Uniform continuity of eigenvalue sequences of regular Sturm-Liouville equations

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9th International Conference in Abstract Harmonic Analysis
Sun Yat-Sen University, Kaohsiung
June 25th, 2018

(joint works with Prof. Jiangang Qi)

A classical *Sturm-Liouville (S-L) theory*, named after J. C. F. Sturm (1803-1855) and J. Liouville (1809-1882), is the theory of a *real second-order linear differential equation* of the form

$$-(p(x)y(x)')' + q(x)y(x) = \lambda\omega(x)y(x).$$

A S-L problem is said to be *regular* if it has separated boundary conditions of the form

$$\begin{cases} R_1(y) = \alpha_1 y(a) + \alpha_2 p(a)y'(a) = 0, & \alpha_1^2 + \alpha_2^2 > 0; \\ R_2(y) = \beta_1 y(b) + \beta_2 p(b)y'(b) = 0, & \beta_1^2 + \beta_2^2 > 0, \end{cases}$$

where p, p', $\frac{1}{p}$, q and ω are integrable functions over the finite interval $[a, b]$, and $p,\ \omega > 0$.

- $q(x)$: potential function.
- $\omega(x)$: weight function or density function.

S-L equations occur very commonly in applied mathematics and physics.

• Dealing with linear PDEs.

Example: wave equation $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$, $t > 0,\ 0 < x < 1$,
$y(0,t) = y(1,t) = 0$, $y(x,0) = f(x)$, $\frac{\partial y(x,0)}{\partial t} = 0$.
Let $y = u(x)\varphi(t)$.
$\Longrightarrow \frac{\partial^2 u}{\partial x^2} = -\lambda u$ and $\frac{\partial^2 \varphi}{\partial t^2} = -\lambda a^2 \varphi$. $u(0) = u(1) = 0$.

• Recasting second-order linear ODEs.

Examples:
1. The Bessel equation $x^2 y'' + xy' + (x^2 - \nu^2)y = 0$
$\Leftrightarrow (xy')' + \left(x - \frac{\nu^2}{x}\right) y = 0$ (S-L form).
2. The Legendre equation $(1 - x^2)y'' - 2xy' + \nu(\nu + 1)y = 0$,
$\Leftrightarrow \left((1 - x^2)y'\right)' + \nu(\nu + 1)y = 0$ (S-L form).

S-L eigenvalue problem:

- Set $\mathcal{L}y := \frac{1}{\omega}\{-(py')' + qy\}$.
- S-L problem can be expressed as

$$\begin{cases} \mathcal{L}y = \lambda y \\ R_1(y) = 0, \quad p,\ \frac{1}{p},\ q,\ \omega \in L^1[a,b] \\ R_2(y) = 0. \quad p,\ \omega > 0 \end{cases} \tag{1.1}$$

- S-L problem corresponds to the formally *self-adjoint differential operator* $\mathcal{L}$ on the Hilbert space $L^2_\omega[a,b] := (L^2[a,b], \omega(x)dx)$.
- λ: an *eigenvalue* of S-L problem, if (1.1) has non-trivial solutions.
- y: the corresponding *eigenfunction*.

• So, those λ are real, and the y to different eigenvalues are orthogonal.

• However, this operator is unbounded, and hence existence of an orthonormal basis of eigenfunctions is not evident.

• Set

$$u = \sqrt{\omega}y, \quad K(x;\xi) = -G(x;\xi)\sqrt{\omega(\xi)\omega(\xi)},$$

where G is the Green's function.

• K is continuous symmetric integral kernel.

