Linearization Trick in Infinitesimal Freeness

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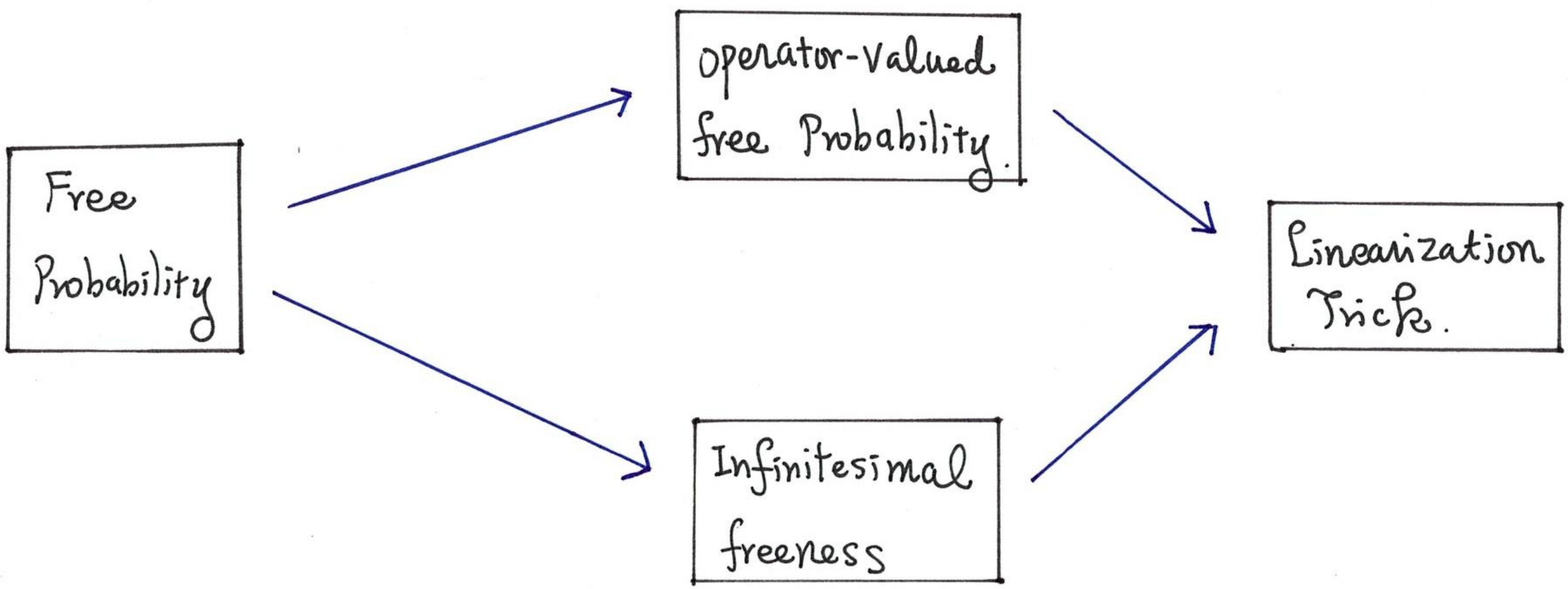
Outline

Question.

Assume that We know the infinitesimal distribution of self-adjoint elements X and Y. Given an self-adjoint polynomial P with variable X and Y. Whether We can write down the Precise formula for the infinitesimal distribution of $P(X,Y)$?.



Free Probability

<u>classical Probability.</u>

$(\Omega, \mathcal{F}, P)$: Probability space

$X : \Omega \to \mathbb{C}$ is a random variable

$\Longrightarrow$

<u>Free Probability</u>

$\mathcal{L}^{\infty-}(\Omega, \mathcal{F}, P) = \bigcap_{1 \le p < \infty} \mathcal{L}^{p}(\Omega, \mathcal{F}, P)$

$\mathbb{E}(X) = \int_{\Omega} X \, dP$. $(\mathbb{E}(1) = 1)$

<u>Framework</u>

- $(\mathcal{A}, \varphi)$: non-commutative Probability space ($*$- Probability space)
  if $\mathcal{A}$ is a unital algebra ($*$- algebra)
  φ is a state. i.e, $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1) = 1$. (φ is positive. i.e, $\varphi(a^* a) \geq 0 \ \forall a \in \mathcal{A}$)

