Hypergroup deformations of semigroups and Ramsey hypergroups

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AHA-2018, Kaohsiung

June 28, 2018

Let K be a discrete space.

- $M(K)$= The space of complex-valued regular Borel measures on K.
- Denote the subset of $M(K)$ consisting of measures with finite support and probability measures by $M_F(K)$ and $M_p(K)$ respectively.
- Set $M_{F,p}(K) := M_F(K) \cap M_p(K)$.
- We begin with a map $* : K \times K \to M_{F,p}(K)$ given by $(\delta_m, \delta_n) \mapsto \delta_m * \delta_n$. Extend '$*$' to a bilinear map called *convolution*, denoted by '$*$' again, from $M(K) \times M(K)$ to $M(K)$.
- A bijective map $\vee : m \mapsto \check{m}$ from K to K is called an *involution* if $\check{\check{m}} = m$. We can extend it to $M(K)$ in a natural way.

Discrete semiconvos [Dunkl, 73 / Jewett, 75/ Specter, 75]

A pair $(K,*)$ is called a *discrete semiconvo* if the following conditions hold.

- The map $*: K \times K \to M_{F,p}(K)$ satisfies the associativity condition

$$(\delta_m * \delta_n) * \delta_k = \delta_m * (\delta_n * \delta_k) \text{ for all } m,n,k \in K.$$

- There exists (necessarily unique) element $e \in K$ such that

$$\delta_m * \delta_e = \delta_e * \delta_m = \delta_m \text{ for all } m \in K.$$

- A discrete semiconvo $(K,*)$ is called *commutative* if $\delta_m * \delta_n = \delta_n * \delta_m$ for all $m,n \in K$.

Discrete hypergroups

A triplet $(K, *, \vee)$ is called a *discrete hypergroup* if

(i) $(K, *)$ is a discrete semiconvo,

(ii) $\vee$ is an involution on K that satisfies

   (a) $(\delta_m * \delta_n)\check{} = \delta_{\check{n}} * \delta_{\check{m}}$ for all $m, n \in K$ and

   (b) $e \in \mathrm{spt}(\delta_m * \delta_{\check{n}})$ if and only if $m = n$.

- A discrete hypergroup $(K, *, \vee)$ is called *hermitian* if the involution on K is the identity map, i.e., $\check{m} = m$ for all $m \in K$.
- Hermitian $\Rightarrow$ Commutative.
- [Jewett] The Haar measure λ on a discrete hermitian hypergroup K is given by: $\lambda(e) = 1$ and for $e \neq n \in K$, $\lambda(n) = \frac{1}{(\delta_n * \delta_n)(e)}$.

- Let K be a commutative discrete hypergroup. For a complex-valued function χ defined on K, we write $\check{\chi}(m) := \overline{\chi(\check{m})}$ and

$$\chi(m*n) = \int_K \chi \, d(\delta_m * \delta_n) \ \text{ for } m,n \in K.$$

Dual objects

Define two dual objects of K :

$$\mathscr{X}_b(K) = \{\chi \in \ell^\infty(K) : \chi \neq 0, \chi(m*n) = \chi(m)\chi(n) \text{ for all } m,n \in K\},$$

$$\widehat{K} = \left\{\chi \in \mathscr{X}_b(K) : \check{\chi} = \chi, \text{ i.e., } \chi(\check{m}) = \overline{\chi(m)} \text{ for all } m \in K\right\}.$$

Each $\chi \in \mathscr{X}_b(K)$ is called a *character* and each $\chi \in \widehat{K}$ is called a *symmetric character*.

Examples of discrete hypergroups

- Every discrete group is a discrete hypergroup with the convolution '$*$' given by $\delta_m * \delta_n = \delta_{mn}$.
- **Polynomial hypergroups** [Lasser, 83/ Bloom and Heyer, 94] This is a wide and important class of hermitian discrete hypergroups in which hypergroup structures are defined on $\mathbb{Z}_+$. This class contains Chebyshev polynomial hypergroups of first kind (CP1), Chebyshev polynomial hypergroups of second kind (CP2), Gegenbauer hypergroups, Hermite hypergroups, Jacobi hypergroups, Laguerre hypergroups etc.

Theorem [Lasser, 83]

Let $\mathbf{P} = (P_n)_{n\in\mathbb{Z}_+}$ be an orthogonal polynomial system such that the linearization coefficients $g(n,m;k)$ occurring in

$$P_n(x)P_m(x) = \sum_{k=|n-m|}^{n+m} g(n,m;k)P_k(x)$$

satisfy

$$g(n,m;k) \geq 0, \;\; n,m \in \mathbb{Z}_+, |n-m| \leq k \leq n+m.$$

Let $* : \mathbb{Z}_+ \times \mathbb{Z}_+ \to M_{F,p}(\mathbb{Z}_+)$ be given by

$$\delta_n * \delta_m = \sum_{k=|n-m|}^{n+m} g(n,m;k)\delta_k.$$

Then $K_{\mathbf{P}} = (\mathbb{Z}_+.*)$ is a hermitian discrete hypergroup, called *polynomial hypergroup* (related to $\mathbf{P} = (P_n)_{n\in\mathbb{Z}_+}$).

