Weak Haagerup property of W^*-crossed products

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Outline

- Γ is a discrete group, e is the identity element of Γ.
- $L(\Gamma)$ is the group von Neumann algebra of Γ.
- $C_0(\Gamma)$ is the space of functions on Γ vanishing at infinity.
- $B_2(\Gamma)$ is the set of Herz-Schur multipliers on Γ. In fact, $B_2(\Gamma)$ is a unital Banach algebra when equipped with the Herz-Schur norm $\|\cdot\|_{B_2}$.
- M is a von Neumann algebra.
- τ is a faithful normal tracial state on M.
- $L_2(M,\tau)$ is the GNS-space associated to τ with the Hilbert norm $\|\cdot\|_{2,\tau}$.
- $M\bar{\rtimes}_\beta\Gamma$ is the W^*-crossed product of the W^*-dynamical system (M,Γ,β), where β is an action of Γ on M.

• $\mathrm{id_M}$ is the identity map of M.

• Suppose that $T : M \to M$ is a normal completely bounded map and there exists $K > 0$ such that $\|T(a)\|_{2,\tau} \leq K\|a\|_{2,\tau}$ for every $a \in M$. Then T can be extended to a bounded operator on $L_2(M, \tau)$ with norm at most K. We say T is L_2-compact if T can be extended to a compact operator on $L_2(M, \tau)$.

• A completely positive map $\Phi : M \to M$ is a completely bounded and $\|\Phi\|_{cb} = \|\Phi(I)\|$.

• A positive definite function is a Herz-Schur multiplier.

Approximation theory is particularly important in group theory and operator algebra theory.

There are many different approximation properties for groups such as amenability, weak amenability, the Haagerup property, property T and so on.

Definition

Γ is amenable if there is a net $\{u_\alpha\}_{\alpha \in I}$ of finitely supported positive definite functions on Γ such that $u_\alpha(g) \to 1$ for every $g \in \Gamma$.

Definition

Γ is weak amenable if there is a constant $C > 0$ and a net $\{u_\alpha\}_{\alpha \in I}$ of finitely supported functions in $B_2(\Gamma)$ such that

(1). $\|u_\alpha\|_{B_2} \leq C$, for every $\alpha \in I$,

(2). $u_\alpha(g) \to 1$ as $\alpha \to \infty$, for every $g \in \Gamma$.

The Cowling-Haagerup constant $\Lambda_{cb}(\Gamma)$ is the infimum of all C for which such a net $\{u_\alpha\}$ exists. We set $\Lambda_{cb}(\Gamma) = \infty$ if Γ is not weakly amenable.

Definition

Γ has the Haagerup property if there is a net $\{u_\alpha\}_{\alpha \in I}$ of positive definite functions in $C_0(\Gamma)$ such that

(1). $u_\alpha(e) = 1$, for every $\alpha \in I$,

(2). $u_\alpha(g) \to 1$ as $\alpha \to \infty$, for every $g \in \Gamma$.

The following are relations between some approximation properties for groups.

(1). Γ is amenable $\Rightarrow$ Γ is weak amenable, but the converse is not true (free groups);

(2). Γ is amenable $\Rightarrow$ Γ has the Haagerup property, but the converse is not true (free groups);

(3). weak amenability does not imply the Haagerup property (see $Sp(1,n)$).

There are also many different approximation properties for von Neumann algebras such as semidiscreteness, W*CBAP (weak* completely bounded approximation property), the Haagerup property and so on.

Definition

We say M is semidiscrete if there exist contractive completely positive maps $\varphi_n : M \to M_{k(n)}(\mathbb{C})$ *and* $\psi_n : M_{k(n)}(\mathbb{C}) \to M$ *such that* $\psi_n \circ \varphi_n \to \mathrm{id_M}$ *in the point-ultraweak topology:*

$$\eta(\psi_n \circ \varphi_n(a)) \to \eta(a)$$

for all $a \in M$ *and all normal functionals* $\eta \in M_*$.

Definition

We say M has the W^*CBAP if there exists a constant $C > 0$ and a net of ultraweakly-continuous finite-rank completely bounded maps $\varphi_i : M \to M$ such that $\varphi_i \to \mathrm{id}_M$ in the point-ultraweak topology for all $a \in M$ and $\sup \|\varphi_i\|_{cb} \leq C$.

The Haagerup constant $\Lambda_{cb}(M)$ is the infimum of all C for which such a net $\{\varphi_i\}$ exists. We set $\Lambda_{cb}(M) = \infty$ if M does not have the W*CBAP.
W*CBAP is weaker than semidiscreteness.

Definition

M is said to have the Haagerup property if there is a net $\{\Phi_i\}_{i \in I}$ of unital normal completely positive maps from M to itself satisfying the following conditions:

1. $\tau \circ \Phi_i \leq \tau$ *for each* $i \in I$;
2. *For any* $a \in M$, $\|\Phi_i(a) - a\|_{2,\tau} \to 0$ *as* $i \to \infty$;
3. *Each* Φ_i *is* L_2*-compact.*

The Haagerup property does not depend on the choice of the faithful normal tracial state.