- $a_N \in (\mathcal{A}_N, \varphi_N)$, $a \in (\mathcal{A}, \varphi)$ for all $N \in \mathbb{N}$
  a_N converges to a in distribution ($a_N \Rightarrow a$ as $N \to \infty$) if
  $\varphi_N(a_N^k) \xrightarrow[N \to \infty]{} \varphi(a^k)$ for all $k \in \mathbb{N}$

- $a_N, b_N \in (\mathcal{A}_N, \varphi_N)$, $a, b \in (\mathcal{A}, \varphi)$ for all $N \in \mathbb{N}$
  (a_N, b_N) converges to (a, b) in distribution ($(a_N, b_N) \Rightarrow (a, b)$ as $N \to \infty$) if
  $\varphi_N(p(a_N, b_N)) \xrightarrow[N \to \infty]{} \varphi(p(a, b))$ for any non-comm poly p in two interderminants X and Y.

$\boxed{\text{Free Probability}}$

Given $(\mathcal{A}, \varphi)$: ncps. $\mathcal{A}_1$ & $\mathcal{A}_2 \subset \mathcal{A}$ are sub-algebras ($1 \in \mathcal{A}_1$, $1 \in \mathcal{A}_2$)

- $\mathcal{A}_1$ and $\mathcal{A}_2$ are free if $\forall n \in \mathbb{N}$, $\forall j = 1, 2, \cdots, n$ $\forall i(j) \in \{1, 2\}$. $i(1) \neq i(2) \neq \cdots \neq i(n)$.

  for all $a_j \in \mathcal{A}_{i(j)}$ with $\varphi(a_j) = 0$, $j = 1, \cdots, n$ $\Rightarrow \varphi(a_1 a_2 \cdots a_n) = 0$.

- $a, b \in \mathcal{A}$ are free if $\mathcal{A}_1 = alg(a, 1)$ & $\mathcal{A}_2 = alg(b, 1)$ are free.

- μ, ν : Probability Measures with compact support on $\mathbb{R}$.

  a, b are self-adjoint elements in some C^*-Prob space with $a \sim \mu$, $b \sim \nu$ and a & b are free. Then the distribution of $a+b$ is called the free convolution of μ and ν, and is denoted by $\mu \boxplus \nu$.

- μ: Probability Measure on $\mathbb{R}$. $G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z - t}$, $z \in \mathbb{C}^+$ is the Cauchy transform of μ

Theorem

Assume that μ & ν are Probability Measures on $\mathbb{R}$, let $\eta = \mu \boxplus \nu$. Then there exists analytic maps $\omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+$ which are uniquely determined by the following Properties:

1°. $G_\mu(\omega_1(z)) = G_\nu(\omega_2(z)) = G_\eta(z)$, $z \in \mathbb{C}^+$.

2° $\omega_1(z) + \omega_2(z) = z + \frac{1}{G_\eta(z)}$

3°. $\lim\limits_{y \to \infty} \frac{\omega_j(iy)}{iy} = \lim\limits_{y \to \infty} \omega_j'(iy) = 1$, $j = 1, 2$.

Links with Random Matrices

$\mathcal{A} = M_N(L^{\infty-}(\Omega, \mathcal{F}, P))$. $\varphi = \mathrm{tr} \otimes E$: normalized trace.

- $a_{ij} \overset{\text{iid}}{\sim} N(0,1)$ for all $i,j \in \{1, \dots, N\}$ & $G = [a_{ij}]_{i,j=1}^{N}$. Let $A_N = \frac{1}{\sqrt{2N}}(G + G^T)$.
  A_N is called the Gaussian Orthogonal Ensemble (GOE)

- (Wigner). A_N : GOE for all $N \in \mathbb{N}$, then $A_N \Longrightarrow S$: semicircle element.
  That is, $\lim_{N \to \infty} \mathrm{tr} \otimes \mathbb{E}(A_N^k) = \varphi(s^k) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \frac{1}{k/2+1}\binom{k}{k/2} & \text{if } k \text{ is even} \end{cases}$

$$= \int_{-2}^{2} t^k \cdot \frac{1}{2\pi}\sqrt{4 - t^2}\, dt.$$

Theorem (Voiculescus, 1991)

$\{A_N\}_N$, $\{B_N\}_N$ GOE + $\{a_{ij}\}_{i,j=1}^{N}$, $\{b_{ij}\}_{i,j=1}^{N}$ classically independent, $N \in \mathbb{N}$ $\Longrightarrow$ $\exists$ semicircle elements s_1, s_2 in a ncps $(\mathcal{A}, \varphi)$ such that $\underbrace{(A_N, B_N) \Longrightarrow (s_1, s_2) \text{ \& } s_1, s_2 \text{ are free.}}_{\text{asymptotically free}}$

Infinitesimal Freeness

A_N : $N\times N$ self-adjoint Matrix, and $\lambda_1(N), \dots, \lambda_N(N)$ are eigenvalues of A_N

The empirical distribution $\eta_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j(N)} \xrightarrow[N\to\infty]{} \eta$.

