Spectral theory of Fourier-Stieltjes algebras

Przemysław Ohrysko
joint work with Mateusz Wasilewski

Chalmers University of Technology and Gothenburg University

Classical motivation I

Let G be a locally compact (non-discrete) Abelian group and let $M(G)$ denote the Banach algebra of all complex-valued Borel regular measures on G equipped with the total variation norm and the convolution product.
The Gelfand space of $M(G)$ (the set of all multiplicative - linear functionals endowed with the weak* topology) will be abbreviated $\triangle(M(G))$.
Since the convolution is transferred to the pointwise product via Fourier-Stieltjes transform it is clear that the dual group of G (denoted Γ) is canonically embedded into $\triangle(M(G))$.

Classical motivation II: Wiener-Pitt phenomenon

As $M(G)$ is a Banach algebra it is natural to consider the spectrum of $\mu \in M(G)$ defined as

$$\sigma(\mu) = \{\lambda \in \mathbb{C} : \mu - \lambda \cdot \delta_0 \text{ is not invertible}\}.$$

For special types of measures (for example, absolutely continuous or discrete ones) we have $\sigma(\mu) = \overline{\widehat{\mu}(\Gamma)}$. However, by the following result it is not true in general.

Wiener-Pitt phenomenon

There exists a measure $\mu \in M(G)$ such that $\overline{\widehat{\mu}(\Gamma)} \subsetneq \sigma(\mu)$.

Note that the above result implies the non-density of Γ in $\triangle(M(G))$ (recall that the spectrum of an element is an image of its Gelfand transform).

Basic definitions

We are going now to change the setting to non-commutative groups. Let G be an infinite discrete group and let $B(G)$ be the Fourier-Stieltjes algebra of G (the linear span of positive definite functions equipped with the norm defined by the duality $(C^*(G))^* = B(G)$). With pointwise multiplication it is a commutative unital Banach algebra. We recall also the standard notion of the spectrum of an element $f \in B(G)$:

$$\sigma(f) = \{\lambda \in \mathbb{C} : f - \lambda \cdot \mathbf{1} \text{ is not invertible}\}.$$

Observe that, in case of Abelian group G the Fourier-Stieltjes transform defines a natural isomorphism between $B(G)$ and $M(\Gamma)$ where Γ is a dual group of G.

Wiener-Pitt phenomenon again

As we are now already on 'the Fourier-Stieltjes side' we have a natural embeeding of G into $\triangle(B(G))$ (via point-evaluations).
Using the fact that one can always extend a positive-definite function from a subgroup of a discrete group to the whole group we obtain the following result.

Wiener-Pitt phenomenon again

If G contains an infinite Abelian subgroup then there exists $f \in B(G)$ such that $\overline{f(G)} \subsetneq \sigma(f)$.

This implies the non-density of G in $\triangle(B(G))$ in the same way as before.

Non-separability of $\triangle(B(G))$

In a similar fashion, but exploiting the results from the recent paper of myself, M. Wojciechowski and C. C. Graham instead of classical Wiener-Pitt phenomenon, we prove the non-separability of $\triangle(B(G))$.

Non-separability of $\triangle(B(G))$

If G contains an infinite Abelian subgroup then $\triangle(B(G))$ contains continuum many pairwise disjoint open subsets. In particular, $\triangle(B(G))$ is not separable.

Elements with a natural spectrum

Following M. Zafran we introduce the following definition.

Natural spectrum

Let $f \in B(G)$. We say that f has a natural spectrum if $\sigma(f) = \overline{f(G)}$. The set of all such elements will be denoted by $NS(G)$.

As in the classical setting we have a rich family of elements with a natural spectrum.

Elements with a natural spectrum

Following M. Zafran we introduce the following definition.

Natural spectrum

Let $f \in B(G)$. We say that f has a natural spectrum if $\sigma(f) = \overline{f(G)}$. The set of all such elements will be denoted by $NS(G)$.

As in the classical setting we have a rich family of elements with a natural spectrum.

Examples of elements with a natural spectrum

If $f \in A(G)$ or $f \in B(G) \cap AP(G)$ then $f \in NS(G)$.

