Idempotents, topologies and ideals

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Weakly almost periodic representations

(G, τ_G) – topological group,

Representation on a Banach space $\mathcal{X}$: strong operator continuous homomorphism $\pi : G \to \mathrm{Is}(\mathcal{X})$ (invertible isometries on $\mathcal{X}$).
Weak operator continuous $\Rightarrow$ strong operator continuous if

- π unitary on Hilbert (folklore), $\mathcal{X}$ reflexive [Megrelishvili '98];
- τ_G locally compact [Johnson '74].

Defintion

A representation $\pi : G \to \mathrm{Is}(\mathcal{X})$ is called weakly almost periodic (w.a.p.) if $\overline{\pi(G)\xi}^{w}$ is weakly compact for each ξ in $\mathcal{X}$.

Equivalently, $\overline{\pi(G)}^{wot}$ is weak operator compact in $\mathcal{B}(\mathcal{X})$.

$\mathcal{X}$ reflexive: any π weakly almost periodic

Weakly almost periodic part

$\pi : G \to \mathrm{Is}(\mathcal{X})$ representation:

$$\mathcal{X}^{\pi}_{\mathcal{W}} = \{\xi : \overline{\pi(G)\xi}^{w} \text{ is weakly compact}\}$$

is a closed subspace of $\mathcal{X}$.

In particular, in bounded continuous functions $\mathcal{CB}(G)$, let

$$\mathcal{LUC}(G) = \{f \in \mathcal{CB}(G) : s \mapsto f(s^{-1}\cdot) \text{ continuous}\}$$
$$\mathcal{W}(G) = \mathcal{LUC}(G)_{\mathcal{W}} \text{ (translation-invariant C*-subalgebra)}$$

(really $\mathcal{W}(G) = \mathcal{CB}(G)_{\mathcal{W}}$, also $\mathcal{W}(G) = \mathcal{RUC}(G)_{\mathcal{W}}$ with right translations), and we call the latter space that of <u>weakly almost periodic</u> functions.

A decomposition theorem

Theorem [Jacobs '54, Dye '65, Bergelson-Rosenblatt '88]

$\pi : G \to \mathrm{Is}(\mathcal{X})$ w.a.p. representation (unitary on Hilbert space)
Then $\mathcal{X}$ decomposes as two π-invariant (hence reducing) subspaces

$$\mathcal{X} = \mathcal{X}^{\pi}_{ret} \oplus \mathcal{X}^{\pi}_{wm}$$

where

$$\mathcal{X}^{\pi}_{ret} = \{\xi \in \mathcal{X} : \xi \in \overline{\pi(G)\eta}^{w} \text{ whenever } \eta \in \overline{\pi(G)\xi}^{w}\}$$
$$\mathcal{X}^{\pi}_{wm} = \{\xi \in \mathcal{X} : 0 \in \overline{\pi(G)\xi}^{w}\}$$

are spaces of "return" and "weakly mixing" vectors.

Decomposition of functions

Theorem [Eberlein '56, de Leeuw-Glicksberg '61]

$$\mathcal{W}(G) = \mathcal{AP}(G) \oplus \mathcal{W}_0(G)$$

where

$$\mathcal{AP}(G) = \{u \in \mathcal{C}_b(G) : \overline{\{f(s^{-1}\cdot) : s \in G\}}^{\|\cdot\|_\infty} \text{ compact}\}$$
$$\mathcal{W}_0(G) = \{u \in \mathcal{W}(G) : m(|u|) = 0\} \lhd \mathcal{W}(G)$$

for the unique invariant mean m on $\mathcal{W}(G)$, with $\mathcal{W}_0(G) \lhd \mathcal{W}(G)$.

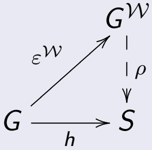
Here, $\mathcal{W}_0(G) = \mathcal{W}(G)_{wm}$.

Semitopological compactifications

$G^{\mathcal{W}}$ – Gelfand spectrum of $\mathcal{W}(G)$, $\varepsilon^{\mathcal{W}} : G \to G^{\mathcal{W}}$ evaluation map

Proposition (folklore)

• $G^{\mathcal{W}}$ semigroup: unique extension of multiplication from dense subgroup $\varepsilon^{\mathcal{W}}(G)$.
• $G^{\mathcal{W}}$ <u>semitopological</u>: $s \mapsto st$, $t \mapsto st$ each continuous.
• Universal property: $h : G \to S$ continuous homo'm into compact semitop'l semigroup S, then $\mathcal{C}(S) \circ h \subseteq \mathcal{W}(G)$, which induces restriction $\rho : G^{\mathcal{W}} \to S$ with



i.e. $\rho \circ \varepsilon^{\mathcal{W}} = h$.