CP1 and CP2 hypergroups

- The condition $g(n.m;k) \geq 0$ is satisfied in all known cases. Also, it is known that $g(n,m;|n-m|) > 0$ and $g(n,m;n+m) > 0$, [Lasser, 83].
- Chebyshev polynomials of first kind define the following convolution '$*$' on $\mathbb{Z}_+$:

$$\delta_m * \delta_n = \frac{1}{2}\delta_{m+n} + \frac{1}{2}\delta_{|m-n|} \text{ for } m,n \in \mathbb{Z}_+.$$

- The *Chebyshev polynomial hypergroup of second kind* $(\mathbb{Z}_+, *)$ arises from the Chebyshev polynomials of second kind and the convolution '$*$' on $\mathbb{Z}_+$ is given by

$$\delta_m * \delta_n = \sum_{k=0}^{\min\{m,n\}} \frac{|m-n|+2k+1}{(m+1)(n+1)} \delta_{|m-n|+2k}.$$

Discrete orbit semiconvos

Theorem [Jewett, 75]

Let G be a discrete group and let H be a finite group with $\#H = c$. Suppose that $(x,s) \mapsto x^s$ is an affine action of H on G. Then the space G^H of orbits x^H given by $x^H = \{x^s : s \in H\}$ equipped with the discrete topology is a semiconvo with respect to the convolution '$*$' defined by

$$\delta_{x^H} * \delta_{y^H} = \frac{1}{c^2} \sum_{s,t \in H} \delta_{(x^s y^t)^H}.$$

Discrete automorphisms orbit hypergroups

Theorem [Jewett, 75]

Let G be a discrete group and let H be a finite subgroup of the group of automorphisms of G with $\#H = c$. Suppose that $(x, s) \mapsto x^s$ is the corresponding action of H on G. Then the space G^H of orbits x^H given by $x^H = \{x^s : s \in H\}$ equipped with the discrete topology is a discrete hypergroup with respect to the convolution '$*$' defined by

$$\delta_{x^H} * \delta_{y^H} = \frac{1}{c} \sum_{s \in H} \delta_{(x^s y)^H}$$

with the identity $e^H = \mathrm{id}_G$ and the involution $(x^H)\check{} = (x^{-1})^H$.

Dunkl-Ramirez example

- [Dunkl and Ramirez, 75 (TAMS)] The one-point compactification $H_a = \{0, 1, 2, \ldots, \infty\}, 0 < a \leq \frac{1}{2}$ of $\mathbb{Z}_+$, can be made it a (hermitian) countable compact hypergroup given by

$$\delta_m * \delta_n := \begin{cases} \delta_{\min(m,n)} & m \neq n \\ \frac{2a-1}{a-1}\delta_n + \sum_{k=n+1}^{\infty} a^{k-n}\delta_k & m = n, \end{cases}$$

  with $\delta_m * \delta_\infty = \delta_\infty * \delta_m := \delta_m$.

- The symmetric dual space $\widehat{H_a}$ of H_a is a hermitian discrete hypergroup.

- The members of $\widehat{H_a}$ are given by $\{\chi_n : n \in \mathbb{Z}_+\}$, where, for $k \in H_a$,

$$\chi_n(k) = \begin{cases} 0 & \text{if } k < n-1, \\ \frac{a}{a-1} & \text{if } k = n-1, \\ 1 & \text{if } k \geq n \ (\text{or } k = \infty). \end{cases}$$

- The convolution '$*$' on $K = \mathbb{Z}_+$ identified with $\widehat{H_a} = \{\chi_n : n \in \mathbb{Z}_+\}$ is dictated by the pointwise product of functions in $\widehat{H_a}$, that is:

$$\begin{aligned} \chi_m \chi_n &= \chi_{\max\{m,n\}} \text{ for } m \neq n, \\ \chi_0^2 &= \chi_0, \ \ \chi_1^2 = \frac{a}{1-a}\chi_0 + \frac{1-2a}{1-a}\chi_1, \\ \chi_n^2 &= \frac{a^n}{1-a}\chi_0 + \sum_{k=1}^{n-1} a^{n-k}\chi_k + \frac{1-2a}{1-a}\chi_n \ \text{ for } n \geq 2. \end{aligned}$$

We call $(K, *)$ a *(discrete) Dunkl-Ramirez hypergroup*.

