Twisted tensor products, and cocycles on Fourier algebras

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For $\alpha \geq 0$ we may define

$$\mathrm{A}_\alpha(\mathbb{T}) = \{f \in C(\mathbb{T})\colon \sum_{n\in\mathbb{Z}} |\hat{f}(n)|(1+|n|)^\alpha < \infty\}.$$

We may define $\mathrm{A}_\alpha(\mathbb{T}^d)$ in an analogous way, using the weight function

$$(n_1,\dots,n_d) \mapsto (1+|n_1|)^\alpha \dots (1+|n_d|)^\alpha \ .$$

This notation is non-standard, but useful here, since

$$\mathrm{A}_\alpha(\mathbb{T}^d) \cong \mathrm{A}_\alpha(\mathbb{T}) \mathbin{\hat{\otimes}_\gamma} \cdots \mathbin{\hat{\otimes}_\gamma} \mathrm{A}_\alpha(\mathbb{T})$$

isometrically as Banach algebras. (Here, $\hat{\otimes}_\gamma$ is the projective tensor product of Banach spaces.)

Remark

These are basic examples of Beurling–Fourier algebras; we can make analogous definitions with more general weight functions.

Reminder

If A is a Banach algebra then a bounded linear map $D : A \to A^*$ is a **derivation** if it satisfies

$$\begin{aligned} D(a_1 a_2)(a_0) &= [a_1 \cdot D(a_2)](a_0) + [D(a_1) \cdot a_2](a_0) \\ &= D(a_2)(a_0 a_1) + D(a_1)(a_2 a_0) \end{aligned}$$

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An example of a derivation

For $f_1, f_0 \in C^1(\mathbb{T})$, define

$$D(f_1)(f_0) = \int_{\mathbb{T}} \frac{\partial f_1}{\partial \theta} f_0$$

Then $D : C^1(\mathbb{T}) \to C^1(\mathbb{T})^*$ is a non-zero derivation.

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If $\alpha \geq 1/2$, then D extends to a non-zero derivation $\mathrm{A}_\alpha(\mathbb{T}) \to \mathrm{A}_\alpha(\mathbb{T})^*$.

Idea of the proof

Use Parseval/Plancherel for the group $\mathbb{T}$: $\langle g, h \rangle_{L^2(\mathbb{T})} = \langle \widehat{g}, \widehat{h} \rangle_{\ell^2(\mathbb{Z})}$.

Another example (Johnson, 1994)

For $f_1, f_0 \in C^1(\mathrm{SU}(2))$, define

$$D(f_1)(f_0) = \int_{\mathrm{SU}(2)} (\partial_\phi f_1) f_0$$

where $(\partial_\phi h)(p) = \frac{\partial}{\partial\phi} h(p s_\phi)\Big|_{\phi=0}$ for $s_\phi = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$.

Then $D : C^1(\mathbb{T}) \to C^1(\mathbb{T})^*$ is a non-zero derivation.

D extends to a non-zero derivation $\mathrm{A}(\mathrm{SU}(2)) \to \mathrm{A}(\mathrm{SU}(2))^*$.

In particular: $\mathrm{A}(\mathrm{SU}(2))$ is not weakly amenable

Idea of one possible proof (C.+Ghandehari, 2014)

Use Parseval/Plancherel for the group $\mathrm{SU}(2)$:

$$\langle g, h \rangle_{L^2(\mathrm{SU}(2))} = \sum_{n \geq 1} n \, \mathrm{Tr}(\pi_n(g) \pi_n(h)^*).$$

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Definition

Let A be a **commutative** *Banach algebra and M a* **symmetric** *Banach A-bimodule, e.g. $M = A^*$. An alternating n-cocycle is: an n-multilinear map $\psi : A \times \cdots \times A \to M$ which satisfies*

- $\psi(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = (-1)^\sigma \psi(a_1, \cdots a_n)$ *for every $\sigma \in S_n$, and*
- $\psi(bc, a_2, \ldots, a_n) = b \cdot \psi(c, a_2, \ldots, a_n) + c \cdot \psi(b, a_2, \ldots, a_n)$,

for all $b, c, a_1, \ldots, a_n \in A$.