$\eta_N = \eta + \frac{1}{N}\eta' + o(\frac{1}{N}) \Longrightarrow$ find η'.

— (N. Enriquez & L. Menard, 2016)

$$A_N : N\times N \text{ GOE. } \quad \eta_N = \underbrace{\left(\frac{1}{2\pi}\sqrt{4-t^2}\,1_{[-2,2]}\,dt\right)}_{\text{semi-circle law}} + \frac{1}{N}\Bigg(\underbrace{\frac{1}{2}\left\{\frac{1}{2}(\delta_{-2}+\delta_2) - \frac{1_{[-2,2]}}{\pi\sqrt{4-t^2}}\,dt\right.}_{\frac{1}{2}(\text{Bernoulli} - \text{Arcsine})}\Bigg) + o\left(\frac{1}{N}\right)$$

Framework

— $(\mathcal{A}, \varphi, \varphi')$: infinitesimal non-commutative Probability space (incps) if $\mathcal{A}$ is a unital algebra & $\varphi, \varphi' : \mathcal{A} \to \mathbb{C}$ are linear functionals such that $\varphi(1) = 1$ and $\varphi'(1) = 0$.

— $\mathcal{A}_1, \mathcal{A}_2 \subset (\mathcal{A}, \varphi, \varphi')$: two unital subalgebras are infinitesimally free if for $a_1, a_2, \dots, a_n \in \mathcal{A}$ are such that $a_k \in \mathcal{A}_{i_k}$, $i_1 \neq i_2 \neq \cdots \neq i_n$ and

$$\varphi(a_k) = 0 \text{ for all } k = 1, 2, \dots, n \Longrightarrow \varphi(a_1 a_2 \cdots a_n) = 0 \;;$$

$$\varphi'(a_1 a_2 \cdots a_n) = \sum_{j=1}^{n} \varphi(a_1 a_2 \cdots a_{j-1} \varphi'(a_j) a_{j+1} \cdots a_n).$$

$\boxed{\text{Infinitesimal freeness}}$

Links with Random Matrices.

$A_N^{(1)}, A_N^{(2)}, \cdots, A_N^{(k)}$: random Matrices of size $N \times N$.

$\mathcal{A}$: the algebra of polynomials on k non-commuting interdeterminates.

$\tau_N : \mathcal{A} \to \mathbb{C}$, $\tau_N(P) = \mathbb{E}\left(\frac{1}{N} \mathrm{Tr}\left(P(A_N^{(1)}, \cdots, A_N^{(k)})\right)\right)$.

Set $\tau = \lim_{N\to\infty} \tau_N$

$\tau' = \lim_{N\to\infty} N(\tau_N - \tau)$. provided that these exist.

Theorem (Belinschi & Shlyakhtenko, 2010)

$X \sim (\mu_1, \nu_1)$
$Y \sim (\mu_2, \nu_2)$ are infinitesimally free. let (μ_3, ν_3) be the infinitesimal law of $X+Y$.

then 1° $\mu_3 = \mu_1 \boxplus \mu_2$;

2° $g_{\nu_3}(z) = g_{\nu_1}(\omega_1(z))\, \omega_1'(z) + g_{\nu_2}(\omega_2(z)) \cdot \omega_2'(z)$ where g_{ν_i} is the Cauchy transform with respect to the ν_i for $i = 1, 2$.

Theorem (Shlyakhtenko, 2015)

A_N : GOE . $B = \begin{bmatrix} \theta_1 & & & & & \\ & \theta_2 & & & & \\ & & \ddots & & & \\ & & & \theta_{N_0} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix} \Rightarrow (A_N, B) \Longrightarrow (s_1, s_2)$

and s_1, s_2 are infinitesimally free.

Operator-Valued Free Probability

Scalar Version		Operator-valued Version
$(\mathcal{A}, \varphi)$: ncps	$\Longrightarrow$	$\mathcal{M} = M_N(\mathcal{A})$, $\mathcal{B} = M_N(\mathbb{C})$, $\mathbb{E} = \varphi \otimes Id_N$
		$(\mathcal{M}, \mathbb{E}, \mathcal{B})$: operator-valued ncps.