Some observations about D-R example

If λ is the haar measure on $K = \widehat{H_a}\,(0 < a \leq \frac{1}{2})$ then

- $\lambda(n) = \frac{1-a}{a^n}$ for all $n \in \mathbb{N}$,
- For $n \in \mathbb{N}$, we have

$$\begin{aligned}
\delta_n * \delta_n(0) &= \frac{a^n}{1-a} = \frac{1}{\lambda(n)}, \\
\delta_n * \delta_n(k) &= a^{n-k} = \frac{\lambda(k)}{\lambda(n)} \text{ for } 1 \leq k < n \text{ and} \\
\delta_n * \delta_n(n) &= \frac{1-2a}{1-a} = 1 - \sum_{0 \leq k < n} (\delta_n * \delta_n)(k) = \frac{\lambda(n) - \lambda(\mathscr{L}_n)}{\lambda(n)}.
\end{aligned}$$

- $(\delta_n * \delta_n)(n) = 0$ for some $n \in \mathbb{N}$ if and only if $a = \frac{1}{2}$ if and only if $(\delta_n * \delta_n)(n) = 0$ for all $n \in \mathbb{N}$. In this case, $\lambda(n) = 2^{n-1}$ for $n \in \mathbb{N}$.

Motivation

We note that a Dunkl-Ramirez hypergroup is a hermitian (hence commutative) discrete hypergroup $K = \widehat{H_a}$ $(0 < a \leq \frac{1}{2})$ and its convolution '$*$' arises as a hypergroup deformation of the semigroup $(\mathbb{Z}_+, <, \cdot)$, where $m \cdot n = \max\{m, n\}$ in the sense that $\delta_m * \delta_n = \delta_{mn}$ for $m \neq n$ or, $m = n = 0$ and for $m = n \neq 0$, we have

$$\begin{aligned} \delta_1 * \delta_1 &= \frac{a}{1-a}\delta_0 + \frac{1-2a}{1-a}\delta_1, \\ \delta_n * \delta_n &= \frac{a^n}{1-a}\delta_0 + \sum_{k=1}^{n-1} a^{n-k}\delta_k + \frac{1-2a}{1-a}\delta_n \text{ for } n \geq 2. \end{aligned}$$

Also, each element of $\mathbb{Z}_+$ is an idempotent in $(\mathbb{Z}_+, \cdot)$.

First step towards abstraction

- We try to replace $(\mathbb{Z}_+, <, \cdot)$ by a discrete “max" semigroup $(S, <, \max)$ and try to deform this discrete semigroup $(S, <, \cdot)$ into a hermitian (hence commutative) discrete hypergroup $(S, *)$ by deforming the product on the diagonal of $S \backslash \{e\}$.
- The convolution product ‘$*$’ defined as follows:

$$\begin{aligned} \delta_m * \delta_n = \delta_n * \delta_m &= \delta_{m \cdot n} (= \delta_{\max\{m,n\}}) \text{ for } m, n \in S \text{ with } m \neq n, \text{ or}, m = n = e, \\ \delta_n * \delta_n &= q_n \quad \text{for } n \in S \backslash \{e\}. \end{aligned}$$

Here, q_n is a probability measure on S with finite support Q_n containing e and has the form $\sum_{j \in Q_n} q_n(j) \delta_j$ with $q_n(j) > 0$ for $j \in Q_n$ and $\sum_{j \in Q_n} q_n(j) = 1$.

Question

Under what condition(s), will $(S,*)$ become a discrete hermitian hypergroup?

Theorem 1 [Kumar, Ross and Singh, preprint]

Let $(S, <, \cdot)$ be a discrete (commutative) "max" semigroup with identity e and ' $*$' and other related symbols as above. Then $(S, *)$ is a hermitian discrete hypergroup if and only if the following conditions hold.

(i) Either S is finite or $(S, <, \cdot)$ is isomorphic to $(\mathbb{Z}_+, <, \max)$.

(ii) For $n \in S \setminus \{e\}$, we have $\mathscr{L}_n \subset Q_n \subset \mathscr{L}_n \cup \{n\}$.

(iii) If $\#S > 2$, then for $e \neq m < n$ in S, we have

- (a) $q_n(e) = q_n(m) q_m(e)$ and
- (b) $q_n(e) \left(1 + \sum_{e \neq k \in \mathscr{L}_n} \frac{1}{q_k(e)}\right) \leq 1$;

or, equivalently, with $v_n = \frac{1}{q_n(e)}$ for $n \in S$,

(iii)' If $\#S > 2$, then for $e \neq m < n$ in S, we have

- (a) $q_n(m) = \frac{v_m}{v_n}$ and
- (b) $\sum_{k \in \mathscr{L}_n} v_k \leq v_n$.

Let $\mathscr{V}$ be the set of sequences $v = (v_j)_{j \in \mathbb{Z}_+}$ in $[1, \infty)$ which satisfy (i) $v_0 = 1$ and (ii) $v_n \geq \sum_{j \in \mathscr{L}_n} v_j$ for $n \in \mathbb{N}$.