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Remark

These objects were already known in commutative algebra/algebraic geometry. (E.g. implicit in HKR theorem for smooth algebras; explicitly named in work of GERSTENHABER–SCHACK, 1987.)

For $\alpha \geq 0$ we may define

$$\mathrm{A}_\alpha(\mathbb{T}) = \{f \in C(\mathbb{T})\colon \sum_{n\in\mathbb{Z}} |\hat{f}(n)|(1+|n|)^\alpha < \infty\}.$$

The multivariable version satisfies

$$\mathrm{A}_\alpha(\mathbb{T}^d) \cong \mathrm{A}_\alpha(\mathbb{T}) \mathbin{\hat{\otimes}_\gamma} \cdots \mathbin{\hat{\otimes}_\gamma} \mathrm{A}_\alpha(\mathbb{T})$$

isometrically as Banach algebras.

For $F_1, F_2, F_0 \in C^1(\mathbb{T}^2)$, define

$$\Psi(F_1, F_2)(F_0) := \int_{\mathbb{T}^2} \left(\frac{\partial F_1}{\partial \theta_1} \frac{\partial F_2}{\partial \theta_2} - \frac{\partial F_2}{\partial \theta_1} \frac{\partial F_1}{\partial \theta_2} \right) F_0$$

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If $\alpha \geq 1/2$, then Ψ extends to a non-zero, alternating 2-cocycle $\mathrm{A}_\alpha(\mathbb{T}^2) \times \mathrm{A}_\alpha(\mathbb{T}^2) \to \mathrm{A}_\alpha(\mathbb{T}^2)^*$.

Motivating example of a (non-inner) 2-cocycle

For $F_1, F_2, F_0 \in C^1(\mathbb{T}^2)$, define

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This construction has something to do with "tensoring derivations".

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Question.

Can we do the same for (some) Fourier algebras?

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Partial answer

Let $\mathcal{Z}^n_{\text{alt}}(A, A^*)$ be the space of alternating n-cocycles for a CBA A.

The number

$$\dim_{\mathsf{JDR}}(A) := \max\{n \colon \mathcal{Z}^n_{\text{alt}}(A, A^*) \neq 0\}$$

is a candidate for measuring some kind of "dimension" of A, which is more tractable than the usual homological dimension.

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Note: $\dim_{\mathsf{JDR}}(A) = 0$ if and only if A is **weakly amenable.**

Example (can be extracted from Johnson, 1997)

$\dim_{\mathsf{JDR}} \mathrm{A}_\alpha(\mathbb{T}^d)$ is d if $\alpha \geq 1/2$ and 0 if $0 \leq \alpha < 1/2$.
In contrast, for **every** $\alpha > 0$, $\mathrm{A}_\alpha(\mathbb{T})$ has non-trivial cohomology in degree 2 (and possibly all higher degrees).

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Remark (paraphrasing results of Johnson)

Results are even more interesting for the algebras $\mathrm{lip}_\alpha(\mathbb{T}^d)$:

$$\dim_{\mathsf{JDR}} \mathrm{lip}_\alpha(\mathbb{T}^d) = \min\left(d, \left\lceil \frac{1}{1-\alpha} \right\rceil - 2\right)$$

So for fixed d: as $\alpha \searrow 0$ we get $\dim_{\mathsf{JDR}} \to 0$; as $\alpha \nearrow 1$ we get $\dim_{\mathsf{JDR}} \to d$.

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Question.

Can we find G with $\dim_{\mathsf{JDR}} \mathrm{A}(G) \geq 2$?

Consider CBAs A and B, and derivations $D_A : A \to A^*$, $D_B : B \to B^*$.