More examples of elements with a natural spectrum

In the classical case (for Abelian group G) the notion of the 'spine' supplies us with more examples of measures with a natural spectrum generalizing absolutely continuous and discrete measures. The spine is the direct sum of 'maximal group subalgebras' which are the L^1 spaces with respect to the original group but with the topology replaced by a finer one (preserving the property of being locally compact group). The analogue of this notion for Fourier-Stieltjes algebras was introduced by N. Spronk and M. Ilie. As in the commutative setting the elements belonging to the spine of the Fourier-Stieltjes algebra also have a natural spectrum.

More examples of elements with a natural spectrum

In the classical case (for Abelian group G) the notion of the 'spine' supplies us with more examples of measures with a natural spectrum generalizing absolutely continuous and discrete measures. The spine is the direct sum of 'maximal group subalgebras' which are the L^1 spaces with respect to the original group but with the topology replaced by a finer one (preserving the property of being locally compact group). The analogue of this notion for Fourier-Stieltjes algebras was introduced by N. Spronk and M. Ilie. As in the commutative setting the elements belonging to the spine of the Fourier-Stieltjes algebra also have a natural spectrum.

Question

What about the structure of $NS(G)$?

The structure of $NS(G)$

Motivated by the theorem of Hatori and Sato we obtained the following result.

Decomposition of $B(G)$

If G is maximally almost periodic discrete group then
$B(G) = NS(G) + NS(G) + B(G) \cap AP(G)$.

When we drop the assumption on the group we are able to prove only a weaker assertion.

By the above theorem and the existence of the Wiener-Pitt phenomenon the set $NS(G)$ is not closed under addition.

The structure of $NS(G)$

Motivated by the theorem of Hatori and Sato we obtained the following result.

Decomposition of $B(G)$

If G is maximally almost periodic discrete group then $B(G) = NS(G) + NS(G) + B(G) \cap AP(G)$.

When we drop the assumption on the group we are able to prove only a weaker assertion.

Weaker variant

Let G be a discrete group containing an infinite Abelian subgroup. Then $B(G) = NS(G) + NS(G) + NS(G)$.

By the above theorem and the existence of the Wiener-Pitt phenomenon the set $NS(G)$ is not closed under addition.

The ideal of Zafran

Let $B_0(G) := B(G) \cap c_0(G)$. For $B_0(G) \cap NS(G)$ the analogue of Zafran's theorem holds true.

Non-commutative Zafran's theorem

Let G be a discrete group. Then

1. $B_0(G) \cap NS(G)$ is a closed ideal of $B(G)$.
2. If $\varphi \in \triangle(B(G)) \setminus G$ then $\varphi(f) = 0$ for every $f \in B_0(G) \cap NS(G)$.
3. $\triangle(B_0(G) \cap NS(G)) = G$.

Mutual singularity and absolute continuity

Let $A = C^*(G)$, $A^* = (C^*(G))^* = B(G) = \mathrm{M}_*$,
$A^{**} = (C^*(G))^{**} = B(G)^* = \mathrm{M} =: W^*(G)$.

Central support

Let $f \in B(G) = \mathrm{M}_*$. We define the central support of f (denoted $\mathrm{zs}(f)$) as the smallest central projection in $W^*(G)$ such that f vanishes on $(1 - \mathrm{zs}(f))\mathrm{M}$.

Mutual singularity and absolute continuity

Let $A = C^*(G)$, $A^* = (C^*(G))^* = B(G) = \mathrm{M}_*$,
$A^{**} = (C^*(G))^{**} = B(G)^* = \mathrm{M} =: W^*(G)$.

Central support

Let $f \in B(G) = \mathrm{M}_*$. We define the central support of f (denoted $\mathrm{zs}(f)$) as the smallest central projection in $W^*(G)$ such that f vanishes on $(1 - \mathrm{zs}(f))\mathrm{M}$.

Mutual singularity and absolute continuity

Let $f, g \in B(G)$. We say that f and g are mutually singular if $\mathrm{zs}(f)\mathrm{zs}(g) = 0$ (notation $f \perp g$). If $\mathrm{zs}(f) \leq \mathrm{zs}(g)$ then we say that f is absolutely continuous with respect to g ($f \ll g$).

Applications

First of all, the definition of mutual singularity and absolute continuity in $B(G)$ extends the classical notions for commutative G. Moreover, proving first the fact $f \ll g$ if and only if f is a norm limit of linear combinations of translates of g we get the next theorem shedding some light on the well-known ideals and subalgebras.

