ISOMETRIES OF REAL HILBERT C^*-MODULES

MING-HSIU HSU AND NGAI-CHING WONG

ABSTRACT. Let $T : V \to W$ be a surjective real linear isometry between full real Hilbert C^*-modules over real C^*-algebras A and B, respectively. We show that the following conditions are equivalent: (a) T is a 2-isometry; (b) T is a complete isometry; (c) T preserves ternary products; (d) T preserves inner products; (e) T is a module map. When A and B are commutative, we give a full description of the structure of T.

1. INTRODUCTION

In this paper, we study surjective isometries T between *real* Hilbert C^*-modules. Replacing T with the map $T - T(0)$, we can assume that T is *real* linear by the Mazur-Ulam theorem. Here is our main result.

Theorem 1.1. *Let A and B be real C^*-algebras, and let V and W be full real Hilbert C^*-modules over A and B, respectively. Let $T : V \to W$ be a surjective real linear isometry. The following conditions are all equivalent:*

(a) *T is a 2-isometry;*
(b) *T is a complete isometry;*
(c) *T preserves ternary products, i.e.,*

$$T(x\langle y, z\rangle) = T(x)\langle T(y), T(z)\rangle, \quad \forall\, x, y, z \in V;$$

(d) *T preserves inner products with respect to a $*$-isomorphism $\alpha : A \to B$, i.e.,*

$$\langle T(x), T(y)\rangle = \alpha(\langle x, y\rangle), \quad \forall\, x, y \in V;$$

(e) *T is a module map with respect to a $*$-isomorphism $\alpha : A \to B$, i.e.,*

$$T(xa) = T(x)\alpha(a), \quad \forall\, x \in V, a \in A.$$

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Since complex C^*-algebras and complex Hilbert C^*-modules are also real C^*-algebras and real Hilbert C^*-modules, respectively, Theorem 1.1 covers both the real and the complex cases. However, we can derive the complex version of Theorem 1.1 with the results established in the literature. In fact for a surjective complex linear isometry T between complex Hilbert C^*-modules, the equivalences (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c) are proved by M. Hamana [22, Proposition 2.1 and comments in page 82]. It is easy to see that (d)$\Rightarrow$(e) (see the corresponding part in the proof of Theorem 1.1 below). The converse, (e)$\Rightarrow$(d), can be found in [35, Lemma 5.10]. The implication (a)$\Rightarrow$(d) is showed by B. Solel [41, Theorem 3.2 and Corollary 3.4]. On the other hand, (d) and (e) together imply (c). Therefore, all five conditions are equivalent to each other.

When A and B are commutative complex C^*-algebras, it is showed in [26, Theorem 1] that every surjective complex linear isometry preserves inner products (in the sense of (d)), and thus all five conditions hold automatically.

Motivated by the works studying real structures (see, e.g., [12, 28, 34, 36, 38, 5, 24, 37, 17, 21, 18, 23]) and inspired by the Mazur-Ulam theorem, which says that every surjective isometry between normed spaces fixing the origin must be real linear, and by the seminal works about isometries between various real or complex structures (see, e.g., [3, 29, 7, 16, 8]), one naturally asks for a real version as stated in Theorem 1.1. However, not everything of the complex Hilbert C^*-module theory carries to real Hilbert C^*-modules. For example, to prove (d) in the complex case, it suffices to show that

$$\langle T(x), T(x)\rangle = \alpha(\langle x, x\rangle), \quad \forall\, x \in V,$$

because the polarization identity

$$\langle y, x\rangle = \sum_{k=0}^{3} \frac{i^k}{4}\langle x + i^k y, x + i^k y\rangle \tag{1.1}$$

holds for every complex Hilbert C^*-module. However, the previous polarization identity does not make sense in the real setting. This induces some technical difficulties to transport the established arguments from the complex case to the real case. We thus have to develop some new tools in this paper.

A milestone result in the theory of JB*-triples asserts that a surjective complex linear map between JB*-triples is an isometry if and only if it is a Jordan triple isomorphism (cf. [29, Proposition 5.5]). It is shown in [27] that every complex Hilbert C^*-module V over a complex C^*-algebra is a JB*-triple with product

$$\{x, y, z\} := \frac{1}{2}(x\langle y, z\rangle + z\langle y, x\rangle) \quad (x, y, z \in V).$$

Thus any surjective complex linear isometry between complex Hilbert C^*-modules preserves the above (symmetric) JB*-triple product. In this paper, we say that a linear map T between Hilbert C^*-modules preserves ternary products when

$$T(x\langle y, z\rangle) = T(x)\langle T(y), T(z)\rangle.$$

Clearly, a linear map between Hilbert C^*-modules preserving ternary products is a JB*-triple homomorphism. We remark, however, that the transpose map $x \mapsto x^t$ is a (real or complex) linear surjective isometry of the (real or complex) C^*-module $B(H)$ over itself preserving symmetric Jordan triple products of the form:

$$\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x),$$

but this isometry does not satisfy any one of the five conditions stated in Theorem 1.1. Moveover, a surjective real linear isometry between real JB*-triples does not necessarily preserve Jordan triple products ([11]; see Example 3.2). Thus results in this paper enrich the current literature about isometries between JB*-structures.

We survey, in Section 2, the complexification theory of various operator systems (see, e.g., [19, 30, 33]). In particular, we can always complexify a real C^*-algebra A to a complex C^*-algebra A_c, and a real Hilbert A-module V to a complex Hilbert A_c-module V_c (Proposition 2.5). Notice that, however, there is a complex Hilbert A_c-module which is not the complexification of any real Hilbert A-module (See, e.g., [9, 39, 2]). Moreover, the complexification of a real linear isometry T might not be isometric (see, e.g., [6, page 227]).

We show in Section 3 that all five conditions in Theorem 1.1 are equivalent again in the real case. Motivated by [25], in Section 4, we represent every real or complex Hilbert C^*-module over a commutative C^*-algebra as a space of continuous sections of a Hilbert bundle (Theorem 4.2). We then show that every surjective real linear isometry T between real Hilbert C^*-modules over commutative real C^*-algebras carries a weighted composition operator form (Theorem 4.5). However, unlike the complex case, such a real linear isometry T might not be a complete isometry (Examples 4.7 and 4.8). In the case when T is a complete isometry, we will have a much better weighted composition operator form (Theorem 4.6) though.

2. Preliminaries

For a complex Banach space E, we can naturally regard E as a real Banach space E_r with the original norm. Denote by E^* and E_r^* the dual space of E and E_r, respectively.

We begin with an elementary result.

Proposition 2.1. *Let E be a complex Banach space. For any f in E^*, we define*

$$(\operatorname{Re} f)(x) = \operatorname{Re} f(x), \quad \forall\, x \in E_r.$$

Then the map $f \mapsto \operatorname{Re} f$ is a real linear surjective isometry from E^* onto E_r^*.

Let φ be a bounded linear functional of a normed space N. The set of peak points of φ is the set

$$\operatorname{Peak}(\varphi) := \{x \in N : \|x\| = 1 \text{ and } \varphi(x) = \|\varphi\|\}.$$

Proposition 2.2 ([31, Theorem 2])**.** *Let N be a nonzero subspace of a normed space M. Let $\varphi \in N^*$ with $\|\varphi\| = 1$. Assume that for all extreme point ψ in the dual unit ball of M we have*

$$\operatorname{Peak}(\varphi) \subseteq \operatorname{Peak}(\psi) \quad \Longrightarrow \quad \psi\,|_N = \varphi.$$

Then φ is an extreme point of the unit ball of N^*.

Let A be a real Banach algebra. Define

$$A_c := A + iA = \{a + ib : a, b \in A\}.$$

There is a unique (up to equivalence) norm on A_c such that A_c is a complex Banach algebra containing A as a real Banach subalgebra and

$$\|a + ib\| = \|a - ib\|, \quad \forall\, a, b \in A$$

(see, e.g. [33, Theorem 2.1.3]). A real Banach algebra A is said to be a *real Banach $*$-algebra* if there is a real involution $*$ on A, i.e.,

$$(\lambda a + b)^* = \lambda a^* + b^*,\ (ab)^* = b^* a^* \ \text{ and } \ a^{**} := (a^*)^* = a, \quad \forall\, a, b \in A, \forall\, \lambda \in \mathbb{R}.$$

We can extend the involution of A to A_c by setting

$$(a + ib)^* := a^* - ib^*, \quad \forall\, a, b \in A.$$

Then A_c is a complex Banach $*$-algebra. A real Banach $*$-algebra A is called a *real C^*-algebra* if we can extend the norm of A to A_c such that A_c is a complex C^*-algebra. On the other hand, complex C^*-algebras are real C^*-algebras when the scalars are restricted to the real field $\mathbb{R}$.

We can always embed a real C^*-algebra A into one with an identity. More precisely, if A does not have an identity, we can define $\widetilde{A} := A \oplus \mathbb{R}$ with norm

$$\|(a, \lambda)\| = \sup\{\|ab + \lambda b\| : b \in A, \|b\| \leq 1\}.$$

Then $\widetilde{A}$ is a unital real C^*-algebra. If A is unital, we set $\widetilde{A} = A$ (see, e.g. [33, page 84]).