Define $\psi : (A \hat{\otimes}_\gamma B) \times (A \hat{\otimes}_\gamma B) \to (A \hat{\otimes}_\gamma B)^*$ by

$$\psi(a_1 \otimes b_1, a_2 \otimes b_2)(a_0 \otimes b_0) = \begin{cases} D_A(a_1)(a_2a_0)D_B(b_2)(b_1b_0) \\ -D_A(a_2)(a_1a_0)D_B(b_1)(b_2b_0) \end{cases}$$

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Then $\psi \in \mathcal{Z}^2_{\mathrm{alt}}(A \hat{\otimes}_\gamma B, (A \hat{\otimes}_\gamma B)^*)$.

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Aide-memoire

Compare this with the earlier motivating example:

$$\Psi(F_1, F_2)(F_0) := \int_{\mathbb{T}^2} \left(\frac{\partial F_1}{\partial \theta_1} \frac{\partial F_2}{\partial \theta_2} - \frac{\partial F_2}{\partial \theta_1} \frac{\partial F_1}{\partial \theta_2} \right) F_0$$

Recall: by Johnson's result/construction, there is a non-zero derivation $D : \mathrm{A}(\mathrm{SU}(2)) \to \mathrm{A}(\mathrm{SU}(2))^*$.

Taking $A = B = \mathrm{A}(\mathrm{SU}(2))$ and $D_A = D_B = D$, we get a non-zero 2-cocycle on $\mathrm{A}(\mathrm{SU}(2)) \hat{\otimes}_\gamma \mathrm{A}(\mathrm{SU}(2))$.

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On the other hand: $\mathrm{A}(\mathrm{SU}(2) \times \mathrm{SU}(2)) \cong \mathrm{A}(\mathrm{SU}(2)) \widehat{\otimes} \mathrm{A}(\mathrm{SU}(2))$ where $\widehat{\otimes}$ denotes the **operator space projective tensor product**.

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Theorem (Spronk, 2002; Samei, 2005)

For ***any*** *locally compact G, the only c.b. derivation $\mathrm{A}(G) \to \mathrm{A}(G)^*$ is the zero map.*

However: if we revisit the derivation for $\mathrm{A}(\mathrm{SU}(2))$, one can show that this map D is c.b. when viewed as a linear map

$$[\mathrm{A}(\mathrm{SU}(2))]^{\sim} \to \mathrm{A}(\mathrm{SU}(2))^{*}$$

where the tilde denotes the opposite operator space structure.

Remark

More concretely, this means that $D : \mathrm{A}(\mathrm{SU}(2)) \to \mathrm{A}(\mathrm{SU}(2))^{*}$ is actually given by an element of $\mathrm{VN}(\mathrm{SU}(2)^{\mathsf{op}} \times \mathrm{SU}(2))$.

[Callback]

Another example (Johnson, 1994)

$$D(f_1)(f_0) = \int_{\mathrm{SU}(2)} (\partial_\phi f_1) f_0$$

where $(\partial_\phi h)(p) = \frac{\partial}{\partial \phi} h(p s_\phi)\Big|_{\phi=0}$ for $s_\phi = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$.
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Idea of one possible proof (C.+Ghandehari, 2014)

Use Parseval/Plancherel for the group $\mathrm{SU}(2)$:

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- The argument in [LLSS16] works by bootstrapping from a finite number of concrete cases, and in all those cases one has explicit non-zero derivations that are c.b. from $\widetilde{\mathrm{A}(G)} \to \mathrm{A}(G)^*$.

  (See also earlier examples of C.+Ghandehari, 2014.)

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- It is natural to wonder if the twisted c.b. property holds in all cases, but it is unclear if the bootstrapping process respects this property. (Work in progress with E. SAMEI.)

Consider CCBAs A and B, and cb maps $T_A : \widetilde{A} \to A^*$, $T_B : \widetilde{B} \to B^*$.

Define $\psi_0 : A \times B \times A \times B \to A^* \otimes B^*$ by

$$\psi_0(a_1, b_1, a_2, b_2) = \begin{cases} \quad [T_A(a_1) \cdot a_2] \otimes [T_B(b_2) \cdot b_1] \\ -[T_A(a_2) \cdot a_1] \otimes [T_B(b_1) \cdot b_2] \end{cases}$$

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Although this looks unappealing, we find after some book-keeping that:

Lemma

The map ψ_0 extends to a bounded bilinear map

$$\psi_1 : (A \mathbin{\widehat{\otimes}} B) \times (A \mathbin{\widehat{\otimes}} B) \to A^* \mathbin{\widehat{\otimes}} \widetilde{B^*}$$

Now what do we do? Remember, we want to land in $(A \mathbin{\widehat{\otimes}} B)^*$!

