On a notion of closeness of groups

Chi-Wai Leung 梁子威 (CUHK), Chi-Keung Ng 吳志強 (Nankai) and Ngai-Ching Wong 黃毅青*

Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, 80424, Taiwan.
台灣 · 中山大學 · 應用數學系
wong@math.nsysu.edu.tw
http:\\www.math.nsysu.edu.tw\ ∼wong

To be or not to be, . . .

the question is how to say two groups G, H are close, or not close.

Question. Is an apple close to a mango?

We need to put G, H into the same context to compare!

To be or not to be, . . .

the question is how to say two groups G, H are close, or not close.

Question. Is an apple close to a mango?

We need to put G, H into the same context to compare!

To be or not to be, . . .

the question is how to say two groups G, H are close, or not close.

Question. Is an apple close to a mango?

We need to put G, H into the same context to compare!

To be or not to be, . . .

the question is how to say two groups G, H are close, or not close.

Question. Is an apple close to a mango?

We need to put G, H into the same context to compare!

Unitary representations of groups

Let $\mathfrak{H}$ be a complex Hilbert space and $U(\mathfrak{H})$ the group of unitary operators on $\mathfrak{H}$.

$\mathfrak{H} = \mathbb{C}^n \implies U(\mathfrak{H}) = SU(n)$, the group of $n \times n$ unitary matrices.

Call $(\mathfrak{H}, \varphi)$ a (unitary) representation of a group G if $\varphi : G \to U(\mathfrak{H})$ is an injective group homomorphism.

Note that there is a metric structure on $U(\mathfrak{H})$, and thus so is $\varphi(G)$.

Unitary representations of groups

Let $\mathfrak{H}$ be a complex Hilbert space and $U(\mathfrak{H})$ the group of unitary operators on $\mathfrak{H}$.

$\mathfrak{H} = \mathbb{C}^n \implies U(\mathfrak{H}) = SU(n)$, the group of $n \times n$ unitary matrices.

Call $(\mathfrak{H}, \varphi)$ a (unitary) representation of a group G if $\varphi : G \to U(\mathfrak{H})$ is an injective group homomorphism.

Note that there is a metric structure on $U(\mathfrak{H})$, and thus so is $\varphi(G)$.

Unitary representations of groups

Let $\mathfrak{H}$ be a complex Hilbert space and $U(\mathfrak{H})$ the group of unitary operators on $\mathfrak{H}$.

$\mathfrak{H} = \mathbb{C}^n \implies U(\mathfrak{H}) = SU(n)$, the group of $n \times n$ unitary matrices.

Call $(\mathfrak{H}, \varphi)$ a (unitary) representation of a group G if $\varphi : G \to U(\mathfrak{H})$ is an injective group homomorphism.

Note that there is a metric structure on $U(\mathfrak{H})$, and thus so is $\varphi(G)$.

Unitary representations of groups

Let $\mathfrak{H}$ be a complex Hilbert space and $U(\mathfrak{H})$ the group of unitary operators on $\mathfrak{H}$.

$\mathfrak{H} = \mathbb{C}^n \implies U(\mathfrak{H}) = SU(n)$, the group of $n \times n$ unitary matrices.

Call $(\mathfrak{H}, \varphi)$ a (unitary) representation of a group G if $\varphi : G \to U(\mathfrak{H})$ is an injective group homomorphism.

Note that there is a metric structure on $U(\mathfrak{H})$, and thus so is $\varphi(G)$.

Distance between groups

Let $(\mathfrak{H}, \varphi)$ be a representation of G.
Let $(\mathfrak{H}, \psi)$ be a representation of H.

The distance between $\varphi(G)$ and $\psi(H)$ in $U(\mathfrak{H})$ is

$$d_{(\varphi,\psi)}(G,H) := \max\left\{ \sup_{g\in G} \inf_{h\in H} \|\varphi(g)-\psi(h)\|, \sup_{h\in H} \inf_{g\in G} \|\psi(h)-\varphi(g)\| \right\}.$$

The distance between two groups G and H is

$$d(G,H) := \inf_{\varphi,\psi} d_{(\varphi,\psi)}(G,H).$$

Distance between groups

Let $(\mathfrak{H}, \varphi)$ be a representation of G.
Let $(\mathfrak{H}, \psi)$ be a representation of H.