Proposition 2.3 ([33, Corollary 5.2.11]). *Let A be a real Banach $*$-algebra. The following statements are equivalent:*

(a) *A is a real C^*-algebra;*
(b) *A is isometrically $*$-isomorphic to a norm closed $*$-subalgebra of $B(H)$ for a real Hilbert space H;*
(c) *$1 + a^*a$ is invertible in $\widetilde{A}$ and $\|a^*a\| = \|a\|^2$, for all a in A.*

Notice that the field $\mathbb{C}$ of complex numbers with the identity involution $z^\sharp = z$ is a real Banach $\sharp$-algebra such that $|z^\sharp z| = |z|^2$. However, $1 + i^\sharp i = 0$ is not invertible.

Definition 2.4. Let A be a real C^*-algebra. A real (right) A-module V is said to be a *real Hilbert C^*-module over A* or a *real Hilbert A-module* if there is an *A-valued inner product* $\langle \cdot, \cdot \rangle$ on V such that

(1) $\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle,\ \forall\, x, y, z \in V,\ \lambda, \mu \in \mathbb{R}$;
(2) $\langle x, ya \rangle = \langle x, y \rangle a,\ \forall\, x, y \in V,\ a \in A$;
(3) $\langle x, y \rangle^* = \langle y, x \rangle,\ \forall\, x, y \in V$;
(4) $\langle x, x \rangle \geq 0,\ \forall\, x \in V$, and the equality holds exactly when $x = 0$;
(5) V is complete with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$.

Roughly speaking, a real Hilbert A-module is the same as a usual complex Hilbert C^*-module except that the underlying field is $\mathbb{R}$. A real Hilbert C^*-module is said to be *full* if the two-sided ideal $\langle V, V \rangle := \operatorname{span}\{\langle v, w \rangle : v, w \in V\}$ is dense in A. We can always assume that Hilbert C^*-modules are full. Otherwise, we can replace the underlying C^*-algebra with $\overline{\langle V, V \rangle}$.

Proposition 2.5. *Let A be a real C^*-algebra and V a real Hilbert A-module. Then there is an A_c-valued inner product on $V_c := V + iV$ extending the original A-valued inner product on V such that V_c is a complex Hilbert A_c-module.*

Proof. For u, v, x, y in V and a, b in A, define the module action

$$(x + iy)(a + ib) := (xa - yb) + i(xb + ya),$$

and inner product

$$\langle u + iv, x + iy \rangle := (\langle u, x \rangle + \langle v, y \rangle) + i(\langle u, y \rangle - \langle v, x \rangle).$$

It is easy to see that $\langle \cdot, \cdot \rangle$ is an A_c-inner product on V_c except possibly the assertion that the self-adjoint elements $\langle x + iy, x + iy \rangle$ are positive in A_c. To see this, we will show that $f(\langle x+iy, x+iy\rangle) \geq 0$ for all positive linear functional f on A_c. Let f be such a functional. Then

$$f(\langle x,y\rangle) = f(\langle y,x\rangle^*) = \overline{f(\langle y,x\rangle)},$$

and f satisfies the Cauchy-Schwarz inequality

$$|f(\langle x,y\rangle)|^2 \leq f(\langle x,x\rangle)f(\langle y,y\rangle),$$

for all x, y in V. It follows that

$$\begin{aligned}
f(\langle x+iy, x+iy\rangle) &= f(\langle x,x\rangle) + f(\langle y,y\rangle) + if(\langle x,y\rangle) - if(\langle y,x\rangle) \\
&= f(\langle x,x\rangle) + f(\langle y,y\rangle) + 2\operatorname{Re} if(\langle x,y\rangle) \\
&\geq f(\langle x,x\rangle) + f(\langle y,y\rangle) - 2|f(\langle x,y\rangle)| \\
&\geq f(\langle x,x\rangle) + f(\langle y,y\rangle) - 2f(\langle x,x\rangle)^{1/2}f(\langle y,y\rangle)^{1/2} \\
&= (f(\langle x,x\rangle)^{1/2} - f(\langle y,y\rangle)^{1/2})^2 \geq 0.
\end{aligned}$$

Therefore, $\langle x+iy, x+iy\rangle \geq 0$ in A_c.

Define a norm on V_c by

$$\|x+iy\| = \|\langle x+iy, x+iy\rangle\|^{1/2}.$$

By Proposition 2.3, A can be embedded into $B(H)$ for some real Hilbert space H. Hence $A_c = A+iA$ can be embedded into $B(H)+iB(H) = B(H_c)$. Since $\|T+iS\| = \|T-iS\|$ for all T, S in $B(H)$, we have

$$\begin{aligned}
\|x+iy\|^2 &= \|(\langle x,x\rangle + \langle y,y\rangle) + i(\langle x,y\rangle - \langle y,x\rangle)\| \\
&= \|(\langle x,x\rangle + \langle y,y\rangle) - i(\langle x,y\rangle - \langle y,x\rangle)\| = \|x-iy\|^2.
\end{aligned}$$

Consequently,

$$\max\{\|x\|, \|y\|\} \leq \|x+iy\| \leq \|x\| + \|y\|.$$

This shows that V_c is complete. □

A real vector space V equipped with a triple product $\{\cdot,\cdot,\cdot\} : V \times V \times V \to V$ is called a *real Jordan triple* if the triple product is real trilinear and symmetric in the outer variables, and satisfies the following identity:

$$\{x,y,\{z,u,v\}\} = \{\{x,y,z\},u,v\} - \{z,\{y,x,u\},v\} + \{z,u,\{x,y,v\}\}. \tag{2.1}$$

A complex vector space V with a triple product $\{\cdot,\cdot,\cdot\}$ which is complex linear and symmetric in the outer variables, conjugate linear in the middle variable, and satisfies (2.1), is called a *complex Jordan triple*.

By restricting to the real scalar field, a complex Jordan triple can be regarded as a real Jordan triple. Conversely, we can complexify a real Jordan triple $(V, \{\cdot,\cdot,\cdot\})$ to form a complex Jordan triple $V_c := V + iV$, which is furnished with the triple product

$$\begin{aligned} &\{x+iu, y+iv, x+iu\}_c \\ =& (\{x,y,x\} - \{u,y,u\} + 2\{x,v,u\}) + \ i(-\{x,v,x\} + \{u,v,u\} + 2\{x,y,u\}). \end{aligned} \tag{2.2}$$

One often makes use of the following identity in real Jordan triples:

$$2\{x,y,z\} = \{x+z, y, x+z\} - \{x,y,x\} - \{z,y,z\}.$$

It follows that the triple product in a real Jordan triple is completely determined by the special Jordan triple product $\{x,y,x\}$. Furthermore, we have the following polarization identity in complex Jordan triples:

$$4\{x,y,x\} = (y+x)^{[3]} + (y-x)^{[3]} - (y+ix)^{[3]} - (y-iz)^{[3]},$$

where $x^{[3]} := \{x,x,x\}$. It follows that the triple product in a complex Jordan triple is completely determined by the cubes $\{x,x,x\}$.

A complex Banach space $(V, \|\cdot\|)$ is called a *JB*-triple* if it is a complex Jordan triple with a continuous triple product and the box operator $a\Box a$, defined by

$$a\Box a : V \to V, \quad b \mapsto \{a,a,b\},$$

for each element a in V, satisfies the following conditions:

(a) $a\Box a$ is a hermitian operator on V, i.e., $\|\exp it(a\Box a)\| = 1$ for all t in $\mathbb{R}$;
(b) $a\Box a$ has non-negative spectrum;
(c) $\|a\Box a\| = \|a\|^2$.

A real Banach space V is called a *real JB*-triple* or a *JB-triple* if it is a real Jordan triple such that its complexification $(V_c, \{\cdot,\cdot,\cdot\}_c)$, defined as in (2.2), can be normed to become a JB*-triple.

Proposition 2.6 ([27, Theorem 1.4])**.** *Every complex Hilbert C^*-module is a* JB**-triple with respect to the Jordan triple product given by*

$$\{x,y,z\} = \frac{1}{2}(x\langle y,z\rangle + z\langle y,x\rangle).$$

Proposition 2.7. *Every real Hilbert C^*-module over a real C^*-algebra is a real* JB**-triple.*

Proof. Let A be a real C^*-algebra and V a real Hilbert A-module. Define

$$\{x, y, z\} = \frac{1}{2}(x\langle y, z\rangle + z\langle y, x\rangle).$$

It is easy to see that $(V, \{\cdot,\cdot,\cdot\})$ is a real Jordan triple. By Proposition 2.5, $V_c = V + iV$ is a complex Hilbert A_c-module. By proposition 2.6, V_c is a JB*-triple. Thus V is a real JB*-triple. □

3. Isometries between real Hilbert C^*-modules

Let V be a real or complex Hilbert A-module. The space $M_n(V)$ of $n \times n$ matrices with entries from V is a Hilbert $M_n(A)$-module with the following module action and inner product:

$$(x_{ij})(a_{ij}) = (z_{ij}), \quad z_{ij} = \sum_{k=1}^{n} x_{ik}a_{kj},$$

$$\langle (x_{ij}), (y_{ij})\rangle = (b_{ij}), \quad b_{ij} = \sum_{k=1}^{n} \langle x_{ki}, y_{kj}\rangle,$$

for all $(x_{ij}), (y_{ij})$ in $M_n(V)$, and (a_{ij}) in $M_n(A)$.