[tumbleweed rolls by]

Theorem (C., 2016 preprint)

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The proof goes via the special case $X = \mathcal{B}(\mathsf{H})$, $Y = \mathcal{B}(\mathsf{K})$. In turn, this special case is a consequence of the following result.

Proposition

Define $\Phi_0 : \mathcal{B}(\mathsf{H}) \otimes \mathcal{B}(\mathsf{K}) \to \mathcal{B}(\mathcal{S}_2(\mathsf{K}, \mathsf{H}))$ by

$$\Phi_0(a \otimes b) : c \mapsto acb$$

Then $\|\Phi_0(w)\| \leq \|w\|_{\mathcal{B}(\mathsf{H}) \widehat{\otimes} \mathcal{B}(\mathsf{K})}$ for all $w \in \mathcal{B}(\mathsf{H}) \otimes \mathcal{B}(\mathsf{K})$.

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Sketch of the proof

WLOG $\|w\|_{\mathcal{B}(\mathsf{H}) \widehat{\otimes} \mathcal{B}(\mathsf{K})} \leq 1$. Can show, by o.s. tensor product arguments, that $\Phi_0(w)$ acts contractively on $\mathsf{H} \widehat{\otimes} \mathsf{K}^* \cong \mathcal{S}_1(\mathsf{K}, \mathsf{H})$ and contractively on $\mathsf{H} \otimes_{\min} \mathsf{K}^* \cong \mathcal{S}_\infty(\mathsf{K}, \mathsf{H})$. **INTERPOLATE.** □

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Remark

If you interpolate in the o.s. category you obtain a complete contraction $\mathcal{B}(\mathsf{H}) \widehat{\otimes} \mathcal{B}(\mathsf{K}) \to \mathcal{CB}(\mathsf{OH})$ where OH is $\mathcal{S}_2(\mathsf{K},\mathsf{H})$ equipped with Pisier's self-dual o.s.s.

Taking $X = Y = \mathrm{VN}(\mathrm{SU}(2))$, and combining this with the earlier calculations:

Corollary

If D is "Johnson's derivation" for $\mathrm{A}(\mathrm{SU}(2))$, and we use it to construct the 2-cocycle ψ on $\mathrm{A}(\mathrm{SU}(2)) \hat{\otimes}_\gamma \mathrm{A}(\mathrm{SU}(2))$, then ψ extends to a 2-cocycle on $\mathrm{A}(\mathrm{SU}(2) \times \mathrm{SU}(2))$.

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Invoking Herz's restriction theorem and standard properties of cocycles:

Corollary

If G is a locally compact group containing a closed copy of $\mathrm{SU}(2) \times \mathrm{SU}(2)$, then $\dim_{\mathsf{JDR}} \mathrm{A}(G) \geq 2$.

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Corollary

If D is "Johnson's derivation" for $\mathrm{A}(\mathrm{SU}(2))$, and we use it to construct the 2-cocycle ψ on $\mathrm{A}(\mathrm{SU}(2)) \hat{\otimes}_\gamma \mathrm{A}(\mathrm{SU}(2))$, then ψ extends to a 2-cocycle on $\mathrm{A}(\mathrm{SU}(2) \times \mathrm{SU}(2))$.

Invoking Herz's restriction theorem and standard properties of cocycles:

Corollary

If G is a locally compact group containing a closed copy of $\mathrm{SU}(2) \times \mathrm{SU}(2)$, then $\dim_{\mathsf{JDR}} \mathrm{A}(G) \geq 2$.

Question.

Is there a group G with $\dim_{\mathsf{JDR}} \mathrm{A}(G) \geq 3$?

Thank you for your attention!