Topologies

Definition: weakly almost periodic topologies

$$\mathcal{T}(G) = \{\tau \subseteq \tau_G : (G,\tau) \text{ top'l group with } \tau = \sigma(G, \mathcal{W}^\tau(G))\}$$

where $\mathcal{W}^\tau(G) = \mathcal{W}(G) \cap \mathcal{C}_b^\tau(G)$ (τ-continuous elements)

Assumption: $\tau_G \in \mathcal{T}(G)$ and is Hausdorff

Warning: not all elements of $\mathcal{T}(G)$ are Hausdorff

e.g. $\tau_{triv} = \{\varnothing, G\}$, often $\tau_{ap} = \sigma(G, \mathcal{AP}(G))$

Examples:

$$\mathcal{T}_{lc}(G) = \{\sigma(G,\{h\}) \,|\, (H,\tau_H) \text{ loc. compact, } h : G \to H \text{ cts. homo'm}\}$$

$$\mathcal{T}_u(G) = \{\sigma(G,\{\pi\}) \,|\, \pi : G \to (\mathrm{Un}(\mathcal{H}), wot) \text{ cts. unitary rep'n}\}$$

$$\mathcal{T}(G) = \{\sigma(G,\{\pi\}) \,|\, \pi : G \to (\mathrm{Is}(\mathcal{X}), wot) \text{ cts. rep'n, } \mathcal{X} \text{ reflexive}\}$$

[Stern '94, Megrelishvili '98]

$$\mathcal{T}_{lc}(G) \subseteq \mathcal{T}_u(G) \subseteq \mathcal{T}(G)$$

On the scope of the classes of topologies

- [Teleman '57] G any topological group: $\varepsilon^{\mathcal{LUC}} : G \to G^{\mathcal{LUC}}$ (spectrum of $\mathcal{LUC}(G)$) homeomorphic embedding onto its range, $G^{\mathcal{LUC}}$ left topological semigroup.
- [Megrelishvili '01] $\mathcal{T}(\mathrm{Homeo}^+[0,1]) = \{\tau_{triv}\}$.
- [Ferri-Galindo '07] $G = (c_0, +)$ (norm topology), $\tau_G \notin \mathcal{T}(G)$.
- [Megrelishvili '02] $G = (L^4[0,1], +)$ (norm topology): $\tau_G \in \mathcal{T}(G) \setminus \mathcal{T}_u(G)$.
- After [Schoenberg '38], $G = (\ell^1, +)$ (norm topology): $\tau_G \in \mathcal{T}_u(G)$ as $e^{-\|\cdot\|_1{}^2} \in \mathrm{P}(G)$.

Conclusions: say $G = \mathbb{Z}^{\oplus \mathbb{N}} = (\mathbb{Z}^2)^{\oplus \mathbb{N}} \hookrightarrow \mathbb{R}^{\oplus \mathbb{N}}$
(i) $\mathcal{T}_{lc}(G) \subsetneq \mathcal{T}_u(G) \subsetneq \mathcal{T}(G) \subsetneq \{\text{group topologies}\}$.
(ii) Quotient groups of unitarizable groups may not be unitarizable.

- [Mayer '97] $G = N \rtimes R$ (certain connected Lie), $\mathcal{T}_{lc}(G) = \mathcal{T}(G)$.