A linear map $T : V \to W$ between Hilbert C^*-modules is said to be an *n-isometry* if the map $T_n : M_n(V) \to M_n(W)$ defined by

$$T_n((x_{ij})_{ij}) = (T(x_{ij}))_{ij}$$

is an isometry. We call T a *complete isometry* if T_n is an isometry for all $n = 1, 2, \ldots$.

Proposition 3.1 ([28, Theorem 4.8])**.** *Let V and W be two real JB*-triples and T a real linear bijective map from V onto W. If T preserves Jordan triple products then T is an isometry. Conversely, if T is an isometry then T preserves cubes, i.e.,*

$$T(\{x, x, x\}) = \{T(x), T(x), T(x)\}, \quad \forall\, x \in V.$$

Example 3.2 ([11]). Unlike the complex case, a surjective real linear isometry between real JB*-triples does not necessarily preserve Jordan triple products. Note that a JB*-triple can be viewed as a real JB*-triple when restricted to real scalar multiplications. Let $M_{1,2}(\mathbb{C})$ be the real JB*-triple with triple product

$$\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x).$$

Let $T : M_{1,2}(\mathbb{C}) \to M_{1,2}(\mathbb{C})$ be defined by

$$T(\alpha + i\beta, \gamma + i\delta) = (\alpha + i\gamma, \beta + i\delta).$$

Then T is a real linear isometry (but it is not complex linear). However, T does not preserve Jordan triple products. For example, let $x = (1+i, 0), y = (0, 1)$. Then

$$T(\{x, y, x\}) = (0, 0) \neq -(i, i) = \{T(x), T(y), T(x)\}.$$

On the other hand, let E be a real JB*-triple such that E^{**} does not contain real or complex rank-one Cartan factors. Then every surjective linear isometry from E onto another real JB*-triple preserves Jordan triple products (see [16, Theorem 3.2]). $\square$

The following is a real version of a known result for complex C^*-algebras (see [32, Lemma 3.4]).

Lemma 3.3. *Suppose that a, b are positive elements of a real C^*-algebra A and that $\|ac\| = \|bc\|$ for all c in A. Then $a = b$.*

Proof. Note that a, b are again positive elements in A_c. Observe that all self-adjoint elements in the commutative C^*-subalgebra $C^*(a^2-b^2)$ of A_c are limits of real coefficient polynomials $p(a^2-b^2)$ in a^2-b^2 with zero constant terms. As such $p(a^2-b^2)$ belonging to A, all self-adjoint elements in $C^*(a^2-b^2)$ belong to A. Therefore, the arguments presented in the book [32, page 25] of Lance for complex C^*-algebras can be applied here to obtain the desired conclusion. $\square$

Proof of Theorem 1.1. We verify the implications (b) $\Rightarrow$ (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (b) and (d) $\Leftrightarrow$ (e).

Clearly, a complete isometry is a 2-isometry. That is (b) $\Rightarrow$ (a).

Suppose (a), that is, T is a 2-isometry. Since real Hilbert C^*-modules are real JB*-triples (Proposition 2.7) and surjective linear isometries between real JB*-triples preserves cubes (Proposition 3.1), we have

$$T_2(u\langle u, u\rangle) = T_2(u)\langle T_2(u), T_2(u)\rangle, \quad \forall\, u \in M_2(V). \tag{3.1}$$

Let $u = \begin{pmatrix} x & 0 \\ y & z \end{pmatrix}$ in $M_2(V)$. It is easy to see that

$$u\langle u, u\rangle = \begin{pmatrix} * & x\langle y, z\rangle \\ * & * \end{pmatrix}.$$

Then equation (3.1) becomes

$$\begin{pmatrix} * & T(x\langle y, z\rangle) \\ * & * \end{pmatrix} = \begin{pmatrix} * & T(x)\langle T(y), T(z)\rangle \\ * & * \end{pmatrix}.$$

We have statement (c), i.e.,

$$T(x\langle y,z\rangle) = T(x)\langle T(y),T(z)\rangle, \quad \forall\, x,y,z \in V.$$

We assume (c). The complexification $T_c : V_c \to W_c$ of $T : V \to W$, defined by

$$T_c(x+iy) := T(x) + iT(y), \quad \forall x,y \in V,$$

is a complex linear bijection satisfying

$$T_c(x_c\langle y_c,z_c\rangle) = T_c(x_c)\langle T_c(y_c),T_c(z_c)\rangle, \quad \forall x_c,y_c,z_c \in V_c. \tag{3.2}$$

Define a map $\alpha_c : \langle V_c,V_c\rangle \to \langle W_c,W_c\rangle$ by

$$\alpha_c(\sum_{i=1}^n \langle x_{ci},y_{ci}\rangle) := \sum_{i=1}^n \langle T_c(x_{ci}),T_c(y_{ci})\rangle, \quad \forall\, x_{ci},y_{ci} \in V_c,\ i=1,\cdots,n.$$

Note that $\sum_{i=1}^n \langle x_{ci},y_{ci}\rangle = 0$ if and only if $z_c(\sum_{i=1}^n \langle x_{ci},y_{ci}\rangle) = 0$ for all z_c in V_c. Since T_c is bijective, the observation

$$T_c(z_c)(\sum_{i=1}^n \langle T_c(x_{ci}),T_c(y_{ci})\rangle) = \sum_{i=1}^n T_c(z_c\langle x_{ci},y_{ci}\rangle) = T_c(z_c(\sum_{i=1}^n \langle x_{ci},y_{ci}\rangle)) = 0, \quad \forall\, z_c \in V_c,$$

implies $\sum_{i=1}^n \langle T_c(x_{ci}),T_c(y_{ci})\rangle = 0$. This argument shows that α_c is well-defined, and also injective. With (3.2) we can see that α_c is a $*$-isomorphism from $A_c = \overline{\langle V_c,V_c\rangle}$ onto $B_c = \overline{\langle W_c,W_c\rangle}$. In particular, the restriction $\alpha := \alpha_c|_A$ of α_c is the desired $*$-isomorphism from A onto B such that statement (d) follows, i.e.,

$$\langle T(x),T(y)\rangle = \alpha(\langle x,y\rangle), \quad \forall\, x,y \in V.$$

We assume (d). Observe that

$$\begin{aligned}
\langle T(w),T(x\langle y,z\rangle)\rangle &= \alpha(\langle w,x\langle y,z\rangle\rangle) = \alpha(\langle w,x\rangle\langle y,z\rangle) \\
&= \alpha(\langle w,x\rangle)\alpha(\langle y,z\rangle) = \langle T(w),T(x)\rangle\langle T(y),T(z)\rangle \\
&= \langle T(w),T(x)\langle T(y),T(z)\rangle\rangle.
\end{aligned}$$

The surjectivity of T ensures that

$$T(x\langle y,z\rangle) = T(x)\langle T(y),T(z)\rangle\rangle, \quad \forall\, x,y,z \in V.$$

Similarly, it is easy to see that

$$T_n(u\langle v,w\rangle) = T_n(u)\langle T_n(v),T_n(w)\rangle, \quad \forall\, u,v,w \in M_n(V), \forall\, n.$$

By Proposition 3.1, each T_n is an isometry, i.e., T is a complete isometry. This gives (b).

We next show that (d) $\Rightarrow$ (e). It follows from (d) that

$$\begin{aligned}\langle T(y), T(xa)\rangle &= \alpha(\langle y, xa\rangle) = \alpha(\langle y, x\rangle)\alpha(a)\\ &= \langle T(y), T(x)\rangle\alpha(a) = \langle T(y), T(x)\alpha(a)\rangle, \quad \forall\, x, y \in V, \forall\, a \in A.\end{aligned}$$

It follows from the surjectivity of T that $T(xa) = T(x)\alpha(a), \forall\, x \in V, \forall\, a \in A$, i.e., (e).

Finally, we suppose (e) and want to derive (d). Similar to [32, page 26], we verify that

$$\langle T(x), T(x)\rangle = \alpha(\langle x, x\rangle), \quad \forall\, x \in V. \tag{3.3}$$

To this end, observe that for any a in A we have

$$\begin{aligned}\|\langle T(x), T(x)\rangle^{1/2}\alpha(a)\|^2 &= \|\alpha(a)^*\langle T(x), T(x)\rangle\alpha(a)\| = \|\langle T(xa), T(xa)\rangle\| = \|T(xa)\|^2\\ &= \|xa\|^2 = \|\langle xa, xa\rangle\| = \|\alpha(\langle xa, xa\rangle)\| = \|\alpha(\langle x, x\rangle)^{1/2}\alpha(a)\|^2.\end{aligned}$$

The desired assertion follows from Lemma 3.3.