For $n \in \mathbb{N}$, let $u_n = v_n - \sum_{j \in \mathscr{L}_n} v_j$. Then simple calculations give the following:

$$\begin{aligned} v_0 &= 1, \\ v_1 &= 1 + u_1, \\ v_2 &= (2 + u_1) + u_2, \\ v_3 &= (2^2 + 2u_1 + u_2) + u_3, \\ \vdots &= \vdots, \end{aligned}$$

to elaborate,

$$v_n = (2^{n-1} + 2^{n-2}u_1 + \ldots + u_{n-1}) + u_n \quad \text{for } n \geq 3.$$

Corollary [Kumar, Ross and Singh, preprint]

For each $v \in \mathscr{V}$ (or the corresponding $u \in \mathscr{U}$), there is one and only one (hermitian) hypergroup deformation $(\mathbb{Z}_+, *)$ of $(\mathbb{Z}_+, <, \max)$ which satisfies $(\delta_n * \delta_n)(0) = \frac{1}{v_n}$, $n \in \mathbb{Z}_+$. Further, for this deformation, the convolution '$*$' and the Haar measure λ satisfy the following conditions.

(i) $\lambda(n) = v_n$ for $n \in \mathbb{Z}_+$ and $\lambda(n) - \lambda(\mathscr{L}_n) = u_n$ for $n \in \mathbb{N}$.

(ii) $\lambda(\mathscr{L}_n) \leq \lambda(n)$ for $n \in \mathbb{N}$.

(iii) For $n \in \mathbb{N}$,

   (a) $\delta_n * \delta_n(m) = \frac{\lambda(m)}{\lambda(n)}$ for $m < n$,

   (b) $\delta_n * \delta_n(n) = \frac{\lambda(n) - \lambda(\mathscr{L}_n)}{\lambda(n)}$,

   (c) $\delta_n * \delta_n(m) = 0$ for $m > n$.

(iv) For $n \in \mathbb{N}$, $\operatorname{spt}(\delta_n * \delta_n) = \begin{cases} \mathscr{L}_n & \text{if } \lambda(n) = \lambda(\mathscr{L}_n), \\ \mathscr{L}_n \cup \{n\} & \text{if } \lambda(n) > \lambda(\mathscr{L}_n). \end{cases}$

Theorem 2 [Kumar, Ross and Singh, preprint]

Let $(S,<,\cdot) \cong (\mathbb{Z}_+,<,\max)$ and other symbols satisfy the conditions (ii)-(iii) of Theorem 1 and let $(S,*)$ be the corresponding deformed hypergroup with the Haar measure λ. Then the dual objects $\mathscr{X}_b(S)$ and $\widehat{S}$ of $(S,*)$ are equal. Equipped with the topology of uniform convergence on compact subsets of S, $\widehat{S}$ can be identified with the one point compactification $\mathbb{Z}_+^* (= \mathbb{Z}_+ \cup \{\infty\})$ of $\mathbb{Z}_+$. More precisely, the identification is given by $k \mapsto \chi_k$, where $\chi_\infty(n) = 1$ for all $n \in \mathbb{Z}_+$, and, for $k \in \mathbb{Z}_+$, χ_k is given by

$$\chi_k(n) = \begin{cases} 1 & \text{if } n \leq k, \\ \beta_k & \text{if } n = k+1, \\ 0 & \text{if } n > k+1, \end{cases}$$

where,

$$\beta_k = \frac{-\lambda(\mathscr{L}_{k+1})}{\lambda(k+1)} = \frac{-\sum_{j \in \mathscr{L}_{k+1}} v_j}{v_{k+1}} = \frac{u_{k+1}}{v_{k+1}} - 1 = \delta_{k+1} * \delta_{k+1}(k+1) - 1 = q_{k+1}(k+1) - 1.$$

Theorem 3 [Kumar, Ross and Singh, preprint]

Let $(S,<,\cdot) \cong (\mathbb{Z}_+,<,\max)$ and other symbols satisfy the conditions (ii)-(iii) of Theorem 1 and let $(S,*)$ be the corresponding deformed hypergroup with the Haar measure λ. The dual space $\widehat{S}$ of $(S,*)$ becomes a countable compact hermitian hypergroup with respect to pointwise multiplication.

More precisely, the convolution '$*$' on $\widehat{S}$ is given by

$$\delta\chi_{_m} * \delta\chi_{_n} = \begin{cases} \delta\chi_{_{\min\{m,n\}}} & \text{for } m,n \in \mathbb{Z}_+ \text{ with } m \neq n \text{ or } m = n = \infty, \\ \sum_{j\in\mathbb{Z}_+} \gamma_j^m \delta\chi_{_j} & \text{otherwise,} \end{cases}$$

where, $\gamma_j^m = 0$ for $j < m$, $\gamma_m^m = 1 + \beta_m \geq 0,$ and for $p \geq 1$, $\gamma_{m+p}^m = \prod_{j=m}^{m+p-1} \frac{-\beta_j}{1-\beta_{j+1}} > 0,$ and, β_j's are as in the previous theorem. Further, we also have $\mathscr{X}_b(\widehat{S}) = \widehat{\widehat{S}} \cong (S,*)$.