- If the entries of two matrices are free w.r.t $(\mathcal{A}, \varphi)$ then the two matrices themselves are free w.r.t $\mathbb{E}$.

- For $x = x^* \in \mathcal{M}$, define the Cauchy transform of x by $G_x(b) = \mathbb{E}[(b-x)^{-1}]$ for all $b \in H^+(\mathcal{B})$: the operator upper half plane (i.e, $\mathrm{Im}(b) > 0$)

- The reciprocal Cauchy transform : $F_x(b) = \mathbb{E}[(b-x)^{-1}]^{-1} = G_x(b)^{-1}$.

Theorem (Belinschi, Mai & Speicher, 2013)

Given $x, y \in \mathcal{M}$ are self-adjoint and they are free. Then there exists a unique pair of Fréchet analytic maps $\omega_1, \omega_2 : H^+(\mathcal{B}) \to H^+(\mathcal{B})$ such that

1° $\mathrm{Im}(\omega_j(b)) \geq \mathrm{Im}(b)$ for all $b \in H^+(\mathcal{B})$, $j = 1, 2$.

2°. $F_x(\omega_1(b)) + b = F_y(\omega_2(b)) + b = \omega_1(b) + \omega_2(b)$ for $b \in H^+(\mathcal{B})$.

3°. $G_x(\omega_1(b)) = G_y(\omega_2(b)) = G_{x+y}(b)$. for all $b \in H^+(\mathcal{B})$.

Operator-Valued Infinitesimal Probability

Scalar Version		Operator-Valued Version
$(\mathcal{A}, \varphi, \varphi')$ incps	$\Longrightarrow$	$\mathcal{M} = M_N(\mathcal{A})$, $\mathcal{B} = M_N(\mathbb{C})$,
		$\mathbb{E} = \varphi \otimes \mathrm{Id}_N$, $\mathbb{E}' = \varphi' \otimes \mathrm{Id}$
		$(\mathcal{M}, \mathbb{E}, \mathbb{E}', \mathcal{B})$: operator-valued incps

Definition

$\mathcal{M}_1$ & $\mathcal{M}_2 \subset \mathcal{M}$ unital subalgebras are infinitesimally free with respect to $(\mathbb{E}, \mathbb{E}')$ if for $i_1, i_2, \dots, i_n \in \{1, 2\}$, $i_1 \neq i_2 \neq \cdots \neq i_n$, $a_j \in \mathcal{M}_{i_j}$ with $\mathbb{E}(a_j) = 0$ for all $j = 1, \dots, n$

then $\mathbb{E}(a_1 a_2 a_3 \cdots a_n) = 0$;

$$\mathbb{E}'(a_1 a_2 \cdots a_n) = \begin{cases} \mathbb{E}\Big(a_1 \mathbb{E}\big(a_2 \mathbb{E}(a_3 \cdots \mathbb{E}(a_{\frac{n-1}{2}} \mathbb{E}'(a_{\frac{n+1}{2}}) a_{\frac{n+3}{2}}) \cdots a_{n-2}) a_{n-1}\big) a_n\Big) \\ \qquad \text{if } n \text{ is odd and } i_1 = i_n,\ i_2 = i_{n-1}, \dots, i_{\frac{n-1}{2}} = i_{\frac{n+3}{2}}. \\ 0 \qquad \text{otherwise.} \end{cases}$$

Theorem (Tseng)

Given $(\mathcal{A}, \varphi, \varphi')$ incps, if $(a_{ij}^{(1)})_{i,j=1}^N$ and $(a_{ij}^{(2)})_{i,j=1}^N$ are infinitesimally free w.r.t (φ, φ'), then $[a_{ij}^{(1)}]_{i,j=1}^N$ and $[a_{ij}^{(2)}]_{i,j=1}^N$ are infinitesimally free w.r.t $(\mathbb{E}, \mathbb{E}')$.