On the other hand, for any x, y in V we have

$$\langle x + y, x + y\rangle - \langle x - y, x - y\rangle = 4(\langle y, x\rangle + \langle x, y\rangle) = 8\mathrm{Re}\langle x, y\rangle.$$

Here, $\mathrm{Re}\, a := (a + a^*)/2$ as usual. It follows from (3.3) that

$$\mathrm{Re}\,\langle T(x), T(y)\rangle = \mathrm{Re}\,\alpha(\langle x, y\rangle), \quad \forall x, y \in V,$$

or

$$\mathrm{Re}\left[\langle T(x), T(y)\rangle - \alpha(\langle x, y\rangle)\right] = 0, \quad \forall x, y \in V.$$

Replacing y with ya, we have

$$\mathrm{Re}\left[(\langle T(x), T(y)\rangle - \alpha(\langle x, y\rangle))\alpha(a)\right] = 0, \quad \forall x, y \in V, a \in A.$$

Since $\alpha(A) = B$, we obtain in particular

$$\mathrm{Re}\left[(\langle T(x), T(y)\rangle - \alpha(\langle x, y\rangle))(\langle T(x), T(y)\rangle - \alpha(\langle x, y\rangle))^*\right] = 0, \quad \forall x, y \in V.$$

It follows the desired assertion that

$$\langle T(x), T(y)\rangle = \alpha(\langle x, y\rangle), \quad \forall\, x, y \in V.$$

This completes the proof of the equivalence among all five conditions (a) to (e). $\square$

4. Isometries between real Hilbert C^*-modules over real commutative C^*-algebras

Let X be a locally compact Hausdorff space. Denote by $C_0(X)$ the complex C^*-algebra of complex-valued continuous functions on X vanishing at infinity. Let $\sigma : X \to X$ be a homeomorphism with period 2, i.e., $\sigma^2(x) = x, \forall x \in X$. Denote by

$$C_0(X, \sigma) = \{f \in C_0(X) : f(\sigma(x)) = \overline{f(x)}\},$$

where $\overline{z}$ is the complex conjugate of a complex number z. The space $C_0(X, \sigma)$ is a commutative real C^*-algebra. It is known that every commutative real C^*-algebra is of this form $C_0(X, \sigma)$ (see, e.g., [33]). Clearly, $C_0(X, \sigma)_c = C_0(X)$.

Example 4.1. Let X and Y be compact Hausdorff spaces, and let σ and τ be two homeomorphisms on X and Y with period 2, respectively. Note that C^*-algebras are Hilbert C^*-modules over itself with inner product

$$\langle a, b\rangle = a^* b.$$

So we can regard $C(X, \sigma), C(Y, \tau)$ as real Hilbert C^*-modules over themselves.

Consider a surjective real linear isometry $T : C(X, \sigma) \to C(Y, \tau)$. Using a result of M. Grzesiak in [20], we have a homeomorphism φ from Y onto X with $\sigma \circ \varphi = \varphi \circ \tau$ and a continuous unimodular function h in $C(Y, \tau)$, i.e., $|h(y)| = 1, \forall y \in Y$, such that

$$T(f) = h \cdot f \circ \varphi, \quad \forall\, f \in C(X, \sigma).$$

We can then define a $*$-isomorphism $\alpha : C(X, \sigma) \to C(Y, \tau)$ by

$$\alpha(g) = g \circ \varphi.$$

For f, g in $C(X, \sigma)$, we have

$$\langle T(f), T(g)\rangle = \langle h \cdot f \circ \varphi, h \cdot g \circ \varphi\rangle = \langle f \circ \varphi, g \circ \varphi\rangle = \langle f, g\rangle \circ \varphi = \alpha(\langle f, g\rangle),$$

i.e., T preserves inner products with respect to α. Also, T is a module map

$$T(fg) = h \cdot (fg) \circ \varphi = h \cdot (f \circ \varphi)(g \circ \varphi) = T(f)\alpha(g),$$

and preserves ternary products

$$T(f\langle g, h\rangle) = h \cdot (f\langle g, h\rangle) \circ \varphi = T(f)\alpha(\langle g, h\rangle) = T(f)\langle T(g), T(h)\rangle.$$

Therefore, all five conditions in Theorem 1.1 hold automatically here. ☐

We are going to study the case for any surjective isometry between arbitrary real Hilbert C^*-modules over commutative real C^*-algebras. To this end, we first represent such Hilbert C^*-modules as continuous sections of real Hilbert bundles.

Let X be a locally compact Hausdorff space. A *Hilbert bundle* (see, e.g. [14, 13, 15]) over X is a pair $\langle \mathcal{H}_X, \pi_X \rangle$. Here $\mathcal{H}_X$ is a topological space and π_X is a continuous open surjective map from $\mathcal{H}_X$ onto X. For all x in X, the fiber $H_x = \pi_X^{-1}(x)$ carries a complex Hilbert space structure. Moreover, the following conditions are satisfied:

(HB1) Scalar multiplication, addition and the norm on $\mathcal{H}_X$ are all continuous wherever they are defined.

(HB2) If $x \in X$ and $\{h_i\}$ is any net in $\mathcal{H}_X$ such that $\|h_i\| \to 0$ and $\pi(h_i) \to x$ in X, then $h_i \to 0_x$ (the zero element of H_x) in $\mathcal{H}_X$.

A *continuous section* f of a Hilbert bundle $\langle \mathcal{H}_X, \pi_X \rangle$ is a continuous function from X into $\mathcal{H}_X$ such that $\pi_X(f(x)) = x$, i.e., $f(x) \in H_x$ for all x in X. Note that the condition (HB2) above ensures that the zero section is continuous. Denote by $C_0(X, \mathcal{H}_X)$ the Banach space of all C_0-sections of $\langle \mathcal{H}_X, \pi_X \rangle$, i.e., those continuous sections f with $\lim_{x\to\infty} \|f(x)\| = 0$.

Note that the space $C_0(X, \mathcal{H}_X)$ is a complex $C_0(X)$-module with pointwise module action and inner product

$$(f\phi)(x) = f(x)\phi(x), \quad \langle f, g\rangle(x) = (f(x), g(x)),$$

for all f, g in $C_0(X, \mathcal{H}_X)$, ϕ in $C_0(X)$, and x in X.

Theorem 4.2. *Let V be a real Hilbert $C_0(X,\sigma)$-module. Then there exists a Hilbert bundle $\langle \mathcal{H}_X, \pi_X \rangle$ over X, and a conjugate linear isometric isomorphism $- : H_x \to H_{\sigma(x)}$ for each x in X, such that V is isomorphic to*

$$C_0(X, \mathcal{H}_X, \sigma, -) := \{f \in C_0(X, \mathcal{H}_X) : f(\sigma(x)) = \overline{f(x)}, \forall x \in X\}.$$

Moreover, V_c is isomorphic to $C_0(X, \mathcal{H}_X)$.

Proof. By Proposition 2.5, V_c is a Hilbert $C_0(X)$-module. Given $x \in X$, let

$$I_x = \{f \in C_0(X) : f(x) = 0\}.$$

By Cohen's factorization theorem (see, e.g., [4, Theorem A.6.2]),

$$\begin{aligned} V_c I_x &= \{v_c f : v_c \in V_c, f \in I_x\} \\ &= \{(u + iv)(g + ih) : u, v \in V, g, h \in C_0(X, \sigma) \text{ with } g(x) + ih(x) = 0\} \end{aligned}$$

is norm closed in V_c. In particular, if $u \in V_c I_x$ and $w \in V_c$ we have $\langle u, w\rangle(x) = \langle w, u\rangle(x) = 0$. Then $V_c/V_c I_x$ is a pre-Hilbert space with inner product

$$\langle (u_1 + iv_1) + V_c I_x, (u_2 + iv_2) + V_c I_x\rangle := \langle u_1 + iv_1, u_2 + iv_2\rangle(x).$$

Denote by H_x the completion of $V_c/V_c I_x$. Let

$$\mathcal{H}_X := \coprod_{x\in X} H_x = \Big\{(z_x)_{x\in X} : z_x \in H_x\Big\}.$$

Each element $u + iv$ in V_c can be regarded as a section from X into $\mathcal{H}_X$ by

$$(u + iv)(x) := (u + iv) + V_c I_x.$$

Let $\pi_X : \mathcal{H}_X \to X$ be the canonical projection. By [15, Theorem 13.18], there is a unique topology on $\mathcal{H}_X$ such that $\langle \mathcal{H}_X, \pi_X\rangle$ is a Hilbert bundle over X with C_0-section space $C_0(X, \mathcal{H}_X) = V_c$.

Let $g, h \in C_0(X, \sigma)$. Since

$$\begin{aligned}(g - ih)(\sigma(x)) &= g(\sigma(x)) - ih(\sigma(x))\\ &= \overline{g(x)} - i\overline{h(x)} = \overline{g(x) + ih(x)} = \overline{(g + ih)}(x),\end{aligned}$$

we have

$$g + ih \in I_x \quad \text{if and only if} \quad g - ih \in I_{\sigma(x)}.$$

This implies that the following map is well-defined:

$$- : V_c/V_c I_x \to V_c/V_c I_{\sigma(x)}, \quad (u + iv) + V_c I_x \mapsto (u - iv) + V_c I_{\sigma(x)}.$$

Moreover, this map can be extended to a conjugate-linear isometric isomorphism from H_x onto $H_{\sigma(x)}$. Then, for u in V, we have

$$\overline{u(x)} = \overline{u + V_c I_x} = u + V_c I_{\sigma(x)} = u(\sigma(x)),$$

for all x in X, i.e., $V \subseteq C_0(X, \mathcal{H}_X, \sigma, -)$. On the other hand, let $f \in C_0(X, \mathcal{H}_X, \sigma, -) \subseteq V_c$, and $f = u + iv$ for some u, v in V. We have

$$u + iv = f = \overline{f \circ \sigma} = \overline{(u + iv) \circ \sigma} = \overline{u \circ \sigma} - i\overline{v \circ \sigma} = u - iv.$$

Hence $v = 0$ and therefore $f = u \in V$. □

By Theorem 4.2, every real Hilbert $C_0(X, \sigma)$-module is of the form $C_0(X, \mathcal{H}_X, \sigma, -)$, which can be considered as a real subspace of $C_0(X, \mathcal{H}_X)_r$. By Proposition 2.1, bounded real linear functionals on $C_0(X, \mathcal{H}_X, \sigma, -)$ are of the form $\mathrm{Re}\, f$ for some f in $C_0(X, \mathcal{H}_X)^*$. We are going to characterize extreme points of the unit ball of $C_0(X, \mathcal{H}_X, \sigma, -)^*$.