Co-Cauchy/co-compact topologies, after [Ruppert '90]

$\tau \in \mathcal{T}(G)$
$G^{\mathcal{W}^\tau}$ – Gelfand spectrum of $\mathcal{W}^\tau(G)$, $\varepsilon^{\mathcal{W}^\tau} : G \to G^{\mathcal{W}^\tau}$ – evaluation

Completion: $G_\tau = G^{\mathcal{W}^\tau}(\varepsilon^{\mathcal{W}^\tau}(e_G))$ – intrinsic group at identity
G_τ complete w.r.t. 2-sided unifomity

$$\tau \subseteq \tau' \text{ in } \mathcal{T}(G) \Rightarrow \mathcal{W}^\tau(G) \subseteq \mathcal{W}^{\tau'}(G), \text{ induces } \rho_\tau^{\tau'} : G^{\mathcal{W}^{\tau'}} \to G^{\mathcal{W}^\tau}$$
$$\Rightarrow \eta_\tau^{\tau'} = \rho_\tau^{\tau'}|_{G_{\tau'}} : G_{\tau'} \to G_\tau \text{ cts. homo'm, dense range}$$

Lemma (after [Ruppert '90] in abelian case)

For $\tau \subseteq \tau'$ in $\mathcal{T}(G)$ TFAE
(co-compact) $\eta_\tau^{\tau'} : G_{\tau'} \to G_\tau$ open with $\ker \eta_\tau^{\tau'}$ compact
(co-Cauchy) each τ-Cauchy filter admits a τ'-Cauchy refinement

Write $\tau \subseteq_c \tau'$, in this case.

Idempotents

Definition

$\mathrm{ZE}(G^{\mathcal{W}}) = \{e \in G^{\mathcal{W}} : e^2 = e \ \& \ e\varepsilon^{\mathcal{W}}(s) = \varepsilon^{\mathcal{W}}(s)e \ \ \forall s \in G\}.$

In $\mathrm{ZE}(G^{\mathcal{W}})$: $e \leq e' \Leftrightarrow ee' = e$

A Galois connection

Theorem (after [Ruppert '90]; he covers abelian case)

There are maps $T : \mathrm{ZE}(G^{\mathcal{W}}) \to \mathcal{T}(G)$ and $E : \mathcal{T}(G) \to \mathrm{ZE}(G^{\mathcal{W}})$ s.t.

$$
\begin{aligned}
&T(e) \subseteq T(e') \text{ if } e \leq e' \\
&E(\tau) \leq E(\tau') \text{ if } \tau \subseteq \tau' \\
&E(\tau) = E(\tau') \text{ if } \tau \subseteq_c \tau' \\
&E \circ T = \mathrm{id}_{\mathrm{ZE}(G^{\mathcal{W}})} \text{ and } \tau \subseteq_c T \circ E(\tau).
\end{aligned}
$$

Thus, if $\overline{\mathcal{T}}(G) = T(\mathrm{ZE}(G^{\mathcal{W}}))$, then $T \circ E|_{\overline{\mathcal{T}}(G)} = \mathrm{id}_{\overline{\mathcal{T}}(G)}$.

- (E, T) is a Galois connection for p.o. sets $(\mathcal{T}(G), \mathrm{ZE}(G^{\mathcal{W}}))$.
- $T \circ E : \mathcal{T}(G) \to \overline{\mathcal{T}}(G)$ is a closure operator.

Idea of proof

- Definition of T. For $e \in \mathrm{ZE}(G^{\mathcal{W}})$ let

$$T(e) = \sigma(G, \{s \mapsto e\varepsilon^{\mathcal{W}}(s) \in G^{\mathcal{W}}(e)\}).$$

- Defintion of E. For $\tau \in \mathcal{T}(G)$ let $\rho_\tau : G^{\mathcal{W}} \to G^{\mathcal{W}^\tau}$, given by restriction to $\mathcal{W}^\tau(G) \subseteq \mathcal{W}(G)$, and

$$S_\tau = \rho_\tau^{-1}(\{\varepsilon^{\mathcal{W}^\tau}(e_G)\}) \subseteq G^{\mathcal{W}}$$

which is a closed subsemigroup. [Ruppert's Book '90]: the minimal ideal $K(S_\tau)$ of S_τ is unique and is a group, with identity $E(\tau)$. I.e.

$$E(\tau) = \min \mathrm{E}(S_\tau) \quad \Rightarrow \quad E(\tau) \in \mathrm{ZE}(G^{\mathcal{W}}).$$

Picture of $G^{\mathcal{W}}$

If $\tau \in \overline{\mathcal{T}}(G)$ then

$$G^{\mathcal{W}^\tau} \cong E(\tau)G^{\mathcal{W}} \text{ (compression of } G^{\mathcal{W}})$$
$$G_\tau = G^{\mathcal{W}}(E(\tau)) \text{ (intrinsic group at } E(\tau))$$