Linearization Trick

Let $P \in \mathbb{C}\langle X_1, X_2, \cdots, X_n\rangle$. A Matrix $\mathcal{L}_P = \begin{array}{cc} & \begin{array}{cc} \overbrace{\ }^{1} & \overbrace{\quad}^{N-1} \end{array} \\ \begin{array}{c} \}1 \\ \}N-1 \end{array} & \begin{bmatrix} 0 & u \\ v & Q \end{bmatrix} \end{array} \in M_N(\mathbb{C}\langle X_1, X_2, \cdots, X_n\rangle)$

where Q is invertible is called a linearization of P if

1° there are Matrices $b_0, b_1, \cdots, b_n \in M_N(\mathbb{C})$ such that $\mathcal{L}_P = b_0 \otimes 1 + b_1 \otimes X_1 + \cdots + b_n \otimes X_n$

2° $P = -uQ^{-1}v$.

- $\mathcal{A}$: unital $*$-algebra. $x_1, x_2, \cdots, x_n \in \mathcal{A}$ are self-adjoint. $P \in \mathbb{C}\langle X_1, \cdots X_n\rangle$ is self-adjoint. Then P admits a self-adjoint linearization $\mathcal{L}_P$. Moreover, $(z-P)^{-1}$ exists in $\mathcal{A}$ $\iff$ $(\Lambda(z) - \mathcal{L}_P)^{-1}$ exists in $M_N(\mathcal{A})$ where $\Lambda(z) = \begin{bmatrix} z & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ \vdots & & \bigcirc & & \\ 0 & & & & \end{bmatrix}$.

- $(\mathcal{A}, \varphi)$: C^*- Prob space. $x_1, x_2, \cdots, x_n \in \mathcal{A}$. $P \in \mathbb{C}\langle X_1, X_2, \cdots, X_n\rangle$ is self-adjoint. For $\varepsilon > 0$, $z \in \mathbb{C}^+$. We have $(z-P)^{-1}$ exists in $\mathcal{A}$ and $(\Lambda_\varepsilon(z) - \mathcal{L}_P)^{-1}$ exists in $M_N(\mathcal{A})$ where $\Lambda_\varepsilon(z) = \begin{bmatrix} z & & & \\ & i\varepsilon & & \\ & & i\varepsilon & \\ & & & \ddots \\ & & & & i\varepsilon \end{bmatrix}$. Also, $\lim_{\varepsilon \downarrow 0} \left[\mathbb{E}\left[(\Lambda_\varepsilon(z) - \mathcal{L}_P)^{-1}\right]\right]_{1,1} = G_P(z)$.

Linearization Trick

For a given $P \in \mathbb{C}\langle X_1, X_2, \dots, X_n\rangle$, how to find a linearization of P ?

— $P = X_j \in \mathbb{C}\langle X_1, X_2, \dots, X_n\rangle \longrightarrow \mathcal{L}_P = \begin{bmatrix} 0 & X_j \\ 1 & -1 \end{bmatrix} \in M_2(\mathbb{C}\langle X_1, \dots, X_n\rangle)$.

— $P = X_{i_1} X_{i_2} \cdots X_{i_k} \in \mathbb{C}\langle X_1, X_2, \dots, X_n\rangle \longrightarrow \mathcal{L}_P = \begin{bmatrix} & & & & X_{i_1} \\ & & & X_{i_2} & -1 \\ & & \iddots & \iddots & \\ & X_{i_{k-1}} & -1 & & \\ X_{i_k} & -1 & & & \end{bmatrix} \in M_k(\mathbb{C}\langle X_1, \dots, X_n\rangle)$

— $P = P_1 + P_2 + \cdots + P_k$ where $P_j \in \mathbb{C}\langle X_1, \dots X_n\rangle$ and $\mathcal{L}_{P_j} = \begin{bmatrix} 0 & u_j \\ v_j & Q_j \end{bmatrix} \in M_{N_j}(\mathbb{C}\langle X_1, \dots, X_n\rangle)$

$$\longrightarrow \mathcal{L}_P = \begin{bmatrix} & u_1 & u_2 & \cdots & u_k \\ v_1 & Q_1 & & & \\ v_2 & & Q_2 & & \\ \vdots & & & \ddots & \\ v_k & & & & Q_k \end{bmatrix} \in M_{(N_1 + \cdots + N_k) - k + 1}(\mathbb{C}\langle X_1, \dots, X_n\rangle)$$

For $P \in \mathbb{C}\langle X_1, X_2, \dots, X_n\rangle$ self-adjoint, let $P = \frac{P}{2} + \left(\frac{P}{2}\right)^*$.

and let $\mathcal{L}_{P/2} = \begin{bmatrix} 0 & u \\ v & Q \end{bmatrix} \in M_N(\mathbb{C}\langle X_1, \dots X_n\rangle)$ with $\frac{P}{2} = -uQ^{-1}v$.

$$\longrightarrow \mathcal{L}_P = \begin{bmatrix} 0 & u & v^* \\ u^* & 0 & Q^* \\ v & Q & 0 \end{bmatrix} \in M_{2N-1}(\mathbb{C}\langle X_1, \dots, X_n\rangle).$$

Linearization Trick (The Algorithm Solution)

$(\mathcal{A}, \varphi)$: C^* Prob space, $x_1, x_2, \cdots, x_n \in \mathcal{A}$ are self-adjoint which are free.

