Fourier inversion and prime ideals on Nilpotent Lie groups

Ying-Fen Lin

Queen's University Belfast
(joint work with Jean Ludwig and Carine Molitor-Braun)

Kaohsiung
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The classical result

Let G be a locally compact *abelian* group.

Theorem (The Fourier inversion theorem)

If $f \in L^1(G)$ and $\hat{f} \in L^1(\widehat{G})$, then $f(x) = (\hat{f})\hat{}(x^{-1})$ for a.e. x in G. If f is continuous, then the above relation holds for every x in G.

For *non-abelian* groups, say if $G = \exp(\mathfrak{g})$ is a nilpotent Lie group, then there is an one-to-one correspondence between $\widehat{G}$ and $\mathfrak{g}^*/G$ (Kirillov, 1962).

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If $f \in \mathcal{S}(G)$ and $\pi_\ell = \mathrm{ind}_{P(\ell)}^{G} \chi_\ell \in \widehat{G}$, then $\pi_\ell(f)$ can be viewed as an integral operator on $L^1(G)$ and is hence determined by an operator kernel F_ℓ. More precisely, for $f \in \mathcal{S}(G)$, $\xi \in \mathcal{H}_\ell$,

$$(\pi_\ell(f)\xi)(g) = \int_{G/P(\ell)} F_\ell(g,u)\xi(u)du,$$

where F_ℓ is the operator kernel given by

$$F_\ell(g,u) = \int_{P(\ell)} f(ghu^{-1})\chi_\ell(h)dh \text{ for } g,u \in G,$$

and $F_\ell \in \mathcal{S}(G/P(\ell) \times G/P(\ell))$.

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Question: Can we recover the original Schwartz function f, starting only with F? Any good correspondence between the functions f and F? That is, can we specify precisely which functions F arise as operator kernels for such $\pi_\ell(f)$'s with Schwartz functions f?

Let G be a connected, simply connected nilpotent Lie group.

Howe (1977)

For any irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of G and any $a \in B^\infty(\mathcal{H}_\pi)$, smooth bounded linear operators on $\mathcal{H}_\pi$, there is $f_a \in \mathcal{S}(G)$ such that $\pi(f_a) = a$. Moreover,

$$a \mapsto f_a : B^\infty(\mathcal{H}_\pi) \mapsto \mathcal{S}(G)$$

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We study a version of the Fourier inversion theorem for nilpotent Lie groups which generalised Howe's result by constructing a *continuous retract* from the space of adapted *smooth kernel functions* defined on a smooth G-invariant manifold of $\mathfrak{g}^*$ with certain property into the space $\mathcal{S}(G)$.

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A subset M of $\mathfrak{g}^*$ is called *G-invariant* if for every $\ell \in M$, the element

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A Fourier inversion thm on nilpotent Lie groups

Theorem (Lin, Ludwig, Molitor-Braun)

Let $\mathfrak{g}$ be a nilpotent Lie algebra and M be a smooth G-invariant submanifold of $\mathfrak{g}^*$. Let π_ℓ be an induced unitary representation for $\ell \in M$. Then there is an open relatively compact subset $\mathcal{M}$ of M such that for any kernel function F supported in $G \cdot \mathcal{M}$, there is $f \in \mathcal{S}(G)$ such that $\pi_\ell(f)$ has $F(\ell, \cdot, \cdot)$ as an operator kernel for all $\ell \in M$. Moreover, the mapping $F \mapsto f$ is continuous w.r.t the corresponding function space topologies.

Remark: We have a similar Fourier inversion theorem for smooth variable nilpotent Lie groups.

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Variable nilpotent Lie groups (Leptin and Ludwig, 1994)

Let $\mathfrak{g}$ be a real vector space with $\dim(\mathfrak{g}) = n$ and $\mathcal{B} \neq \emptyset$. Then $(\mathcal{B}, \mathfrak{g})$ is called a *variable nilpotent Lie algebra* if

- for $\beta \in \mathcal{B}$, there is $[\cdot, \cdot]_\beta$ defined on $\mathfrak{g}$ such that $\mathfrak{g}_\beta := (\mathfrak{g}, [\cdot, \cdot]_\beta)$ is a nilpotent Lie algebra; and
- there is Jordan-Hölder basis $\{Z_1, \dots, Z_n\}$ for $\mathfrak{g}_\beta$. That is, $\exists$ a fixed basis $\{Z_1, \dots, Z_n\}$ of $\mathfrak{g}$ such that the constants $a_{ij}^k(\beta)$ defined by
$$[Z_i, Z_j]_\beta := \sum_{k=1}^{n} a_{ij}^k(\beta) Z_k$$
has the property that $a_{ij}^k(\beta) = 0$ for $\beta \in \mathcal{B}$ and $k \leq \max\{i, j\}$.