Complete characterization

Main theorem [Kumar, Ross and Singh, preprint]

Let $(S, \cdot)$ be a commutative discrete semigroup with identity e such that S is action-free. Let '$*$' and other related notation and concepts be as above. Then $(S, *)$ is a commutative discrete semiconvo if and only if the following conditions hold.

(i) $E(S)$ is finite or $E(S)$ is isomorphic to $(\mathbb{Z}_+, <, \max)$, where the order on $E(S)$ is defined by $m < n$ if $mn = n \neq m$.

(ii) $(\widetilde{S}, \cdot)$ is an ideal of $(S, \cdot)$.

(iii) $Q_n \subset E(S)$ for $n \in E_0(S)$.

Continue...

(iv) If $n \in E_0(S)$ and $m \in \widetilde{S}$ then $Q_n \cdot m = \{nm\}$.

(v) For $n \in E_0(S)$, we have $\mathscr{L}_n \subset Q_n \subset \mathscr{L}_n \cup \{n\}$, where for $n \in E(S)$, $\mathscr{L}_n := \{j \in E(S) : j < n\}$.

(vi) If $\#E(S) > 2$, then for $e \neq m < n$ in $E(S)$, we have the following:

(α) $q_n(e) = q_n(m)q_m(e)$ and

(β) $q_n(e)\left(1 + \sum_{e \neq k \in \mathscr{L}_n} \frac{1}{q_k(e)}\right) \leq 1$.

Further, under these conditions, $E(S)$ is a hermitian discrete hypergroup. Moreover, S is a hermitian discrete hypergroup if and only if $S = E(S)$.

Begining of Ramsey theory

- Ramsey theory, now a well-developed branch of combinatorics, has a long history dating back to 1892 starting with David Hilbert.
- D. Hilbert, Ueber die Irreducibilität ganzer rationaler Functionen mit ganzzahligen Coefficienten, (German) *J. Reine Angew. Math.* 110 (1892) 104-129.
- For a discrete semigroup $(S, \cdot)$, the algebra structure of Stone-Čech compactification βS of S has been utilized with a great advantage to study the Ramsey theory.
- A good account of all this can be found in the book by N. Hindman and D. Strauss, *Algebra in Stone-Čech compactification: theory and application*, 2nd edition, De Gruyter, Berlin (2012).

Celebrated Hindman's Theorem

- Given a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$,

$$\mathrm{FS}(\langle x_n \rangle_{n=1}^{\infty}) = \{ \sum_{n \in F} x_n : F \text{ is non-empty finite subset of } \mathbb{N} \}.$$

Finite Sums Theorem [Hindman, 74 (J. Combinatorial Theory)]

Let $r \in \mathbb{N}$ and let $\mathbb{N} = \bigcup_{i=1}^{r} A_i$ be a partition of $\mathbb{N}$. There exist $i \in \{1, 2, ..., r\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $\mathrm{FS}(\langle x_n \rangle_{n=1}^{\infty}) \subset A_i$.

Semigroup version of Hindman's Theorem

For a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in a semigroup S, set

- $\mathrm{FP}(\langle x_n \rangle_{n=1}^{\infty}) = \{\prod_{n \in F} x_n : F \text{ is non-empty finite subset of } \mathbb{N}\}$, if S is commutative.
- $\mathrm{FP}(\langle x_n \rangle_{n=1}^{\infty}) = \{\prod_{j=1}^{k} x_{n_j} : k \in \mathbb{N}, 1 < n_1 < n_2 < \ldots < n_k\}$, if S is non-commutative.

Finite products theorem for semigroups [Hindman and Strauss]

Let S be a semigroup, let $r \in \mathbb{N}$ and let $S = \bigcup_{i=1}^{r} A_i$ be a partition of S. There exist $i \in \{1, 2, \ldots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $\mathrm{FP}(\langle x_n \rangle_{n=1}^{\infty}) \subset A_i$.

Galvin-Glazer-Hindman Theorem

- [Golan and Tsaban, 13] A semigroup S is called *moving* if it is infinite and, for each infinite $A \subset S$ and each finite $F \subset S$, there exist $x_1, x_2, \ldots, x_k \in A$ such that $\{x_1 s, x_2 s, \ldots, x_k s\} \not\subseteq F$ for all but finitely many $s \in S$.
- It is clear that every right cancellative infinite semigroup is moving. In particular, the semigroup $(\mathbb{Z}_+, +)$ and every infinite group is moving.

Galvin-Glazer-Hindman Theorem [Golan and Tsaban,13]

Let S be a moving semigroup. For each finite colouring of S, there is a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with distinct terms such that $\mathrm{FP}(\langle x_n \rangle_{n=1}^{\infty})$ is monochromatic.

Ramsey semigroups

Ramsey semigroups or groups

(i) An infinite semigroup S is called a *Ramsey semigroup* if the conclusion of Galvin-Glazer-Hindman Theorem holds for S.

(ii) A group which is a Ramsey semigroup will be called a *Ramsey group*.