$p \in \mathbb{C}\langle X_1, \cdots, X_n \rangle$ is self-adjoint.

Step 1.

$P = p(x_1, x_2, \cdots, x_n)$

$\downarrow$

$L_p = b_0 \otimes 1 + \cdots + b_n \otimes x_n$

where b_i is self-adjoint for $i = 0, 1, 2, \cdots, n$.

$\Rightarrow$

Step 2

$x_1, x_2, \cdots, x_n$ free

$\downarrow$

$b_1 \otimes x_1, \cdots, b_n \otimes x_n$ free w.r.t $(M_N(\mathcal{A}), \mathbb{E})$

and

$$G_{b_j \otimes x_j}(b) = \lim_{\varepsilon \downarrow 0} \frac{-1}{\pi} \int (b - t b_j)^{-1} \operatorname{Im} G_{x_j}(t + i\varepsilon)\, dt$$

for all $b \in H^+(M_N(\mathbb{C}))$.

$\Rightarrow$

Step 3.

$L_p - b_0 \otimes 1 = b_1 \otimes x_1 + \cdots + b_n \otimes x_n$

and

$$G_{L_p}(b) = G_{L_p - b_0 \otimes 1}(b - b_0) = G_{b_1 \otimes x_1 + \cdots + b_n \otimes x_n}(b - b_0)$$

for all $b \in H^+(M_N(\mathbb{C}))$

$\Downarrow$

Step 4.

$$G_p(z) = \lim_{\varepsilon \downarrow 0} \left[G_{L_p}(\Lambda_\varepsilon(z)) \right]_{1,1}$$

$\downarrow$

Stieltjes Inversion Formula

Theorem (Tseng)

Given $(\mathcal{A}, \varphi, \varphi')$ incps, if $(a_{ij}^{(1)})_{i,j=1}^{N}$ and $(a_{ij}^{(2)})_{i,j=1}^{N}$ are infinitesimally free w.r.t (φ, φ') then $[a_{ij}^{(1)}]_{i,j=1}^{N}$ and $[a_{ij}^{(2)}]_{i,j=1}^{N}$ are infinitesimally free w.r.t $(\mathbb{E}, \mathbb{E}')$.

(Sketch of Proof).

- Define the operator-valued infinitesimal cumulants $\kappa_n' : M_N(\mathcal{A}) \to M_N(\mathbb{C})$ by

$$\kappa_n'(a_1, a_2, \cdots, a_n) = \sum_{\pi \in NC(n)} \mu(\pi, 1) \cdot \sum_{V \in \pi} \mathbb{E}'_{\pi, V}(a_1, a_2, \cdots, a_n)$$

- $\mathcal{A}_1, \mathcal{A}_2, \cdots \mathcal{A}_K \subseteq M_N(\mathcal{A})$ are infinitesimally free w.r.t $(\mathbb{E}, \mathbb{E}')$

  if and only if the mixed cumulants κ_n and mixed infinitesimal cumulants κ_n' both vanish, $n \geq 2$.

- Let $A_1 = [a_{ij}^{(1)}]_{i,j=1}^{N}$, $A_2 = [a_{ij}^{(2)}]_{i,j=1}^{N}$. Then A_1, A_2 are free $\Rightarrow$ mixed cumulants $\kappa_n = 0$.

$$\left.\begin{array}{l} [\kappa_n'(A_{i_1}, \cdots, A_{i_K})]_{ij} = \displaystyle\sum_{j_2 \cdots j_n = 1}^{n} \kappa_n'\left(a_{ij_2}^{(i_1)}, a_{j_2 j_3}^{(i_2)}, \cdots, a_{j_K j}^{(i_n)}\right) \\ (a_{ij}^{(1)})_{i,j=1}^{N} \ \& \ (a_{ij}^{(2)})_{i,j=1}^{N} \text{ are inf. free} \end{array}\right\} \Rightarrow \text{mixed inf. cumulants } \kappa_n' = 0.$$