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A retract for smooth variable nilpotent Lie groups

Theorem (Lin, Ludwig, Molitor-Braun)

Let $\mathcal{B} \times G$ be a smooth variable nilpotent Lie group and M be a smooth G-invariant submanifold of $\mathcal{B} \times \mathfrak{g}^*$ contained in $(\mathcal{B} \times \mathfrak{g}^*)_{\leq I}$ such that $M_I := M \cap (\mathcal{B} \times \mathfrak{g}^*)_I \neq \emptyset$. Let $\pi(\beta, l)$ be the corresponding family of induced unitary representations for $(\beta, l) \in M$. Then there exists an open relatively compact subset $\mathcal{M}$ of M_I such that the following holds: for any adapted kernel function F supported in $G \cdot \mathcal{M}$, there is a function $f \in \mathcal{S}(\mathbb{R}^r, \mathcal{B}, G)$ such that $\pi_{(\beta,l)}(f(\alpha, \beta, \cdot))$ has $F(\alpha, (\beta, l), \cdot, \cdot)$ as an operator kernel for all $(\alpha, (\beta, l)) \in \mathbb{R}^r \times M$. Moreover, the mapping $F \mapsto f$ is continuous w.r.t the corresponding function space topologies.

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Prime ideals in $L^1(G)$

As an application, we can study the **G**-prime ideals of $L^1(G)$, where **G** is a Lie subgroup of $\mathrm{Aut}(G)$.

Definition

A two-sided closed ideal $\mathfrak{I}$ in $L^1(G)$ is called ***G**-prime* if $\mathfrak{I}$ is **G**-invariant and for all **G**-invariant ideals $\mathfrak{I}_1, \mathfrak{I}_2$ in $L^1(G)$ with the property that $\mathfrak{I}_1 * \mathfrak{I}_2 \subset \mathfrak{I}$, then either $\mathfrak{I}_1 \subset \mathfrak{I}$ or $\mathfrak{I}_2 \subset \mathfrak{I}$.

Note: the kernel of each **G**-orbit is a **G**-prime ideal.

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Known results

Ludwig (1983)

Closed prime ideals of $L^1(G)$ coincide with the kernels of irreducible unitary representations.

Poguntke (1984)

characterised the K-prime ideals as kernel of K-orbits, where K is an abelian compact group acting on a nilpotent Lie group M

Lahiani and Molitor-Braun (2011)

If I is a proper closed K-prime ideal, where K is a compact subgroup of $\mathrm{Aut}(G)$, then there is K-orbit Ω_ℓ such that

$$I \cap \mathcal{S}(G) = \ker \Omega_\ell \cap \mathcal{S}(G).$$

Prime ideals in $L^1(G)$

Theorem (Lin, Ludwig, Molitor-Braun)

Let G be a simply connected, connected nilpotent Lie group and $\mathbf{G}$ be a Lie group of automorphisms of G containing the inner automorphisms such that every $\mathbf{G}$-orbit in $\mathfrak{g}^*$ is locally closed.
Then every $\mathbf{G}$-prime ideal of $L^1(G)$ is the kernel of an $\mathbf{G}$-orbit.

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Notations and remarks

Let $\mathrm{Prim}^*(G) := \{\ker\pi : \pi \in \widehat{G}\}$. For $\mathfrak{J} \subset L^1(G)$, the hull of $\mathfrak{J}$ is given by

$$h(\mathfrak{J}) := \{P \in \mathrm{Prim}^*(G) : \mathfrak{J} \subset P\}.$$

For connected, simply connected nilpotent Lie group G,

$$\pi \mapsto \ker\pi : \widehat{G} \mapsto \mathrm{Prim}^*(G)$$

is a homeomorphism.

Note that for any closed orbit Ω in $\mathfrak{g}^*$,

- $\overline{\mathcal{S}(G) \cap \ker(\Omega)}^{L^1(G)} = \ker(\Omega)$ (from the Inversion Theorem);
- there is a minimal ideal $J(\Omega)$ in $\mathcal{S}(G)$ such that $h(J(\Omega)) = \Omega$ and $J(\Omega) \subset \mathfrak{J}$ for all ideal $\mathfrak{J}$ of $\mathcal{S}(G)$ with $h(\mathfrak{J}) \subset \Omega$.

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Proof

Let $\mathfrak{I}$ be a proper **G**-prime ideal of $L^1(G)$. Then $\mathfrak{I}_{\mathcal{S}} = \mathfrak{I} \cap \mathcal{S}(G)$ is a proper **G**-prime ideal of $\mathcal{S}(G)$ which is closed in the $\|\cdot\|_1$-norm.
By Molitor-Braun, there is an orbit $\Omega \in \mathrm{Prim}^*(G)$ such that $\mathfrak{I}_{\mathcal{S}} = \ker \Omega \cap \mathcal{S}(G)$. Hence,

$$h(\mathfrak{I}) = h(\mathfrak{I}_{\mathcal{S}}) = h(\ker \Omega \cap \mathcal{S}(G)) = h(\ker \Omega) = \overline{\Omega}$$

and $\mathfrak{I} \subset \ker \Omega$. On the other hand, since $\mathcal{S}(G) \cap \ker \Omega$ is dense in $\ker \Omega$, we have

$$(\ker \Omega)^N \subset J(\Omega) \subset \mathfrak{I}$$

for some $N \in \mathbb{N}$. Since $\mathfrak{I}$ is **G**-prime, we have that $\mathfrak{I} = \ker \Omega$.

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Theorem

Let G be a simply connected, connected nilpotent Lie group and **G** *be a Lie group of automorphisms of G containing the inner automorphisms such that every* **G***-orbit in $\mathfrak{g}^*$ is locally closed. Then every* **G***-prime ideal of $L^1(G)$ is the kernel of an* **G***-orbit.*

THANK YOU for YOUR ATTENTION!!