• To overcome this problem above, define a *compact integral operator*

$$\mathcal{T}u := \int_a^b K(x;\xi)u(\xi)\, d\xi.$$

• we can get an equivalent relation:

$$\mathcal{T}u = \frac{1}{\lambda}u \iff \mathcal{L}y = \lambda y.$$

Recall S-L problem (1.1):

$$\begin{cases} -(py')' + qy = \lambda\omega y \\ R_1(y) = 0, & p,\ \frac{1}{p},\ q,\ \omega \in L^1[a,b] \\ R_2(y) = 0. & p,\ \omega > 0 \end{cases}$$

Theorem (Theorem of Eigenvalues)

- *S-L problem (1.1) has countable real eigenvalues, which can be ordered such that*

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots \to \infty.$$

- *There exists* $c,\ C > 0$*, such that, when* n *is big enough,*

$$cn^2 \leqslant \lambda_n \leqslant Cn^2.$$

Theorem (Theorem of Eigenfunctions)

- *The eigenfunction $\varphi_n(x)$ corresponding to λ_n is a unique (up to a normalization constant).*
- *$\varphi_n(x)$ is called the "n-th fundamental solution".*
- *φ_n has exactly $n-1$ zeros in (a, b).*
- *The normalized $\{\varphi_n\}_{n=1}^{\infty}$ form an orthonormal basis*

$$\langle \varphi_n \ , \ \varphi_m \rangle_\omega := \int_a^b \varphi_n(x)\varphi_m(x)\omega(x)\, dx = \delta_{mn},$$

in the Hilbert space $L^2_\omega[a, b]$. Here δ_{mn} is the Kronecker delta.

Asymptotic formula:

- In the case that $p = \omega = 1$:

$$\sqrt{\lambda_n} = n\pi + \frac{\alpha}{n\pi} + \frac{K_n}{n\pi}, \quad \varphi_n(x;\lambda) = \cos n\pi x + \frac{\xi_n(x)}{n},$$

where α is a constant that depends on boundary condition and q, $K_n \to 0$ as $n \to +\infty$, and ξ_n is a function constructed by q, α, K_n and trianglefunctions.

- In the case that $p = 1$ and ω is absolutely continuous:

$$\sqrt{\lambda_n} = \frac{n\pi}{\int_a^b \sqrt{\omega(x)}\,dx} + O(1).$$

- In the general case: $\lambda_n \sim \dfrac{n^2\pi^2}{\left(\int_a^b \sqrt{\frac{\omega(x)}{p(x)}}\,dx\right)^2}$. *Just an equivalence!*

Sturm-Liouville equation and its spectral theory:

• the existence and asymptotic behavior of the eigenvalues

• continuity of eigenvalues and eigenfunctions with respect to q, ω, parameters and initial values.

• qualitative theory of the eigenfunctions and their completeness in a suitable function space

• spectral theory (point spectrum, essential spectrum, continuous spectrum)

• inverse spectral theory

To making the proof simpler, using Liouville transform, we can let $p = 1$.

More precisely, we may turn to study the following form:

$$-y(x)' + qy(x) = \lambda\omega y(x),$$
$$y(0) = 0,\ y(1) = 0,$$

where $q,\ \omega \in L^1[0,1]$ and $x \in [0,1]$.

Definition:

The eigenvalue sequence $\{\lambda_n(q)\}$ is s. t. b. *uniform continuous with respect to q*, if it satisfies that

for any bounded subset $\Omega \subset L^1[a,b]$, $\exists C(\Omega) > 0$, such that,

$$|\lambda_n(q+\Delta q) - \lambda_n(q)| \leqslant C(\Omega) \, \|\Delta q\|_{L^1} \, ,$$

for any $q \in \Omega$, $\Delta q \in L^1[a,b]$ and for all $n \geqslant 1$.

Analysis:

Setting $q_t(x) = q(x) + \cdot \Delta q(x)$ as well as $\tilde{\lambda}(t) = \lambda_n(q_t)$. we obtain that

$$\begin{aligned}
\lambda(q+\Delta q) - \lambda(q) &= \tilde{\lambda}_n(1) - \tilde{\lambda}_n(0) = \int_0^1 \frac{dq_n(t)}{dt}\, dt \\
&= \int_0^1 \frac{\partial \lambda_n(q_t)}{\partial q_t} \cdot \frac{d(q_t)}{dt}\, dt \\
&= \int_0^1 \frac{\partial \lambda_n(q_t)}{\partial q_t}|_{q=q_t} \cdot \Delta q\, dt,
\end{aligned}$$

where $\frac{\partial \lambda_n(q_t)}{\partial q_t}$ is its Frechet partial derivative of $\lambda_n(q)$ with respect to q at $q = q_t$.