Let $x \in X$ and $\mu \in H_x$, denote by $\delta_{x,\mu}$ the evaluation map of $C_0(X, \mathcal{H}_X)$ given by $\delta_{x,\mu}(f) := \langle f(x), \mu\rangle$. Let

$$P_X = \{(x, \mu) \in X \times \mathcal{H}_X : \mu \in H_x, \|\mu\| = 1, \text{ and } \overline{\mu} = \mu \text{ in case } \sigma(x) = x\}.$$

Since $- : H_x \to H_{\sigma(x)}$ is a conjugate linear isometry, for all $(x, \mu) \in P_X$ and $f \in C_0(X, \mathcal{H}_X, \sigma, -)$ we have

$$\operatorname{Re} \delta_{\sigma(x),\overline{\mu}}(f) = \operatorname{Re}\langle f(\sigma(x)), \overline{\mu}\rangle = \operatorname{Re}\langle \overline{f(x)}, \overline{\mu}\rangle = \operatorname{Re} \overline{\langle f(x), \mu\rangle} = \operatorname{Re} \delta_{x,\mu}(f). \tag{4.1}$$

Therefore, $\operatorname{Re} \delta_{\sigma(x),\overline{\mu}} = \operatorname{Re} \delta_{x,\mu}$ when considered as norm one linear functionals of $C_0(X, \mathcal{H}_X, \sigma, -)$.

Lemma 4.3. (*a*) *For all (x, μ) in P_X, there is an f in $C_0(X, \mathcal{H}_X, \sigma, -)$ with $\|f\| = 1$ and $f(x) = \mu$.*

(*b*) *For all y in $X \backslash \{x, \sigma(x)\}$, there is a ϕ in $C_0(X, \sigma)$ with $\|\phi\| = 1$, $\phi(x) = 1$ and $\phi(y) = 0$.*

(*c*) *Let (y, ν) and (x, μ) be in P_X. If $(y, \nu) \neq (x, \mu)$ and $(y, \nu) \neq (\sigma(x), \overline{\mu})$, there is an f in $C_0(X, \mathcal{H}_X, \sigma, -)$ with $\|f\| = 1$ and $\operatorname{Re} \delta_{x,\mu}(f) = 1$, but $\operatorname{Re} \delta_{y,\nu}(f) \neq 1$.*

Proof. (*a*) Let $(x, \mu) \in P_X$. If $\sigma(x) \neq x$ then choose g and h from $C_0(X, \mathcal{H}_X)$ with disjoint supports, $\|g\| = 1$ and $\|h\| = 1$ such that $g(x) = \mu$ and $h(\sigma(x)) = \overline{\mu}$. Clearly, $\|g + h\| = 1$. Let

$$f = \frac{1}{2}(g + h + \overline{(g + h) \circ \sigma}).$$

Then $f \in C_0(X, \mathcal{H}_X, \sigma, -)$, $f(x) = \mu$ and $\|f\| = 1$.

If $\sigma(x) = x$, and thus $\mu = \overline{\mu}$, take a g from $C_0(X, \mathcal{H}_X)$ with $\|g\| = 1$ and $g(x) = \mu$. Let

$$f = \frac{1}{2}(g + \overline{g \circ \sigma}).$$

Then $f \in C_0(X, \mathcal{H}_X, \sigma, -)$, $f(x) = \mu$ and $\|f\| = 1$.

(*b*) The sets $\{x, \sigma(x)\}$ and $\{y, \sigma(y)\}$ are disjoint closed subsets of X. By Urysohn's Lemma there is a u in $C_0(X)$, $0 \leq u \leq 1$, $u(x) = 1 = u(\sigma(x))$ and $u(y) = 0 = u(\sigma(y))$. Let

$$\phi = \frac{1}{2}(u + \overline{u \circ \sigma}).$$

Then $\phi \in C_0(X, \sigma)$, $\|\phi\| = 1$, $\phi(x) = 1$ and $\phi(y) = 0$.

(*c*) By (a), let f be in $C_0(X, \mathcal{H}_X, \sigma, -)$ such that $\|f\| = 1$ and $f(x) = \mu$. Then $\operatorname{Re} \delta_{x,\mu}(f) = 1$. We verify the assertion in three different cases below.

If $y = x$ and $\operatorname{Re} \delta_{y,\nu}(f) = 1$, then

$$1 = \operatorname{Re}(f(y), \nu) = \operatorname{Re}(f(x), \nu) = \operatorname{Re}(\mu, \nu) \leq |(\mu, \nu)| \leq 1.$$

This implies that $\nu = \mu$.

If $y = \sigma(x)$ and $\operatorname{Re}\delta_{y,\nu}(f) = 1$, then

$$1 = \operatorname{Re}(f(y), \nu) = \operatorname{Re}(f(\sigma(x)), \nu) = \operatorname{Re}(\overline{\mu}, \nu) \leq |(\overline{\mu}, \nu)| \leq 1.$$

This implies that $\nu = \overline{\mu}$.

Suppose that $y \notin \{x, \sigma(x)\}$. Let ϕ be in $C_0(X, \sigma)$ such that $\|\phi\| = 1$, $\phi(x) = 1$ and $\phi(y) = 0$ by (b). Then $\|f\phi\| = 1$, $f\phi(x) = f(x)\phi(x) = \mu$ and

$$\operatorname{Re}\delta_{x,\mu}(f\phi) = \operatorname{Re}(f(x)\phi(x), \mu) = 1.$$

But

$$\operatorname{Re}\delta_{y,\nu}(f\phi) = \operatorname{Re}(f(y)\phi(y), \nu) = 0.$$

□

Lemma 4.4. *Let $N = C_0(X, \mathcal{H}_X, \sigma, -)$ and its complexification $M = C_0(X, \mathcal{H}_X)$. The set of extreme points of the unit ball of $C_0(X, \mathcal{H}_X, \sigma, -)^*$ is*

$$\{\operatorname{Re}\delta_{x,\mu}|_N : (x, \mu) \in P_X\}.$$

Moreover, $\operatorname{Re}\delta_{x,\mu}|_N = \operatorname{Re}\delta_{y,\nu}|_N$ *if and only if* $(y, \nu) = (x, \mu)$ *or* $(y, \nu) = (\sigma(x), \overline{\mu})$

Proof. Let φ be an extreme point of the unit ball of N^*. Then φ has an extension to an extreme point of the unit ball of M_r^* (see, for example, [1, Proposition 3.3]). By Proposition 2.1, this extension is the real part of an extreme point of the unit ball of M^*. It is known that the set of extreme points of the unit ball of M^* is $\{\delta_{x,\mu} : x \in X, \mu \in H_x, \|\mu\| = 1\}$ (see, e.g., [40, page 564], or the original result [10]). Hence $\varphi = \operatorname{Re}\delta_{x,\mu}|_N$ for some (x, μ) in $X \times \mathcal{H}_X$ such that $\mu \in H_x$ and $\|\mu\| = 1$.

We show that $(x, \mu) \in P_X$. This holds automatically when $\sigma(x) \neq x$. Assume $\sigma(x) = x$. In this case, $- : H_x \to H_x$ is a conjugate-linear isomorphism from H_x onto itself, and $\overline{f(x)} = f(x)$ for all f in N. Moreover, since $\langle f, g\rangle$ is in $C_0(X, \sigma)$ for all f and g in N, we have

$$\overline{(f(x), g(x))} = \overline{\langle f, g\rangle(x)} = \langle f, g\rangle(\sigma(x)) = \langle f, g\rangle(x) = (f(x), g(x)),$$

i.e., $(f(x), g(x)) \in \mathbb{R}$ for all f and g in N. Choose $g + ih$ in $C_0(X, \mathcal{H}_X)$ with g, h in N such that $\|g + ih\| = 1$ and $(g + ih)(x) = \mu$. Observe that

$$\begin{aligned} |\operatorname{Re}\delta_{x,\mu}(f)| &= |\operatorname{Re}(f(x), \mu)| = |\operatorname{Re}(f(x), g(x) + ih(x))| \\ &= |(f(x), g(x))| \leq \|f\|\|g(x)\|. \end{aligned}$$

This implies that $1 = \|\operatorname{Re}\delta_{x,\mu}\| \le \|g(x)\| \le \|g\| \le \|g+ih\| = 1$. Since $(g(x), h(x)) \in \mathbb{R}$, we have

$$\begin{aligned}\|g(x)\|^2 = 1 &\ge \|g(x)+ih(x)\|^2\\ &= \|g(x)\|^2 + \|h(x)\|^2 + 2\operatorname{Re} i(g(x), h(x)) = \|g(x)\|^2 + \|h(x)\|^2\end{aligned}$$

Hence $h(x) = 0$ and therefore $\mu = g(x) = \overline{g(x)} = \overline{\mu}$. Therefore, we again have $(x, \mu) \in P_X$ in this case.