Further, if $\tau \in \mathcal{T}(G)$

$$K_\tau = K(S_\tau) \cong \ker \eta_\tau^{T \circ E(\tau)}$$

is centric in $G^{\mathcal{W}}$, and letting m_{K_τ} be normalized Haar measure we have in convolution on $\mathcal{W}(G)^* \cong \mathrm{M}(G^{\mathcal{W}})$ that $m_{K_\tau} \leq E(\tau)$ and

$$G^{\mathcal{W}^\tau} \cong E(\tau)G^{\mathcal{W}}/K_\tau \cong m_{K_\tau} * G^{\mathcal{W}} \text{ (averaged over } K_\tau)$$
$$G_\tau = G^{\mathcal{W}}(E(\tau))/K_\tau \cong m_{K_\tau} * G^{\mathcal{W}}(E(\tau))$$

Ideals

Definition

An ideal $\mathcal{J}$ of $\mathcal{W}(G)$ is called an Eberlein-de Leeuw-Glicksberg (E-dL-G) ideal provided

- $\mathcal{J}$ is translation invariant; and
- $\mathcal{J}$ admits a linear complement $\mathcal{A}$, a C*-subalgebra of $\mathcal{W}(G)$.

Main Theorem on Ideals

(i) Let $\tau \in \overline{\mathcal{T}}(G)$, then $\mathcal{W}^{\tau}(G) = E(\tau) \cdot \mathcal{W}(G)$ and

$$\mathcal{I}(\tau) = \{u \in \mathcal{W}(G) : E(\tau) \cdot u = 0\}$$

is an E-dL-G ideal. Further

$$\mathcal{W}(G) = \mathcal{W}^{\tau}(G) \oplus \mathcal{I}(\tau).$$

(ii) Any E-dL-G ideal of $\mathcal{W}(G)$ is of the form $\mathcal{I}(\tau)$, as above.

Some decompositions

Lemma

Given $\tau \in \overline{\mathcal{T}}(G)$, $e_G \in U \in \tau$, $\varepsilon > 0$ and $u_1, \ldots, u_n$ in $\mathcal{I}(\tau)$, there is $s \in U$ s.t. $|u_j(s)| < \varepsilon$ for $j = 1, \ldots, n$.

Theorem

Given a w.a.p. rep'n $\pi : G \to \mathrm{Is}(\mathcal{X})$, $\tau \in \overline{\mathcal{T}}(G)$, the spaces

$$\mathcal{X}^{\pi}_{\tau} = \{\xi \in \mathcal{X} : \pi(\cdot)\xi \text{ is } \tau\text{-continuous}\}$$
$$\mathcal{X}^{\pi}_{\tau\perp} = \{\xi \in \mathcal{X} : 0 \in \overline{\pi(U)\xi}^{w} \text{ for each } e \in U \in \tau\}$$

are π-reducing with $\mathcal{X} = \mathcal{X}^{\pi}_{\tau} \oplus \mathcal{X}^{\pi}_{\tau\perp}$.

Corollary (refinement of Jacobs, Dye, Bergelson-Rosenblatt)

$$\mathcal{X}^{\pi}_{wm} = \mathcal{X}^{\pi}_{\tau_{ap}\perp} = \{\xi \in \mathcal{X} : 0 \in \overline{\pi(U)\xi}^{w} \text{ for each } e \in U \in \tau_{ap}\}$$

Some more decompositions

If $\tau \in \mathcal{T}(G) \setminus \overline{\mathcal{T}}(G)$, we can average π over K_τ to get:

Theorem

Given a w.a.p. rep'n $\pi : G \to \mathrm{Is}(\mathcal{X})$ the space

$$\mathcal{X}^\pi_\tau = \{\xi \in \mathcal{X} : \pi(\cdot)\xi \text{ is } \tau\text{-continuous}\}$$

is π-reducing.

Examples

- [Segal-von Neumann '50] If $\pi : G_d \to \mathrm{Is}(\mathcal{X})$ is w.a.p., then $\mathcal{X}^\pi_{\tau_G}$ is reducing in $\mathcal{X}$; e.g. $\mathcal{W}(G)$ reducing in $\mathcal{W}(G_d)$.
- (after [Lau-Losert '90]) If $N \lhd G$ (and is closed)

$$\mathcal{X}^\pi_{\tau_{G:N}} = \{\xi \in \mathcal{X} : \pi(n)\xi = \xi \text{ for } n \text{ in } N\}$$

is π-reducing in $\mathcal{X}$.