- Let $(S, <, \cdot)$ be an infinite “max" semigroup with $m \cdot n = \max\{m, n\}$. Then $(S, <, \cdot)$ is a Ramsey semigroup.
- Galvin-Glazer-Hindman Theorem can be restated as: every moving semigroup is a Ramsey semigroup. In particular, every infinite group is a Ramsey group.
- If an infinite semigroup S contains a copy of $(\mathbb{N}, +)$ then S is a Ramsey semigroup. This follows immediately from the observation that a partition of S induces a partition of $\mathbb{N}$.

GGH Theorem is not true for arbitrary semigroups

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- [Golan and Tsaban, 13] Let $k \in \mathbb{N}$, let $S_k := \{0, 1, 2, \ldots, k-1\} \cup k\mathbb{N} + 1$ be the commutative semigroup with the operation of addition modulo k. It can be easily seen that S_k is not Ramsey semigroup by assigning each $s \in S_k$ the colour $s \bmod k$.

Theorem [Golan and Tsaban, 13]

Let S be an infinite semigroup. For each finite colouring of S, there exist a sequence $\langle x_n \rangle_{n=1}^{\infty}$ with distinct terms and a finite subset F of $\mathrm{FP}(\langle x_n \rangle_{n=1}^{\infty})$ such that $\mathrm{FP}(\langle x_n \rangle_{n=1}^{\infty}) \setminus F$ is monochromatic.

Theorem [Golan and Tsaban, 13]

For each finite colouring of $\bigoplus_n \mathbb{Z}_2$, there is an infinite subgroup H of $\bigoplus_n \mathbb{Z}_2$ such that $H \setminus \{0\}$ is monochromatic.

Almost-strong Ramsey semigroups or groups

Definition

(i) An infinite semigroup S is called an *almost-strong Ramsey semigroup* if given any finite colouring of S there exists an infinite almost-monochromatic subsemigroup T of S.

(ii) An *almost-strong Ramsey group* can be defined by replacing semigroup and subsemigroup by group and subgroup respectively in (i) above.

- If T is an almost-strong Ramsey semigroup (group) then S is an almost-strong Ramsey semigroup (group). In particular, if $\bigoplus_n \mathbb{Z}_2$ is contained in a group G as subgroup of G, then G is an almost-strong Ramsey group.
- If $S \backslash T$ is finite then the converse is also true.

Examples

- [Golan and Tsaban, 13] Let $k \in \mathbb{N}$. Consider the commutative semigroup $S_k := \{0, 1, 2, \ldots, k-1\} \cup k\mathbb{N} + 1$ with the operation of addition modulo k . Then S_k is an almost -strong Ramsey semigroup.
- [Lemma (Folklore), Golan and Tsaban, 13] The semigroup $(\mathbb{N}, +)$ is not almost-strong Ramsey semigroup . It can be seen by considering the 2-colouring of $\mathbb{N}$ given by

  1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 ...

  where the length of the intervals of elements of identical colours are $1, 2, 3, \ldots$.

Ramsey theory for hypergroups

From sums to convolutions

- It seems natural that the finite sums in $\mathbb{Z}_+$ will be replaced by the supports of the convolution of finitely many unit point mass measures on $K_{\mathbf{P}}$.
- Let $\mathbf{x} = \langle x_n \rangle_{n=1}^{\infty}$ be an injective sequence in $\mathbb{Z}_+$ with the range B. For a non-empty finite subset F of B i.e., $F = \{x_{n_j} : 1 \leq j \leq m\}$, we set $\delta_F = \delta_{x_{n_1}} * \delta_{x_{n_2}} * \cdots * \delta_{x_{nm}}$.
- Let $\{A_i\}_{i=1}^{r}$ be a partition of $\mathbb{Z}_+$. We would like that there must be an injective sequence $\langle x_n \rangle_{n=1}^{\infty}$ with the range B and an $i \in \{1, 2, \ldots, r\}$ such that $\sup(\delta_F) \subset A_i$ for all finite subsets F of B.

Motivation through CP2

Consider the Chebyshev polynomial hypergroup of second kind (CP2).

Take the finite partition $\{A_i\}_{i=1}^3$ where $A_i := \{n \in \mathbb{Z}_+ : n \equiv i-1 \bmod 3\}$. Take any injective sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$. Since x_n's are distinct, we may take it to be strictly increasing. Then, for $k \in \mathbb{N}$, $1 \leq x_k < x_k + 1 \leq x_{k+1}$. By choosing $F := \{x_k, x_{k+1}\}$, we get $\mathrm{spt}(\delta_F) \nsubseteq A_i$ for any i as the support $\mathrm{spt}(\delta_{x_k} * \delta_{x_{k+1}})$ contains two or more elements staring from $x_{k+1} - x_k$ to $x_{k+1} + x_k$ with the consecutive differences of 2 while every A_i contains elements with the difference of multiples of 3. Therefore, the situation is different in the setting of hypergroups.