Let $\varphi_n(x;t)$ be the unique eigenfunction of $\tilde{\lambda}_n(t)$ satisfying

$$\int_0^1 \omega(x)\varphi_n^2(x;t)\,dx = 1.$$

A well known result:

$$\frac{\partial \lambda_n(q_t)}{\partial q_t} = \varphi_n^2(x;t) \text{ as a linear functional on } L^1[0,1].$$

So, we deduce that

$$\begin{aligned} |\lambda_n(q+\Delta q) - \lambda_n(q)| &= \int_0^1 \int_0^1 \varphi_n^2(x;\tilde{\lambda}_n(t))\Delta q(t)\,dxdt \\ &\leqslant \int_0^1 \int_0^1 \varphi_n^2(x;\tilde{\lambda}_n(t))\,|\Delta q(t)|\,dxdt. \end{aligned}$$

So, we need to prove that

For any bounded subset $\Omega \subset L^1[a,b]$, $\exists M(\Omega) > 0$, such that,

$$\left|\varphi_n(x;\tilde{\lambda}_n(t))\right| \leqslant M(\Omega), \quad \forall n \geqslant 1.$$

Let $\varphi_n(x;\lambda)$ be the unique eigenfunction of λ_n satisfying $\|\varphi_n\|_{L^1_\omega} = 1$.

Aim:

For any bounded subset $\Omega \subset L^1[a,b]$, $\exists M(\Omega) > 0$, such that,

$$|\varphi_n(x;\lambda)| \leqslant M(\Omega), \quad \forall n \geqslant 1.$$

for any $q \in \Omega$ and all λ.

In fact, we only need to consider a simple form:

$$-y(x)'' = \lambda\omega y(x),$$
$$y(0) = \alpha,\ y'(0) = \beta,$$

where $x \in [0,1]$, $\alpha,\ \beta \in \mathbb{R}$ and $\omega \in L^1[0,1]$.

So, original problem is transformed into the following problem:

For any solution $\varphi(x;\lambda)$ of (15) with $\int_0^1 \omega\,|\varphi(x;\lambda)|^2\,dx = 1$,
? $\exists$ uniform bound of $\varphi(x;\lambda)$?

Proposition

Suppose that

(1) $\omega \in L^1[0,1] \cap L^\infty[0,1]$;

(2) ω *is increasing;*

(3) $\omega(0) > 0$.

Then $\exists M > 0$, *such that for any solution* $\varphi(x;\lambda)$ *of the initial value problem above with* $\int_0^1 \omega \left|\varphi(x;\lambda)\right|^2 \, dx = 1$, *it holds that*

$$|\varphi(x;\lambda)| \leqslant M$$

for all $x \in [0,1]$ *and all* $\lambda \geqslant 0$.

Outline of the proof:

Using the polar coordinate

$$\sqrt{\lambda}y = \rho \sin\theta,\ y' = \rho\cos\theta$$

on the initial value problem above, we obtain the Prüfer system

$$\begin{aligned}\theta'(x;\lambda) &= \sqrt{\lambda}(\cos^2\theta(x;\lambda) + \omega(x)\sin^2\theta(x;\lambda)),\\ \rho'(x;\lambda) &= \frac{\sqrt{\lambda}}{2}\rho(x;\lambda)(1-\omega(x))\sin(2\theta(x;\lambda)).\end{aligned}$$

Consequently, the solution $y(x;\lambda)$ has the following expression:

$$\rho(x;\lambda) = \rho(0)\cdot e^{\frac{\sqrt{\lambda}}{2}\int_0^1 (1-\omega)\sin 2\theta(x;\lambda)\,dx},\ y(x;\lambda) = \rho(x;\lambda)\sin\theta(x;\lambda).$$

Set

$$I_h(x;\lambda) = \int_0^x h(t)\sin 2\theta(t;\lambda)\,dt,$$
$$I_w(x;\lambda) = \int_0^x \omega(t)\cos 2\theta(t;\lambda)\,dt,$$

where $h(t) = 1 - \omega(t)$ and is decreasing.