The following are relations between approximation properties for groups and approximation properties for von Neumann algebras.

(1). Γ is amenable $\Leftrightarrow$ $L(\Gamma)$ is semidiscrete;

(2). Γ is weak amenable $\Leftrightarrow$ $L(\Gamma)$ has the W*CBAP, and $\Lambda_{cb}(\Gamma) = \Lambda_{cb}(L(\Gamma))$.

(3). Γ has the Haagerup property $\Leftrightarrow$ $L(\Gamma)$ has the Haagerup property.

In order to study the relation between weak amenability and the Haagerup property, Knudby introduced the weak Haagerup property for discrete groups.

Definition

Γ has the weak Haagerup property if there is a constant $C > 0$ and a net $\{u_\alpha\}_{\alpha \in I}$ in $B_2(\Gamma) \cap C_0(\Gamma)$ such that

(1). $\|u_\alpha\|_{B_2} \leq C$, for every $\alpha \in I$,

(2). $u_\alpha(g) \to 1$ as $\alpha \to \infty$, for every $g \in \Gamma$.

The weak Haagerup constant $\Lambda_{WH}(\Gamma)$ is defined as the infimum of those C for which such a net $\{u_\alpha\}_{\alpha \in I}$ exists, and if no such net exists we write $\Lambda_{WH}(\Gamma) = \infty$.

In 2016, Knudby introduced the weak Haagerup property for von Neumann algebras.

Definition

M has the weak Haagerup property if there is a constant $C > 0$ and a net $\{T_\alpha\}_{\alpha\in I}$ of normal completely bounded maps on M such that

(1). $\|T_\alpha\|_{cb} \leq C$ *for every* $\alpha \in I$,

(2). $\langle T_\alpha(x), y\rangle = \langle x, T_\alpha(y)\rangle$ *for every* $x, y \in M$,

(3). *each* T_α *is* L_2*-compact,*

(4). $\|T_\alpha(x) - x\|_{2,\tau} \to 0$ *for every* $x \in M$.

The weak Haagerup constant $\Lambda_{WH}(M)$ is defined as the infimum of those C for which such a net $\{u_\alpha\}_{\alpha\in I}$ exists, and if no such net exists we write $\Lambda_{WH}(M) = \infty$.

The weak Haagerup property does not depend on the choice of the faithful normal tracial state.
Γ has the weak Haagerup property $\Leftrightarrow$ $L(\Gamma)$ has the weak Haagerup property.

Motivated by the above results, we study the weak Haagerup property of W^*-crossed products.
First, we have the following result.

Theorem

If $M\bar{\rtimes}_\beta\Gamma$ has the weak Haagerup property, then both M and Γ have the weak Haagerup property and

$$\Lambda_{WH}(M) \leq \Lambda_{WH}(M\bar{\rtimes}_\beta\Gamma), \;\; \Lambda_{WH}(\Gamma) \leq \Lambda_{WH}(M\bar{\rtimes}_\beta\Gamma).$$

In general, the weak Haagerup property of M and Γ does not imply that of $M\bar{\rtimes}_\beta\Gamma$. For example, both groups $\mathbb{Z}^2$ and $\mathrm{SL}(2,\mathbb{Z})$ enjoy the Haagerup property and hence also the weak Haagerup property, but $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ does not have the weak Haagerup property.
Next, we show that if Γ be a amenable group, then the weak Haagerup property of M implies that of $M\bar{\rtimes}_\beta\Gamma$.

Theorem

Let Γ be an amenable group. If M has the weak Haagerup property, then $M\bar{\rtimes}_\beta\Gamma$ has the weak Haagerup property and

$$\Lambda_{WH}(M\bar{\rtimes}_\beta\Gamma) = \Lambda_{WH}(M).$$

Finally, we give a condition under which the weak Haagerup property of M and Γ implies that of $M\bar{\rtimes}_\beta\Gamma$.

Theorem

Let Γ be a countable group. Then the following statements are equivalent:

(1). Γ *has the weak Haagerup property and* M *has the weak Haagerup property with the approximating maps* $T_i : M \to M$ *satisfying* $T_i \circ \beta_t = \beta_t \circ T_i$ *for all* $t \in \Gamma$.

(2). $M\bar{\rtimes}_\beta\Gamma$ *has the weak Haagerup property and the approximating maps* $\Phi_i : M\bar{\rtimes}_\beta\Gamma \to M\bar{\rtimes}_\beta\Gamma$ *satisfy* $\mathcal{E} \circ \Phi_i \circ \beta_t(x) = \beta_t \circ \mathcal{E} \circ \Phi_i(x)$ *for all* $t \in \Gamma$ *and* $x \in M$.

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Thank you!