Motivation through hypergroup deformations of semigroups

Consider a hypergroup deformations $(S,*)$ of semigroup $(S,\cdot) = (\mathbb{Z}_+,<,\max)$.

Take a partition $(A_i)_{i=1}^r$ of $\mathbb{Z}_+$. Then at least one of the A_i's is infinite. In case A_i has identity $e=0$, we replace A_i by $\widetilde{A}_i = A_i \backslash \{e\}$, otherwise we redesignate A_i by $\widetilde{A}_i$. Then $\widetilde{A}_i$ has an injective sequence $\langle x_n \rangle_{n=1}^{\infty}$ with the range B so that all finite products from $\langle x_n \rangle_{n=1}^{\infty}$ in $(\mathbb{Z}_+,<,\max)$ are in A_i. Now, for this set and sequence; for any finite subset $F = \{x_{n_j} : 1 \le j \le m\}$ of B, δ_F becomes δ_y where $y = \prod_{1\le j\le m} x_{n_j} = \max_{1\le j\le m} x_{n_j}$ and thus $\mathrm{spt}(\delta_F) \subseteq A_i$.

Formulation of new concepts in hypergroups

- Let $(K, *)$ be an infinite discrete semiconvo. Let $\mathbf{x} = \langle x_n \rangle_{n=1}^{\infty}$ be an injective sequence in $K \backslash \{e\}$. We denote its range by B. For a non-empty finite subset F of B, we first write it in its increasing indices form, i.e., $F = \{x_{n_j} : 1 \leq j \leq m\}$ with $1 \leq n_1 < n_2 < \ldots < n_m$. Next, we set $\delta_F = \delta_{x_{n_1}} * \delta_{x_{n_2}} * \cdots * \delta_{x_{n_m}}$.
- Let $\mathbf{x} = \langle x_n \rangle_{n=1}^{\infty}$ be an injective sequence in $K \backslash \{e\}$ with range B. Set

$$\begin{aligned} \mathrm{SFC}(\langle x_n \rangle_{n=1}^{\infty}) &:= \{\mathrm{spt}(\delta_{x_{n_1}} * \delta_{x_{n_2}} * \cdots * \delta_{x_{n_m}}) : n_1 < n_2 < \ldots < n_m, m \geq 1\} \\ &= \{\mathrm{spt}(\delta_F) : F \text{ is a non-empty finite subset of } B\}. \end{aligned}$$

Ramsey semiconvos or hypergroups

Definition [Kumar, Ross and Singh]

- Let $(K,*)$ be an infinite discrete semiconvo. $(K,*)$ will be called a *Ramsey semiconvos* if for every partition $K = \bigcup_{i=1}^{r} C_i$, there exist i and an injective sequence $\mathbf{x} = \langle x_n \rangle_{n=1}^{\infty}$ in $K \backslash \{e\}$ such that $\mathrm{spt}(\delta_F) \subset C_i$, i.e., $\delta_F(C_i) = 1$ for every non-empty finite subset $F \subset B$. In other words, $\mathrm{SFC}(\langle x_n \rangle_{n=1}^{\infty}) \subset \mathscr{P}(C_i)$.
- If $(K,*,\vee)$ is an infinite discrete hypergroup such that $(K,*)$ is a Ramsey semiconvo then $(K,*,\vee)$ will be called a *Ramsey hypergroup*.

Lemma [Kumar, Ross and Singh]

If an infinite discrete subsemiconvo L of semiconvo K is Ramsey then K is Ramsey semiconvo.

Theorem [Kumar, Ross and Singh]

Let $(S, \cdot)$ be an infinite commutative discrete action-free semigroup with the identity e satisfying the conditions (i)-(vi) of main theorem. Then the semiconvo $(S, *)$ is a Ramsey semiconvo.

Theorem [Kumar, Ross and Singh]

Let K be a commutative discrete hypergroup and let H be a finite subgroup of $Z(K)$. If K is a Ramsey hypergroup then the hypergroup $K//H$ is a Ramsey hypergroup.

Example

We may take $K = (S, *)$ for any hypergroup deformation of $(\mathbb{Z}_+, <, \max)$ with $q_1(1) = 0$. Then $Z(K) = \{0, 1\}$. We take $H = Z(K)$. We note that

$$K//H = \{\{0,1\}, \{m\} : m \geq 2\}$$

and

$$\delta_{\{m\}} * \delta_{\{n\}} = \begin{cases} \delta_{\{\max\{m,n\}\}} & \text{for } m \neq n \text{ with } m, n \geq 2, \\ (q_m(0) + q_m(1))\,\delta_{\{0,1\}} + \sum_{n \geq 2} q_m(n) \delta_{\{n\}} & \text{for } m = n \geq 2. \end{cases}$$

Then, by above Theorem $K//H$ is a Ramsey hypergroup.