Conversely, let $(x, \mu) \in P_X$. To show that $\operatorname{Re}\delta_{x,\mu}|_N$ is an extreme point, we make use of Proposition 2.2. Let $\operatorname{Re}\delta_{y,\nu}(f) = 1$, for all f in N with $\operatorname{Re}\delta_{x,\mu}(f) = 1 = \|f\|$. By Lemma 4.3 (c), $(y, \nu) = (x, \mu)$ or $(y, \nu) = (\sigma(x), \overline{\mu})$. If $(y, \nu) = (x, \mu)$, then $\operatorname{Re}\delta_{y,\nu}|_N = \operatorname{Re}\delta_{x,\mu}|_N$. If $(y, \nu) = (\sigma(x), \overline{\mu})$, for each f in N,

$$\begin{aligned}\operatorname{Re}\delta_{y,\nu}(f) &= \operatorname{Re}(f(y), \nu) = \operatorname{Re}(f(\sigma(x)), \overline{\mu})\\ &= \operatorname{Re}\overline{(f(x), \mu)} = \operatorname{Re}(f(x), \mu) = \operatorname{Re}\delta_{x,\mu}(f),\end{aligned}$$

i.e., $\operatorname{Re}\delta_{y,\nu}|_N = \operatorname{Re}\delta_{x,\mu}|_N$. By Proposition 2.2, $\operatorname{Re}\delta_{x,\mu}|_N$ is an extreme point of the unit ball of N^*.

The last assertion follows from Lemma 4.3 (c). □

Now we consider the real Hilbert $C_0(X, \sigma)$-module $C_0(X, \mathcal{H}_X, \sigma, -)$. Let

$$K_x = \{f(x) : f \in C_0(X, \mathcal{H}_X, \sigma, -)\}. \tag{4.2}$$

It follows from Lemma 4.3(a) that

$$H_x = K_x \quad \text{whenever } x \neq \sigma(x).$$

In the other case,

$$K_x = \{k \in H_x : \overline{k} = k\}$$

is a real closed subspace of H_x such that

$$H_x = K_x + iK_x \quad \text{whenever } x = \sigma(x).$$

Theorem 4.5. *Let $T : C_0(X, \mathcal{H}_X, \sigma, -) \longrightarrow C_0(Y, \mathcal{H}_Y, \tau, -)$ be a surjective real linear isometry. Then there exist a (not necessarily bijective or continuous) map $\varphi : Y \to X$ and, for each y in Y, a surjective real linear isometry $h_y : K_{\varphi(y)} \to K_y$ such that*

$$T(f)(y) = h_y(f(\varphi(y))), \quad \forall\, f \in C_0(X, \mathcal{H}_X, \sigma, -).$$

Proof. Let

$$P_X = \{(x,\mu) \in X \times \mathcal{H}_X : \mu \in H_x, \|\mu\| = 1, \text{ and } \overline{\mu} = \mu \text{ in case } \sigma(x) = x\},$$
$$P_Y = \{(y,\nu) \in Y \times \mathcal{H}_Y : \nu \in H_y, \|\nu\| = 1, \text{ and } \overline{\nu} = \nu \text{ in case } \tau(y) = y\}.$$

Since T is a surjective linear isometry, its dual map T^* is a surjective linear isometry from $C_0(Y,\mathcal{H}_Y,\tau,-)^*$ onto $C_0(X,\mathcal{H}_X,\sigma,-)^*$ sending the set of extreme points of the unit ball onto the set of extreme points of the unit ball. By Lemma 4.4, for all $(y,\nu) \in P_Y$ we have

$$T^*(\operatorname{Re}\delta_{y,\nu}|_{C_0(Y,\mathcal{H}_Y,\tau,-)}) = \operatorname{Re}\delta_{x,\mu}|_{C_0(X,\mathcal{H}_X,\sigma,-)}, \tag{4.3}$$

for some (x,μ) in P_X.

Claim 1. Let $(y,\nu_1),(y,\nu_2)$ be in P_Y, and $(x_1,\mu_1),(x_2,\mu_2)$ in P_X such that

$$T^*(\operatorname{Re}\delta_{y,\nu_1}|_{C_0(Y,\mathcal{H}_Y,\tau,-)}) = \operatorname{Re}\delta_{x_1,\mu_1}|_{C_0(X,\mathcal{H}_X,\sigma,-)}$$

and

$$T^*(\operatorname{Re}\delta_{y,\nu_2}|_{C_0(Y,\mathcal{H}_Y,\tau,-)}) = \operatorname{Re}\delta_{x_2,\mu_2}|_{C_0(X,\mathcal{H}_X,\sigma,-)}.$$

Then $x_1 = x_2$ or $x_1 = \sigma(x_2)$.

The assumptions indicate that

$$\operatorname{Re}(T(f)(y),\nu_i) = \operatorname{Re}(f(x_i),\mu_i), \quad \forall f \in C_0(X,\mathcal{H}_X,\sigma,-). \tag{4.4}$$

Suppose that $x_1 \neq x_2$ and $x_1 \neq \sigma(x_2)$. Let U_1 and U_2 be two disjoint open neighborhoods of x_1 and x_2, respectively. By Lemma 4.3(a), for $i = 1,2$, there is f_i in $C_0(X,\mathcal{H}_X,\sigma,-)$ such that the support of f_i is contained in U_i and

$$(f_i(x_i),\mu_i) = \|f_i\| = 1.$$

By (4.4), we have $\|T(f_i)(y)\| = 1$. Because f_i's have disjoint supports, $\|f_1 \pm f_2\| = 1$ and this implies $\|T(f_1 \pm f_2)(y)\| \leq 1$. Since

$$\begin{aligned} 2 = 2\|T(f_1)(y)\| &= \|T(f_1+f_2)(y) + T(f_1-f_2)(y)\| \\ &\leq \|T(f_1+f_2)(y)\| + \|T(f_1-f_2)(y)\| \leq 2, \end{aligned}$$

we have $\|T(f_1 \pm f_2)(y)\| = 1$. It follows from the uniform convexity of the Hilbert space vector norm, one of

$$\|T(f_1+f_2)(y) \pm T(f_1-f_2)(y)\| < 2.$$

This is a contradiction.

Define an equivalence relation on X by $x_1 \sim x_2$ if and only if $x_1 = x_2$ or $x_1 = \sigma(x_2)$. Let $\tilde{X} = X/\sim$. Similarly, we define $y_1 \sim y_2$ in Y if and only if $y_1 = y_2$ or $y_1 = \tau(y_2)$, and let $\tilde{Y} = Y/\sim$. Denote by $[x]$ and $[y]$ the equivalence classes represented by x and y, respectively. In view of (4.1), we can define a map $\tilde{\varphi}$ from $\tilde{Y}$ into $\tilde{X}$ by

$$\tilde{\varphi}([y]) = [x] \quad \text{if} \quad T^*(\operatorname{Re}\delta_{y,\nu}|_{C_0(Y,\mathcal{H}_Y,\tau,-)}) = \operatorname{Re}\delta_{x,\mu}|_{C_0(X,\mathcal{H}_X,\sigma,-)}, \tag{4.5}$$

where $(x,\mu) \in P_X$ and $(y,\nu) \in P_Y$. A new application of the claim to $(T^{-1})^* = (T^*)^{-1}$ shows that the mapping $\tilde{\varphi}$ is a bijection. Let its inverse be $\tilde{\psi}$. In other words,

$$\tilde{\psi}([x]) = [y] \quad \text{if} \quad (T^{-1})^*(\operatorname{Re}\delta_{x,\mu}|_{C_0(X,\mathcal{H}_X,\sigma,-)}) = \operatorname{Re}\delta_{y,\nu}|_{C_0(Y,\mathcal{H}_Y,\tau,-)}. \tag{4.6}$$

By the Axiom of Choice, we can choose a subset Y' of Y such that $Y' \cap \{y, \tau(y)\} = \{y'\}$ is a singleton for each equivalence class $[y] = \{y, \tau(y)\}$ in $\tilde{Y}$. Similarly, we can choose a subset X' of X such that $X' \cap \{x, \sigma(x)\} = \{x'\}$ is a singleton for each equivalence class $[x] = \{x, \sigma(x)\}$ in $\tilde{X}$. Define a (not necessarily bijective or continuous) map $\varphi : Y \to X$ such that

$$\varphi(y') = x' \quad \text{whenever} \quad x' \in X', y' \in Y', \text{ and } \tilde{\varphi}([y']) = [x'].$$

If $y \notin Y'$, then $\tau(y) \in Y'$ and we define $\varphi(y) = \sigma(\varphi(\tau(y)))$. We define $\psi : X \to Y$ in a similar way. Clearly, $[\varphi(y)] = \tilde{\varphi}([y])$ and $[\psi(x)] = \tilde{\psi}([x])$.

