}$, the length of the subinterval
$E_{n,j}$ will be very small, and the above scaling would be computationally ill-advised. Hence the choice of the 1-norm is not appropriate.)

Since $\chi_{n,j}(t) \geq 0$ and $\sum_{j=1}^{n} \chi_{n,j}(t) = 1$ for all $t \in [a, b]$, since the kernel $k(\cdot, \cdot)$ is uniformly continuous on $[a, b] \times [a, b]$, and since the multiplier $\xi$ is uniformly continuous on $[a, b]$, we see that $\|F - T_n\| \to 0$. Since the spectral set $\Lambda$ of $F$ is of finite type and since it is disjoint from the closed set range ($\xi$), it follows, as in the case of the singularity subtraction, that there is a spectral set $\Lambda_n$ of $(T_n)$ such that $\text{dist}(\text{range} (\xi), \Lambda_n) \geq \epsilon$ for some $\epsilon > 0$, and if $m_n$ denotes the dimension of the associated spectral space $M(T_n, \Lambda_n)$, then $m_n = m$ for all large $n \in \mathbb{N}$. (See [1, Theorem 2.12].) Further, since range $(\xi_n) \subseteq \text{range} (\xi)$, we see that $\epsilon_n := \text{dist}(\text{range} (\xi_n), \Lambda_n) \geq \epsilon$.

As an alternative, we may consider an approximation $\tilde{S}_n + U_n$ of the operator $F$, where $\tilde{S}_n$ is the degenerate kernel approximation based on piecewise constant interpolation in both variables, namely,

$$
(\tilde{S}_n x)(s) := \sum_{j=1}^{n} \left( \sum_{i=1}^{n} k(t_{n,i}, t_{n,j}) \chi_{E_{n,i}}(s) \right) \int_{E_{n,j}} x(t) d\mu(t)
$$

for $x \in X$ and $s \in [a, b]$. Then $\tilde{S}_n = \tilde{K}_n \tilde{L}_n$, where $\tilde{K}_n = K_n$ and $\tilde{L}_n : C^{n \times 1} \to X$ is defined by

$$
(\tilde{L}_n u)(s) := \sum_{j=1}^{n} u(j) \left( \sum_{i=1}^{n} k(t_{n,i}, t_{n,j}) \chi_{E_{n,i}}(s) \right) \quad \text{for} \quad x \in X \text{ and } s \in [a, b].
$$

With respect to the 1-norm on $C^{n \times 1}$, we again obtain $\|\tilde{L}_n\| \leq \sup \{ |k(s, t)| : s, t \in [a, b] \}$. (Compare Proposition 2.5 (ii).) Hence the earlier estimates go through with $\tilde{S}_n$ in place of $S_n$. The advantage of $\tilde{S}_n$ over $S_n$ lies in the fact that the entries of the matrix $[k(t_{n,i}, t_{n,j})]$ representing the map $A_n$ corresponding to $\tilde{S}_n$ are much easier to calculate than the entries of the matrix $[\int_{E_{n,i}} k(s, t_{n,j})ds]$ representing the map $A_n$ corresponding to $S_n$. This was pointed out by the authors of [16] elsewhere.

To summarize, in both situations described in subsections 5.1 and 5.2, the sequences $(\|K_n\|)$, $(\|L_n\|)$, $(m_n)$, $(\|\xi\|)$ and $(\|\xi\|_{\sup})$ are bounded, while the sequence $(\text{dist}(\text{range} (\xi_n), \Lambda_n))$ is bounded away from 0, that is, all the requirements stated in Remark 4.4 are fulfilled. If we let $A_n := K_n L_n$ and $B_n := A_n + C_n$ for $n \in \mathbb{N}$ for $n \in \mathbb{N}$, then we may choose a sequence $u_n := (u_n)$ of uniformly conditioned ordered bases of the spectral subspaces $M(B_n, \Lambda_n)$. Let $B_n u_n = \Theta_n u_n$ with $\text{sp}(\Theta_n) = \Lambda_n$, and define $\varphi_n := L_n u_n Z_n^{-1}$ for all large $n \in \mathbb{N}$, where $Z_n := \Theta_n - \text{diag}(\xi_n, \ldots, \xi_n)$. Then the sequence $\varphi := (\varphi_n)$ of ordered bases of the spectral subspaces $M(T_n, \Lambda_n)$ is uniformly conditioned, and it serves as a sequence of approximate ordered bases of the spectral subspace $M(F, \Lambda)$ associated with $F$ and $\Lambda$.

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