Suppose that

$$I_h(x;\lambda) = O(\frac{1}{\sqrt{\lambda}}); \qquad \lim_{\lambda\to+\infty} I_w(x;\lambda) = 0,$$

for any $x \in [0,1]$ and all $\lambda \geqslant 0$.

On the one hand, since $\int_0^x h(t)\sin 2\theta(t;\lambda)\,dt = O(\frac{1}{\sqrt{\lambda}})$, we have

$$H(x;\lambda) = \frac{\sqrt{\lambda}}{2}\int_0^x h(t)\sin 2\theta(t;\lambda)\,dt = O(1),$$

for any $x \in [0,1]$ and all $\lambda \geqslant 0$.

So, we can find a $H_0 > 0$ such that

$$|H(x;\lambda)| \leqslant \frac{H_0}{2} < +\infty,$$

for any $x \in [0,1]$ and all $\lambda \geqslant 0$.
Then

$$|y(x;\lambda)| \leqslant \rho(0)e^{\frac{H_0}{2}}.$$

On the other hand, let $\varphi(x;\lambda)$ be a solution corresponding to λ satisfying

$$\int_0^1 \omega(x)\left|\varphi(x;\lambda)\right|^2\,dx = 1.$$

For given $y(x;\lambda)$ and λ above, we can find a real number $\beta(\lambda)$ such that $\varphi(x;\lambda) = \beta(\lambda)y(x;\lambda)$.

Hence,

$$\alpha^2(\lambda)\int_0^1 \omega(x)e^{H(x;\lambda)}\sin^2\theta(x;\lambda)\,dx = 1,$$

where $\alpha(\lambda) = \beta(\lambda)\rho(0)$.

Since $e^{H(x;\lambda)} \geqslant e^{-\frac{H_0}{2}}$, we have

$$\alpha^2(\lambda) \int_0^1 \omega(x) \sin^2 \theta(x;\lambda)\, dx \leqslant e^{\frac{H_0}{2}},$$

that is,

$$\alpha^2(\lambda) \int_0^1 \omega(x) \frac{1 - \cos 2\theta(x;\lambda)}{2}\, dx \leqslant e^{\frac{H_0}{2}}.$$

$$\because \quad \lim_{\lambda \to +\infty} \int_0^1 \omega(x) \cos 2\theta(x;\lambda)\, dx = 0.$$

$\therefore \forall\, \gamma \in (0,1)$, $\exists \lambda_\gamma > 0$, such that, for any $\lambda > \lambda_\gamma$,

$$\int_0^1 \omega(x) \cos 2\theta(x;\lambda)\, dx < \gamma \int_0^1 \omega,$$

$$\therefore \quad |\alpha(\lambda)| < \sqrt{\frac{2e^{\frac{H_0}{2}}}{(1-\gamma)\,\|\omega\|_{L^1}}}, \text{ for any } \lambda > \lambda_\gamma.$$

Set

$$M := \max\{e^{\frac{3H_0}{4}}\sqrt{\frac{2}{(1-\gamma)\,\|\omega\|_{L^1}}},\ sup\{\|\varphi(x;\lambda)\|_\infty \mid \lambda \leqslant \lambda_0\}\}$$

.

Hence

$$|\varphi(x;\lambda)| = |\alpha(\lambda)y(x;\lambda)| \leqslant M,\ \text{for all } \lambda \geqslant 0,$$

Recall these two oscillatory integral

$$\begin{aligned} I_h(x;\lambda) &= \int_0^x h(t)\sin 2\theta(t;\lambda)\,dt, \\ I_w(x;\lambda) &= \int_0^x \omega(t)\cos 2\theta(t;\lambda)\,dt, \end{aligned}$$

where $h(t) = 1 - \omega(t)$ and is decreasing.

The key point is to show that

$$I_h(x;\lambda) = O(\frac{1}{\sqrt{\lambda}}); \quad \lim_{\lambda\to+\infty} I_w(x;\lambda) = 0,$$

for any $x \in [0,1]$ and all $\lambda \geqslant 0$.