Unitarizable topologies

$$\mathcal{T}_u(G) = \{\tau \in \mathcal{T}(G) : \tau = \sigma(G, \mathrm{P}^\tau(G))\}$$

where $\mathrm{P}^\tau(G) = \{u \in \mathcal{C}_b^\tau(G) : u \text{ positive definite}\}$.

Let $\varpi_\tau = \bigoplus_{u \in \mathrm{P}^\tau(G)} \pi_u$ (GNS), so $\sigma(G, \mathrm{P}^\tau(G)) = \sigma(G, \{\varpi_\tau\})$.

Assume: $\tau_G \in \mathcal{T}_u(G)$.

Let $\varpi = \varpi_{\tau_G}$. $G^\varpi = \overline{\varpi(G)}^{wot}$ is a semitopological semigroup.

Theorem (Galois connection, revisited)

There are two order preserving maps

$$P : \mathcal{T}_u(G) \to \mathrm{ZE}(G^\varpi), \quad T_u : \mathrm{ZE}(G^\varpi) \to \mathcal{T}_u(G)$$

so $\tau \subseteq_c T_u \circ P(\tau)$ for each τ in $\mathcal{T}_u(G)$.

Let $\overline{\mathcal{T}}_u(G) = T_u \circ P(\mathcal{T}_u(G))$.

E-dL-G ideals in Fourier-Steiltjes algebras

$\mathrm{B}(G) = \mathrm{span}\mathrm{P}(G) \cong \mathrm{W}^*(G) = \varpi(G)''$.

$$\varpi(s) \mapsto \varpi(s) \otimes \varpi(s) \text{ extends to } \mathrm{W}^*(G) \to \mathrm{W}^*(G)\overline{\otimes}\mathrm{W}^*(G).$$

Preadjoint makes $\mathrm{B}(G)$ Banach algebra of continuous functions on G; see also [Lau-Ludwig '12].

Theorem

If $\tau \in \overline{\mathcal{T}}_u(G)$ then

$$\mathrm{B}^\tau(G) := P(\tau) \cdot \mathrm{B}(G) = \{u \in \mathrm{B}(G) : u \text{ is } \tau\text{-continuous}\}$$
$$\mathrm{I}(\tau) := (I - P(\tau)) \cdot \mathrm{B}(G) \lhd \mathrm{B}(G).$$

Moreover

$$\mathrm{B}(G) = \mathrm{B}^\tau(G) \oplus_{\ell^1} \mathrm{I}(\tau)$$

is the direct sum of a translation-invariant subalgebra and a translation invariant ideal.

Application: Operator amenability of $\mathrm{B}(G)$

Operator amenability ... is a certain "averaging property" for a Banach algebra with cooperative operator space structure.

Theorem

For locally comapct G, TFAE:
(i) G is amenable;
(ii) [Johnson] $\mathrm{L}^1(G)$ is (operator) amenable; and
(iii) [Ruan] $\mathrm{A}(G)$ is operator amenable.

Theorem [Dales, Ghahramani & Helemskiĭ]

For locally compact G:

$\mathrm{M}(G)$ is (operator) amenable $\Leftrightarrow$ G is discrete and amenable.

Naïve conjecture

$\mathrm{B}(G)$ is operator amenable $\Leftrightarrow$ G is compact.

On operator amenability of $\mathrm{B}(G)$

Theorem [Runde-S.]

$\mathrm{B}(\mathbb{Q}_p \rtimes \mathbb{O}_p^\times)$ is operator amenable.

Theorem

$\mathrm{B}(G)$ operator amenable $\Rightarrow |\overline{\mathcal{T}}_u(G)| < \infty$.

Theorem

If G locally compact and connected, then

$$\mathrm{B}(G) \text{ is operator amenable } \Leftrightarrow \; G \text{ is compact.}$$

Shameless advertising

N.S., Weakly almost periodic topologies, idempotents and ideals, arXiv:1805.09892

N.S., On operator amenability of Fourier-Stieltjes algebras, arXiv:1806.08421

Happy birthday, Tony!
(To the next 75 years ...)

Xie! Xie!

(Thank-you!)