Almost-Ramsey semiconvos or hypergroups

Definition [Kumar, Ross and Singh]

- Let $(K,*)$ be an infinite discrete semiconvo. $(K,*)$ will be called an *almost-Ramsey semiconvo* if for every partition $K = \bigcup_{i=1}^{r} C_i$, there exist i, an injective sequence $\mathbf{x} = \langle x_n \rangle_{n=1}^{\infty}$ in $K \setminus \{e\}$ and a finite subset $\mathscr{F}$ of $\mathrm{SFC}(\langle x_n \rangle_{n=1}^{\infty})$ such that $\mathrm{SFC}(\langle x_n \rangle_{n=1}^{\infty}) \setminus \mathscr{F} \subset \mathscr{P}(C_i)$.
- If $(K,*,\vee)$ is an infinite discrete hypergroup such that $(K,*)$ is an almost-Ramsey semiconvo then $(K,*,\vee)$ will be called an *almost-Ramsey hypergroup*.

Almost-strong Ramsey semiconvos or hypergroups

Definition [Kumar, Ross and Singh]

- Let $(K,*)$ be an infinite discrete semiconvo. $(K,*)$ will be called an *almost-strong Ramsey semiconvo* if for every partition $K=\bigcup_{i=1}^{r} C_i$, there exist an $i\in\{1,2,\ldots,r\}$, an infinite subsemiconvo of L of K and a finite subset D of L such that $L\backslash D\subset C_i$.
- An *almost-strong Ramsey hypergroup* can be defined by replacing semiconvo and subsemicovo by hypergroup and subhypergroup respectively in above definition.

- If an infinite discrete subsemiconvo L of semiconvo K is an almost-Ramsey semiconvo or an almost-strong Ramsey semiconvo then K is also an almost-Ramsey semiconvo or an almost-strong Ramsey semiconvo respectively.

Examples

- The Chebyshev polynomial hypergroup of second kind (CP2) is not an almost-Ramsey hypergroup.
- The hypergroup deformaations $(S,*)$ of $(S,\cdot) := (\mathbb{Z}_+, <, \max)$ as above are not almost-strong Ramsey hypergroups as $(S,*)$ does not have any proper infinite subhypergroup.

Theorem [Kumar, Ross and Singh]

Let K be a commutative discrete hypergroup and let H be a finite subgroup of $Z(K)$. If K is an almost-Ramsey hypergroup then the hypergroup $K//H$ is an almost-Ramsey hypergroup.

Theorem [Kumar, Ross and Singh]

No polymomial hypergroup $K_{\mathbf{P}}$ is almost-strong Ramsey hypergroup.

A variant of Ramsey principle for hypergroups

Definition [Kumar, Ross and Singh]

- Let $(K,*)$ be an infinite discrete semiconvo. $(K,*)$ will be called a *recurrent semiconvos* if for every partition $K = \bigcup_{i=1}^{r} C_i$, there exist i and an injective sequence $\mathbf{x} = \langle x_n \rangle_{n=1}^{\infty}$ in $K \setminus \{e\}$ such that $\delta_F(C_i) > 0$, (i.e., $\mathrm{spt}(\delta_F) \cap C_i \neq \emptyset$) for every non-empty finite subset F of the range of $\langle x_n \rangle_{n=1}^{\infty}$.
- If $(K,*,\vee)$ is an infinite discrete hypergroup such that $(K,*)$ is a recurrent semiconvo then $(K,*,\vee)$ will be called a *recurrent hypergroup*.

- If an infinite discrete sub-semiconvo L of a discrete semiconvo K is recurrent then K is recurrent.

Theorem [Kumar, Ross and Singh]

Every polynomial hypergroup $K_{\mathbf{P}}$ is a recurrent hypergroup.

Theorem [Kumar, Ross and Singh]

Let G be an infinite discrete group and let H be a finite group with $\#H = c$. Suppose that $(x,s) \mapsto x^s$ is an affine action of H on G. Then the discrete orbit semiconvo G^H is a recurrent semiconvo. In particular, we have the following facts.

(i) If H is a finite subgroup of G with $\#H = c$ then discrete coset semiconvo G/H is a recurrent semiconvo.

(ii) If H is a finite subgroup of G with $\#H = c$ then discrete doube coset hypergroup $G//H$ is a recurrent hypergroup.

(iii) If H is a finite subgroup of the group of automorphisms of G with $\#H = c$ then discrete automorphism orbit hypergroup G^H is a recurrent hypergroup.

Theorem [Kumar, Ross and Singh]

Let $(S,\cdot)$ be an infinite discrete Ramsey semigroup with identity e. Let H be a finite group of automorphisms of $(S,\cdot)$ with $\#H = c$. Then the space S^H of orbits s^H given by $s^H = \{\alpha(s) : \alpha \in H\}$ equipped with the discrete topology can be made into a recurrent semiconvo by defining '$*$' as follows:

$$\delta_{s^H} * \delta_{t^H} = \frac{1}{c} \sum_{\alpha \in H} \delta_{(\alpha(s)\cdot t)^H}.$$

Thank you for your attention !!!

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Thank You !!!