Let $f \in C_0(X, \mathcal{H}_X, \sigma, -)$. Let $\nu = \dfrac{T(f)(y)}{\|T(f)(y)\|}$ if $T(f)(y) \neq 0$. Then $(T(f)(y), \nu) = \|T(f)(y)\|$. In the case that $\tau(y) = y$, we have $\overline{T(f)(y)} = T(f)(\tau(y)) = T(f)(y)$. Hence, $(y,\nu) \in P_Y$. By the definition of φ, there is a norm one vector μ in $H_{\varphi(y)}$ such that

$$\|T(f)(y)\| = \operatorname{Re}(T(f)(y), \nu) = \operatorname{Re}(f(\varphi(y)), \mu).$$

Hence, $f(\varphi(y)) = 0$ implies $Tf(y) = 0$. It then follows from (4.2) that we can define a real linear map h_y from $K_{\varphi(y)}$ into K_y such that

$$T(f)(y) = h_y(f(\varphi(y))), \quad \forall\, f \in C_0(X, \mathcal{H}_X, \sigma, -),\ y \in Y. \tag{4.7}$$

In view of (4.5) we have

$$\|h_y(f(\varphi(y)))\| = \|T(f)(y)\| = (T(f)(y), \nu) = \operatorname{Re}(f(\varphi(y)), \mu) \le \|f(\varphi(y))\|.$$

This shows that

$$\|h_{y|K_{\varphi(y)}}\| \le 1, \quad \forall y \in Y. \tag{4.8}$$

We apply the same arguments to T^{-1} and ψ. Then there is a real linear map k_x from $K_{\psi(x)}$ into K_x such that

$$T^{-1}(g)(x) = k_x(g(\psi(x))), \quad \forall g \in C_0(Y, \mathcal{H}_Y, \tau, -), \forall x \in X.$$

We have

$$\begin{aligned} f(x) &= T^{-1}T(f)(x) = k_x(T(f)(\psi(x))) = k_x h_{\psi(x)}(f(\varphi(\psi(x))) \\ &= k_x h_{\psi(x)}(f(x)) \text{ or } k_x h_{\psi(x)}(f(\sigma(x))) \end{aligned}$$

and

$$\begin{aligned} g(y) &= TT^{-1}(g)(y) = h_y(T^{-1}(g)(\varphi(y))) = h_y k_{\varphi(y)}(g(\psi(\varphi(y))) \\ &= h_y k_{\varphi(y)}(g(y)) \text{ or } h_y k_{\varphi(y)}(g(\tau(y))). \end{aligned}$$

This implies that all h_y and k_x are surjective. Since $\|g(y)\| = \|\overline{g(y)}\| = \|g(\tau(y)\|$, with (4.8), we have

$$\begin{aligned} \|g(y)\| &= \|h_y k_{\varphi(y)}(g(y))\| \text{ or } \|h_y k_{\varphi(y)}(g(\tau(y)))\| \\ &\leq \|k_{\varphi(y)}(g(y))\| \text{ or } \|k_{\varphi(y)}(g(\tau(y)))\| \\ &\leq \|g(y)\|, \quad \forall g \in C_0(Y, \mathcal{H}_Y, \tau, -), \forall x \in X. \end{aligned}$$

Therefore, all inequalities above are exactly equalities. This forces that each $k_{\varphi(y)}$ is a surjective real linear isometry from $K_{\psi(\varphi(y))}$ onto $K_{\varphi(y)}$. It turns out that each h_y is also a surjective real linear isometry. $\square$

With routine arguments, one can find in the proof of Theorem 4.5 that the map $\tilde{\varphi}: Y/\sim \longrightarrow X/\sim$ induced from a surjective real linear isometry $T : C_0(X, \mathcal{H}_X, \sigma, -) \longrightarrow C_0(Y, \mathcal{H}_Y, \tau, -)$ is indeed a homeomorphism between the quotient spaces. However, we will see in Example 4.7 that the induced symbol map φ can be not bijective. Moreover, the identity $\sigma \circ \varphi = \varphi \circ \tau$ might not hold in general. When the real linear isometry is a 2-isometry, we will have a much better weighted composition form.

Theorem 4.6. *Let $T : C_0(X, \mathcal{H}_X, \sigma, -) \longrightarrow C_0(Y, \mathcal{H}_Y, \tau, -)$ be a surjective real linear 2-isometry. Then there exist a homeomorphism $\varphi : Y \to X$ such that $\sigma \circ \varphi = \varphi \circ \tau$, and a surjective complex linear isometry $h_y : H_{\varphi(y)} \to H_y$ for each y in Y, such that*

$$h_{\tau(y)}(\mu) = \overline{h_y(\overline{\mu})}, \quad \forall \mu \in H_{\varphi(y)},$$

and

$$T(f)(y) = h_y(f(\varphi(y))), \quad \forall f \in C_0(X, \mathcal{H}_X, \sigma, -).$$

Proof. By Theorem 1.1, we know that T satisfies all the equivalent conditions stated there. In particular, there is a $*$-isomorphism $\alpha : C_0(Y, \tau) \to C_0(X, \sigma)$ between the underlying C*-algebras. It is well known that there exists a homeomorphism $\varphi : Y \to X$ such that $\sigma \circ \varphi = \varphi \circ \tau$ and $\alpha(f) = f \circ \varphi$ for all f in $C_0(Y, \tau)$ (see, e.g., [33]).

For each y in Y, for all a in $C_0(X,\sigma)$ such that $a(\varphi(y)) = 0$ we have

$$(Tfa)(y) = (Tf)(y)\alpha(a)(y) = (Tf)(y)a(\varphi(y)) = 0, \quad \forall f \in C_0(X, \mathcal{H}_X, \sigma, -).$$

By Uryshon's Lemma and the continuity of T, we see that

$$f(\varphi(y)) = 0 \quad \Longrightarrow \quad Tf(y) = 0, \quad \forall f \in C_0(X, \mathcal{H}_X, \sigma, -),$$

This provides a real linear map $h_y : K_{\varphi(y)} \to K_y$ such that

$$Tf(y) = h_y(f(\varphi(y))), \quad \forall f \in C_0(X, \mathcal{H}_X, \sigma, -). \tag{4.9}$$

It is routine to see that all fiber maps h_y's are of norm not greater than one.

Let $\psi = \varphi^{-1} : X \to Y$ and notice that

$$T(T^{-1}(g)b \circ \psi) = T(T^{-1}(g))b \circ \psi \circ \varphi = gb,$$

or equivalently,

$$T^{-1}(gb) = T^{-1}(g)b \circ \psi = T^{-1}(g)\alpha^{-1}(b), \quad \forall g \in C_0(Y, \mathcal{H}_Y, \tau, -), \forall b \in C_0(Y, \tau).$$

Hence T^{-1} also satisfies the equivalent conditions stated in Theorem 1.1. Using above arguments, we have for each x in X a real linear map $k_x : K_{\psi(x)} \to K_x$ of norm not greater one such that

$$T^{-1}g(x) = k_x(g(\psi(x))), \quad \forall g \in C_0(Y, \mathcal{H}_Y, \tau, -).$$

This together with (4.9) gives

$$\begin{aligned} f(x) &= T^{-1}(Tf)(x) = k_x(Tf(\psi(x))) = k_x h_{\psi(x)} f(\varphi(\psi(x))) = k_x h_{\psi(x)} f(x), \\ g(y) &= T(T^{-1}g)(y) = h_y(T^{-1}g(\varphi(y))) = h_y k_{\varphi(y)} g(\psi(\varphi(y))) = h_y k_{\varphi(y)} g(y), \end{aligned}$$

for all f in $C_0(X, \mathcal{H}_X, \sigma, -)$ and g in $C_0(Y, \mathcal{H}_Y, \tau, -)$. Since both $h_y, k_{\varphi(y)}$ have norms not greater than one, $h_y = k_{\varphi(y)}^{-1}$ is a surjective real linear isometry from $K_{\varphi(y)}$ onto K_y.

It follows from the identity $\sigma \circ \varphi = \varphi \circ \tau$ that $y = \tau(y)$ if and only if $\varphi(y) = \sigma(\varphi(y))$. When $y \neq \tau(y)$ and thus $\varphi(y) \neq \sigma(\varphi(y))$, we have $K_y = H_y$ and $K_{\varphi(y)} = H_{\varphi(y)}$. When $y = \tau(y)$ and thus $\varphi(y) = \sigma(\varphi(y))$, we have $H_y = K_y + iK_y$ and $H_{\varphi(y)} = K_{\varphi(y)} + iK_{\varphi(y)}$. In the latter case, we can extend h_y to a surjective complex linear isometry from H_y onto $H_{\varphi(y)}$ such that $h_y(\xi + i\eta) = h_y(\xi) + ih_y(\eta)$. To see h_y is complex linear in the case when $y \neq \tau(y)$, we observe that for any a in $C_0(X,\tau)$ such that $a(\varphi(y)) = i$, it follows from Theorem 1.1 that

$$\begin{aligned} h_y(if(\varphi(y))) &= h_y((fa)(\varphi(y))) = T(fa)(y) = Tf(y)\alpha(a)(y) \\ &= Tf(y)a(\varphi(y)) = iTf(y) = ih_y(f(\varphi(y))), \quad \forall f \in C_0(X, \mathcal{H}_X, \sigma, -). \end{aligned}$$

This gives the complex linearity of h_y. Hence we can say that h_y is a surjective complex linear isometry from $H_{\varphi(y)}$ onto H_y for each y in Y.