Outline of the proof: (Only for I_h)

We only need to show that

$$\sqrt{\lambda}\int_0^x h(t)\sin 2\theta(t;\lambda)\,dt = O(1), \ \ \forall\, x\in[0,1] \text{ and } \lambda\geqslant 0..$$

Assume that $\omega(0)\leqslant\omega\leqslant 1$, and then $0\leqslant h(t)\leqslant 1-\omega(0)<1$.

Recall that

$$\theta'(x;\lambda)=\sqrt{\lambda}(\cos^2\theta(x;\lambda)+\omega(x)\sin^2\theta(x;\lambda)),$$

which implies $\theta(1;\lambda)\to+\infty$, as $\lambda\to+\infty$.

Fix an arbitrary $x \in (0, 1]$.

For any λ, we can find two finite sequences

$$\{x_i\}_{i=0}^{m}, \ \{s_i\}_{i=0}^{m} \subseteq [0, x],$$

such that

$$0 = x_0 < s_0 < x_1 < s_1 < \cdots < x_m \leqslant x,$$

and

$$\theta(x_i; \lambda) = i\pi, \quad 0 \leqslant i \leqslant m;$$

$$\theta(s_{j-1}; \lambda) = (j-1)\pi + \frac{\pi}{2}, \quad 1 \leqslant j \leqslant m;$$

$$m\pi \leqslant \theta(\bar{x}; \lambda) \leqslant (m+1)\pi.$$

For any integer $j \in \{1, 2, ..., m\}$, since h is decreasing, we know that
for all $x \in [x_{j-1}, s_{j-1}]$,

$$h(s_{j-1}) \leqslant h(x) \leqslant h(x_{j-1}),$$

$$\omega(x_{j-1}) \leqslant \omega(x) \leqslant \omega(s_{j-1}),$$

$$\sin 2\theta(x; \lambda) \geqslant 0;$$

for all $x \in [s_{j-1}, x_j]$,

$$h(x_j) \leqslant h(x) \leqslant h(s_{j-1}),$$

$$\omega(s_{j-1}) \leqslant \omega(x) \leqslant \omega(x_j),$$

$$\sin 2\theta(x; \lambda) \leqslant 0.$$

$$\sqrt{\lambda}\int_0^x h(x)\sin 2\theta(x)\,dx$$

can be decomposed as follows:

$$\sqrt{\lambda}\left(\sum_{j=1}^{m}\left(\int_{x_{j-1}}^{s_{j-1}}+\int_{s_{j-1}}^{x_j}\right)+\int_{x_m}^{x}\right)h(x)\sin 2\theta(x)\,dx$$

that is,

$$\sqrt{\lambda}\left(\sum_{j=1}^{m}\left(\int_{x_{j-1}}^{s_{j-1}}+\int_{s_{j-1}}^{x_j}\right)+\int_{x_m}^{x}\right)\frac{h(x)\sin 2\theta(x)}{\cos^2\theta(x)+\omega(x)\sin^2\theta(x)}\,d\theta(x).$$

Define a function as follows:

$$f(t) = \int_0^{\frac{\pi}{2}} \frac{h(t)\sin 2u}{\cos^2 u + \omega(t)\sin^2 u}\, du, \ \forall t \in [0,1].$$

Then, substituting θ for u, by the periodicity of $\sin 2u$, we have that

$$\int_{x_{j-1}}^{s_{j-1}} \frac{h(t)\sin 2\theta}{\cos^2\theta + \omega(t)\sin^2\theta}\, d\theta(x) = f(t),$$

and

$$\int_{s_{j-1}}^{x_j} \frac{h(t)\sin 2\theta}{\cos^2\theta + \omega(t)\sin^2\theta}\, d\theta(x) = -f(t).$$

Moreover, $f(t) \geqslant 0$ and $f(t)$ is bounded on $[0,1]$.