L-subspaces

Let $X \subset B(G)$ be a closed linear subspace. The following conditions are equivalent.

1. X is an L-subspace ($g \in X$, $f \ll g \Rightarrow f \in X$).
2. X is bi-G-invariant, i.e. $\mathbb{C}[G] \cdot X \subset X$ and $X \cdot \mathbb{C}[G] \subset X$.
3. There exists a central projection $z \in W^*(G)$ such that $X = zB(G)$.

In particular, $A(G)$, $B_0(G)$ and $B_0(G) \cap NS(G)$ are L-ideals and $B(G) \cap AP(G)$ is an L-subalgebra.

Problems

Working with the notion of central support is convenient but there are some unexpected obstacles.

For example, it is not true in general that $f_1 \ll g_1,\ f_2 \ll g_2 \Rightarrow f_1 f_2 \ll g_1 g_2$ even if all f_1, f_2, g_1 and g_2 are positive definite (there is an easy counterexample for $G = S_3$) so we need one technical assumption.

GNS faithfulness

Let f be a positive definite function. We say that f is *GNS faithful* if f is a faithful functional on the von Neumann algebra generated by f.

Equivalently, a positive definite function f is GNS faithful, if the standard support of f (treated as a functional on $W^*(G) = (B(G))^*$) is equal to the central support.

The previous assertion holds true provided both g_1 and g_2 are GNS faithful.

New applications

Equipped with the theorem on L-subspaces and the notion of GNS faithfulness we are prepared to prove two nice results (also counterparts of commutative ones).

Criterion for non-naturality of the spectrum

Let $f \in B_0(G) \setminus \{0\}$ and suppose that $f^n \perp A(G)$ for all $n \in \mathbb{N}$. Then $f \notin B_0(G) \cap NS(G)$.

New applications

Equipped with the theorem on L-subspaces and the notion of GNS faithfulness we are prepared to prove two nice results (also counterparts of commutative ones).

Criterion for non-naturality of the spectrum

Let $f \in B_0(G) \setminus \{0\}$ and suppose that $f^n \perp A(G)$ for all $n \in \mathbb{N}$. Then $f \notin B_0(G) \cap NS(G)$.

Orthogonality to $B_0(G) \cap NS(G)$

Let $f \in B(G)$ be positive, GNS faithful and such that $f^n \perp A(G)$ for all $n \in \mathbb{N}$. Then $f \perp B_0(G) \cap NS(G)$.

Example: the Haagerup function

We will examine the spectral properties of two examples of positive definite functions on free groups.
Let $f_r : \mathbb{F}_k \to \mathbb{R}$ for $r \in (0,1)$ be the Haagerup function defined by the formula $f_r(x) = r^{|x|}$ where $|\cdot|$ is the lenght of the reduced word in $\mathbb{F}_k$. Of course, $f_r^m = f_{r^m}$ and calculating the l^2 norm of f_r we get

The naturality of the spectrum of the Haagerup function

For every $r \in (0,1)$ there exists $n \in \mathbb{N}$ such that
$f_r^n \in l^2(G) \subset A(G) \subset NS(G)$.

Example: free products

Let $\mathbb{F}_\infty$ be the free group on infinitely many generators $x_k, k \in \mathbb{N}$, considered as a free product of $\mathbb{Z}^{(k)} =< x_k >\simeq \mathbb{Z}$.

Free product of M. Bożejko

If $f_k : \mathbb{Z}^{(k)} \to \mathbb{C}$ is a sequence of normalized positive definite functions then we define their free product $f := \bigcirc_{k=1}^{\infty} f_k$ by the following recipe: $f(e) = 1$ and if x is reduced word in the alphabet $\{x_k\}_{k\in\mathbb{N}}$ then to each $x_k^{\pm m}$ in the word x we apply the function f_k and finally multiply the outcomes.
It is not trivial to check the positive-definiteness of f (theorem of M. Bożejko).

Example: free Riesz products

It is easy to check that for α_k satisfying $0 < |\alpha_k| \leq \frac{1}{2}$ the function $v_k = \delta_e + \alpha_k \delta_{x_k} + \overline{\alpha_k} \delta_{x_k^{-1}}$ is positive definite.