Let f be in $C_0(X, \mathcal{H}_X, \sigma, -)$. Then $\overline{f \circ \sigma} = f$ and $\overline{T(f) \circ \tau} = T(f)$. Observe that

$$\overline{h_{\tau(y)}(f(\varphi(\tau(y))))} = \overline{T(f)(\tau(y))} = T(f)(y) = h_y(f(\varphi(y))) = h_y(\overline{f(\sigma(\varphi(y)))}) \quad (4.10)$$

It then follows from (4.10) and the identity $\sigma \circ \varphi = \varphi \circ \tau$ that $\overline{h_{\tau(y)}(\mu)} = h_y(\overline{\mu})$, and thus,

$$h_{\tau(y)}(\mu) = \overline{h_y(\overline{\mu})}, \quad \forall \mu \in K_{\varphi(y)}.$$

Note that $H_{\varphi(y)} = K_{\varphi(y)}$ whenever $y \neq \tau(y)$. The above equality also extends to $H_{\varphi(y)}$ due to the construction of the map h_y whenever $y = \tau(y)$. □

Following the proof of Theorem 4.2, one can show that the quotient spaces $X/\sim$ and $Y/\sim$ are homeomorphic. However, it does not assure that X and Y are homeomorphic, unless the isometry is a 2-isometry. Unlike the complex case, a surjective real linear isometry between real Hilbert C^*-modules over commutative real C^*-algebras might not be a 2-isometry. Moveover, the underlying locally compact spaces X and Y might not be homeomorphic, as shown in the following counter example.

Example 4.7. Let $X = \{-1, 1\}$ and $Y = \{0\}$ in discrete topology. Define $\sigma : X \to X$ by $\sigma(\pm 1) = \mp 1$ and $\tau : Y \to Y$ by $\tau(0) = 0$. Let $A = C(X, \sigma)$ and $B = C(Y, \tau)$. In other words, A is the *real* C*-algebra $\mathbb{C}$ and B is the real C*-algebra $\mathbb{R}$.

Let $H_{-1} = H_1 = \mathbb{C}$ be the one dimensional complex Hilbert space. Consider the complex Hilbert bundle $H_{-1} \coprod H_1$ with base space X. Define two conjugate linear surjective isometries $h \mapsto \overline{h}$ from $H_{\pm 1}$ onto $H_{\mp 1}$ by the usual complex conjugation. Then the real Hilbert C*-module $C(X, H_{-1} \coprod H_1, \sigma, -)$ is isomorphic to $\mathbb{C}$. Indeed, if $f \in C(Y, H_0, \tau, -)$ with $f(1) = a + bi$ then $f(-1) = \overline{f(1)} = a - bi$.

On the other hand, let $H_0 = \mathbb{C} \oplus_2 \mathbb{C}$ be the two dimensional complex Hilbert space. Define a conjugate linear surjective isometry $(h, k) \mapsto (\overline{h}, \overline{k})$ from H_0 onto H_0 with the usual complex conjugation. In this way, the real Hilbert C*-module $C(Y, H_0, \tau, -)$ is isomorphic to $\mathbb{R} \oplus_2 \mathbb{R}$.

Consider the map $T : C(X, H_{-1} \coprod H_1, \sigma, -) \to C(Y, H_0, \tau, -)$ defined by

$$f \mapsto (a, b), \quad \forall f \in C(X, H_{-1} \coprod H_1, \sigma, -) \text{ with } f(1) = a + bi.$$

It is easy to see that T is a real linear surjective isometry. Since X and Y are not homeomorphic, it is no hope to get a homeomorphism symbol φ in the weighted composition operator form of T in this case. Moreover, there is no hope the identity $\sigma \circ \varphi = \varphi \circ \tau$ to be held.

Finally, we remark that T does not satisfy any of the equivalent conditions stated in Theorem 1.1. For example, T is not a module map with respect to any $*$-isomorphism between A and B, as there is simply none of them. $\square$

In the following example, we see that the fiber maps h_y might not be complex linear or conjugate linear if T is not a 2-isometry.

Example 4.8. Let X be the discrete space $\{-1, 1\}$ with an involution $\sigma(\pm 1) = \mp 1$. Let H be the two dimensional complex Hilbert space $\mathbb{C}^2$. Let $\mathcal{H}$ be the complex Hilbert bundle $H_{-1} \coprod H_1$ with both $H_{\pm 1} = H$. Define two surjective conjugate linear isometries $- : H_{\pm 1} \to H_{\mp 1}$ by $(u, v) \mapsto (\bar{u}, \bar{v})$.

Consider the real Hilbert A-module $V = C(X, \mathcal{H}, \sigma, -)$ over the commutative real C^*-algebra $A = C(X, \sigma)$. We consider an element f in V as a function from $X = \{-1, 1\}$ into $H = \mathbb{C}^2$, and write $f = (f_1, f_2)$ for f in V. Note that $f_k(\mp 1) = f_k(\sigma(\pm 1)) = \overline{f_k(\pm 1)}$ in $\mathbb{C}$ for $k = 1, 2$.

Let $T : V \to V$ be defined by $Tf(\pm 1) = (f_1(\pm 1), \overline{f_2(\pm 1)}) = (f_1(\pm 1), f_2(\mp 1))$. Then T is a surjective real linear isometry of $C(X, \mathcal{H}, \sigma, -)$. We can write T in the weighted composition form $Tf(y) = h(y)f(\varphi(y))$ with $\varphi(\pm 1) = \pm 1$ and surjective real, but not complex, linear isometries $h(1) = h(2) = h : H \to H$ defined by $h(u, v) = (u, \bar{v})$. Alternatively, we can use $\varphi(\pm 1) = \mp 1$ and $h(u, v) = (\bar{u}, v)$. However, T does not satisfy any one of the five equivalent conditions stated in Theorem 1.1.

For example, if $\beta \in C(X, \sigma)$ with $\beta(\pm 1) = \pm i$ then $T(f\beta) = (Tf)\alpha(\beta)$ cannot hold for any *-automorphism α of $C(X, \sigma)$, as such α can only be either the one $\beta \mapsto \beta$ or the one $\beta \mapsto \beta \circ \sigma$.

On the other hand, T is not a 2-isometry. To see this, observe that an element F in $M_2(V)$ carries the form $F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$. Here, f_{ij}'s come from $C(X, \mathcal{H}, \sigma, -)$. Consider F in $M_2(V)$ with $F(1) = \begin{bmatrix} (i, -i) & (1, 1) \\ (1, 1) & (0, 0) \end{bmatrix}$ and $F(-1) = \overline{F(1)} = \begin{bmatrix} (-i, i) & (1, 1) \\ (1, 1) & (0, 0) \end{bmatrix}$. Then $T_2(F)(1) = \begin{bmatrix} (i, i) & (1, 1) \\ (1, 1) & (0, 0) \end{bmatrix}$ and $T_2(F)(-1) = \begin{bmatrix} (-i, -i) & (1, 1) \\ (1, 1) & (0, 0) \end{bmatrix}$. Their inner products

$$\langle F, F \rangle(\pm 1) = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \langle T_2(F), T_2(F) \rangle(\pm 1) = \begin{bmatrix} 4 & \mp 2i \\ \pm 2i & 2 \end{bmatrix}.$$

It follows $\|F\| = 2$ while $\|T_2 F\| = \sqrt{3+\sqrt{5}}$. Thus, T_2 is not an isometry. $\square$

In general, let T be a surjective real linear isometry between real Hilbert C^*-modules over commutative real C^*-algebras. We try to check whether T preserves the ternary products. Assuming the weighted composition form of T as stated in Theorem 4.5, we compute

$$T(f_1\langle f_2, f_3\rangle)(y) = h(y)(f_1\langle f_2, f_3\rangle)(\varphi(y)) = \langle f_2(\varphi(y)), f_3(\varphi(y))\rangle h(y)(f_1(\varphi(y))),$$

and

$$T(f_1)\langle T(f_2), T(f_3)\rangle(y) = \langle h(y)(f_2(\varphi(y)), h(y)(f_3(\varphi(y)))\rangle h(y)(f_1(\varphi(y))).$$

Hence the above quantities always equal exactly when

$$\langle h(y)(f_2(\varphi(y))), h(y)(f_3(\varphi(y)))\rangle = \langle f_2(\varphi(y)), f_3(\varphi(y))\rangle.$$

Since $h(y)$ is a surjective real linear isometry, we always have

$$\mathrm{Re}\,\langle h(y)u, h(y)v\rangle = \mathrm{Re}\,\langle u, v\rangle, \quad \forall u, v \in K_{\varphi(y)}.$$

However, $h(y)$ might not preserve inner products, e.g., the one given in Example 4.8. We note that a surjective real linear isometry $h : H \to K$ between two complex Hilbert spaces preserves inner products exactly when h is complex linear. Indeed, complex linear isometries preserve inner products due to the polarization identity (1.1). Conversely, suppose h preserves inner products. Then

$$\langle h(\lambda u), hv\rangle = \langle \lambda u, v\rangle = \overline{\lambda}\langle u, v\rangle = \overline{\lambda}\langle hu, hv\rangle = \langle \lambda hu, hv\rangle,$$

for all u, v in H and complex numbers λ. Since h is onto, we see that $h(\lambda u) = \lambda hu$ as asserted.

Let T be a surjective complex linear isometry between complex Hilbert C^*-modules over complex commutative C^*-algebras. Then from the construction all fibre maps $h(y)$ are also complex linear. This explains why we can show that the surjective complex linear isometry T preserves inner products in [26].

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(Hsu) Department of Mathematics, National Central University, Chung-Li, 32054, Taiwan.

E-mail address: `hsumh@math.ncu.edu.tw`

(Wong) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan.

E-mail address: `wong@math.nsysu.edu.tw`