For $[x_{j-1}, s_{j-1}]$ $(1 \leqslant j \leqslant m)$,

$$f(s_{j-1}) \leqslant \int_{x_{j-1}}^{s_{j-1}} \frac{h(x)\sin 2\theta(x)}{\cos^2\theta(x) + \omega(x)\sin^2\theta(x)}\, d\theta(x) \leqslant f(x_{j-1}).$$

For $[s_{j-1}, x_j]$,

$$-f(s_{j-1}) \leqslant \int_{s_{j-1}}^{x_j} (\cdots) \leqslant -f(x_j).$$

For $[x_m, x]$,

$$f(x_m) \geqslant \int_{x_m}^{x} (\cdots) \geqslant \begin{cases} 0, & \text{if } m\pi \leqslant \theta(x) \leqslant m\pi + \frac{\pi}{2}; \\ -f(x), & \text{if } m\pi + \frac{\pi}{2} < \theta(x) \leqslant (m+1)\pi. \end{cases}$$

Combining the three inequalities above, we have that

if $m\pi \leqslant \theta(x) \leqslant m\pi + \frac{\pi}{2}$,

$$-f(x) \leqslant \sqrt{\lambda} \int_0^x h(t) \sin 2\theta(t;\lambda)\, dt \leqslant f(x_0);$$

if $m\pi + \frac{\pi}{2} < \theta(x) \leqslant (m+1)\pi$,

$$0 \leqslant \sqrt{\lambda} \int_0^x h(t) \sin 2\theta(t;\lambda)\, dt \leqslant f(x_0).$$

Hence, $\sqrt{\lambda} \int_0^x h(t) \sin 2\theta(t;\lambda)\, dt$ is bounded by $\|f\|_\infty$.
And then,

$$\int_0^x h(t) \sin 2\theta(t;\lambda)\, dt = O(\frac{1}{\sqrt{\lambda}}),\ \forall x \in [0,1],\ \lambda \geqslant 0.$$

(a) The above results also can hold for the case that ω is decreasing.

(b) The upper bound $C(\Omega)$ is essentially controlled by H_0. As H_0 becomes bigger or smaller, the bound $C(\Omega)$ also will become bigger or smaller accordingly.

(c) Observe that

$$0 \leqslant f(t) \leqslant \int_0^{\frac{\pi}{2}} \frac{\sin 2u}{\cos^2 u + \omega(0)\sin^2 u}\, du < +\infty,$$

Note that H_0 equals to $+\infty$ when $\omega(0) = 0$.

So, it is necessary that $\omega(0)$ is away from 0.

For (c), we can take a special weight function as a concrete example.

Example

Suppose that $\omega \in L^1[0,1]$ *satisfying the following conditions:*

(1) ω *is of bounded variation;*

(2) ω *is right continuous;*

(3) $\omega(0) = \epsilon$ *and* $\omega(1) = \frac{1}{\epsilon}$, *where* $\epsilon \in (0,1)$.

Then

$$\left| \int_0^x h(t) \sin 2\theta(t;\lambda)\, dt \right| \leqslant \frac{1}{\sqrt{\lambda}} \left(\frac{1}{\epsilon^2} + \frac{\|\omega'\|_{L^1} - 2}{2\epsilon} \right) := \frac{1}{\sqrt{\lambda}} U(\epsilon),$$

for all $x \in [0,1],\ \lambda \geqslant 0$.

(1) We can know that

$$-\frac{U(\epsilon)}{2} \leqslant H(x;\lambda) \leqslant \frac{U(\epsilon)}{2},$$

And then we also can figure out the corresponding $C(\Omega)$, and thereby obtain the uniform continuity of eigenvalue sequence $\{\lambda_n(q)\}$.

(2) It is obvious that $U(\epsilon)$ tend to $+\infty$, as $\epsilon \to 0$. Consequently, if we let $\epsilon \to 0$, which implies the rang of the weight function ω approximates $[0, +\infty)$, then both of H_0 and $C(\Omega)$ tend to $+\infty$ accordingly.

(3) We also can find that, the integrability of the derivative ω' of ω, is a sufficient condition for the boundedness of $H(x;\lambda)$.

Further,

(1) Counterexample?

(2) Asymptotic formula of eigenvalues ?

(3) Inverse spectrum theory.

Thank you for your attention!

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