Free Riesz products

Let $(\alpha_k)_{k=1}^{\infty}$ be a sequence of complex numbers satisfying $0 < |\alpha_k| \leq \frac{1}{2}$. Then $R = \bigcirc_{k=1}^{\infty} v_k$ is called a free Riesz product.

It is elementary to check that $R \in B_0(\mathbb{F}_\infty)$ iff $(\alpha_k) \in c_0$.

Example: free Riesz products

It is easy to check that for α_k satisfying $0 < |\alpha_k| \leq \frac{1}{2}$ the function $v_k = \delta_e + \alpha_k \delta_{x_k} + \overline{\alpha_k} \delta_{x_k^{-1}}$ is positive definite.

Free Riesz products

Let $(\alpha_k)_{k=1}^{\infty}$ be a sequence of complex numbers satisfying $0 < |\alpha_k| \leq \frac{1}{2}$. Then $R = \bigcirc_{k=1}^{\infty} v_k$ is called a free Riesz product.

It is elementary to check that $R \in B_0(\mathbb{F}_\infty)$ iff $(\alpha_k) \in c_0$.

Theorem on free Riesz products (M. Bożejko)

If $\sum_k |\alpha_k|^2 = \infty$ then $R \perp B_\lambda(G)$. In particular $R \perp A(G)$.

Corollary on free Riesz products

One checks easily that the power of a free Riesz product is again a Riesz product so basing on on our criterion on the non-naturality of the spectrum in $B_0(G)$ and the last result we obtain the corrolary.

Corollary on free Riesz products

If $(\alpha_k) \in c_0$ is such that $\sum_k |\alpha_k|^n = \infty$ for every $n \in \mathbb{N}$ then the corresponding free Riesz product does not have a natural spectrum.

Using completely different method (functional calculus + idempotent theorem) it is also possible to show that $R = \bigcirc(\delta_e + \frac{1}{2}\delta_{x_k} + \frac{1}{2}\delta_{x_k^{-1}})$ does not have a natural spectrum.

Non-naturality of the standard free Riesz product I

Let $R = \bigcirc(\delta_e + \frac{1}{2}\delta_{x_k} + \frac{1}{2}\delta_{x_k^{-1}})$. In order to prove that $R \notin NS(\mathbb{F}_\infty)$ we are going to mimic the approach of the classical Riesz product proof of the Wiener-Pitt phenomenon by C.C. Graham.

Suppose $R \in NS(\mathbb{F}_\infty)$, towards the contradiction. Then,

$$\sigma(R) = \overline{R(\mathbb{F}_\infty)} = R(\mathbb{F}_\infty) = \{0\} \cup \left\{\frac{1}{2^n}\right\}_{n \in \mathbb{N}} \cup \{1\}.$$

Let us take two disjoint open sets $A, B \subset \mathbb{C}$ such that $\sigma(R) \subset A \cup B$ and $A \cap \sigma(R) = \left\{\frac{1}{2}\right\}$ and define a function $F : A \cup B \to \{0, 1\}$ by the conditions: $F|_A = 1$ and $F|_B = 0$.

Non-naturality of the standard free Riesz product II

Clearly, F is an analytic function on $A \cup B$. Thus we are allowed to use functional calculus obtaining an element $P := F(R) \in B(G)$. We see immediately that P is an idempotent. Moreover, as $R(x) = \frac{1}{2}$ iff $x \in \left\{x_k, x_k^{-1}\right\}_{k \in \mathbb{N}} := X$ we get $P = \chi_X$.
By the Host idempotent theorem X belongs to the coset ring of $\mathbb{F}_\infty$. We show that this is not possible using an elementary theorem valid for **all groups**.

Non-naturality of the standard free Riesz product II

Clearly, F is an analytic function on $A \cup B$. Thus we are allowed to use functional calculus obtaining an element $P := F(R) \in B(G)$. We see immediately that P is an idempotent. Moreover, as $R(x) = \frac{1}{2}$ iff $x \in \left\{x_k, x_k^{-1}\right\}_{k\in\mathbb{N}} := X$ we get $P = \chi_X$.
By the Host idempotent theorem X belongs to the coset ring of $\mathbb{F}_\infty$. We show that this is not possible using an elementary theorem valid for **all groups**.

Theorem on coset ring

Let X be an infinite member of the coset ring of any group. Then X contains a coset of infinite subgroup with a possible exception of finitely many elements.

Thank You for your attention!