The distance between $\varphi(G)$ and $\psi(H)$ in $U(\mathfrak{H})$ is

$$d_{(\varphi,\psi)}(G, H) := \max\left\{ \sup_{g\in G} \inf_{h\in H} \|\varphi(g) - \psi(h)\|, \sup_{h\in H} \inf_{g\in G} \|\psi(h) - \varphi(g)\| \right\}.$$

The distance between two groups G and H is

$$d(G, H) := \inf_{\varphi,\psi} d_{(\varphi,\psi)}(G, H).$$

Distance between groups

Let $(\mathfrak{H}, \varphi)$ be a representation of G.
Let $(\mathfrak{H}, \psi)$ be a representation of H.

The distance between $\varphi(G)$ and $\psi(H)$ in $U(\mathfrak{H})$ is

$$d_{(\varphi,\psi)}(G,H) := \max\left\{ \sup_{g\in G} \inf_{h\in H} \|\varphi(g)-\psi(h)\|, \sup_{h\in H} \inf_{g\in G} \|\psi(h)-\varphi(g)\| \right\}.$$

The distance between two groups G and H is

$$d(G,H) := \inf_{\varphi,\psi} d_{(\varphi,\psi)}(G,H).$$

Distance between groups

Let $(\mathfrak{H}, \varphi)$ be a representation of G.
Let $(\mathfrak{H}, \psi)$ be a representation of H.

The distance between $\varphi(G)$ and $\psi(H)$ in $U(\mathfrak{H})$ is

$$d_{(\varphi,\psi)}(G, H) := \max\left\{ \sup_{g \in G} \inf_{h \in H} \|\varphi(g) - \psi(h)\|, \sup_{h \in H} \inf_{g \in G} \|\psi(h) - \varphi(g)\| \right\}.$$

The distance between two groups G and H is

$$d(G, H) \ := \inf_{\varphi,\psi} d_{(\varphi,\psi)}(G, H).$$

1. G is δ-contained in H if $\exists\ (\mathfrak{H}, \phi), (\mathfrak{H}, \psi)$ s.t.

$$\sup_{g \in G} \inf_{h \in H} \|\varphi(g) - \psi(h)\| \leq \delta.$$

2. G is δ-close to H if $\exists\ (\mathfrak{H}, \phi), (\mathfrak{H}, \psi)$ s.t.

$$d_{(\varphi,\psi)}(G, H) \leq \delta.$$

In other words, G is δ-contained in (resp. δ-close to) H if

$$\phi(G) \subseteq \psi(H) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})}$$

(resp.

$$\phi(G) \subseteq \psi(H) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})} \quad \text{and} \quad \psi(H) \subseteq \phi(G) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})})$$

for some representations $(\mathfrak{H}, \varphi), (\mathfrak{H}, \psi)$ of G, H.

1. G is δ-contained in H if $\exists$ $(\mathfrak{H}, \phi), (\mathfrak{H}, \psi)$ s.t.

$$\sup_{g \in G} \inf_{h \in H} \|\varphi(g) - \psi(h)\| \leq \delta.$$

2. G is δ-close to H if $\exists$ $(\mathfrak{H}, \phi), (\mathfrak{H}, \psi)$ s.t.

$$d_{(\varphi,\psi)}(G, H) \leq \delta.$$

In other words, G is δ-contained in (resp. δ-close to) H if

$$\phi(G) \subseteq \psi(H) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})}$$

(resp.

$$\phi(G) \subseteq \psi(H) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})} \quad \text{and} \quad \psi(H) \subseteq \phi(G) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})})$$

for some representations $(\mathfrak{H}, \varphi), (\mathfrak{H}, \psi)$ of G, H.

1. G is δ-contained in H if $\exists\ (\mathfrak{H}, \phi), (\mathfrak{H}, \psi)$ s.t.

$$\sup_{g\in G} \inf_{h\in H} \|\varphi(g) - \psi(h)\| \leq \delta.$$

2. G is δ-close to H if $\exists\ (\mathfrak{H}, \phi), (\mathfrak{H}, \psi)$ s.t.

$$d_{(\varphi,\psi)}(G, H) \leq \delta.$$

In other words, G is δ-contained in (resp. δ-close to) H if

$$\phi(G) \subseteq \psi(H) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})}$$

(resp.

$$\phi(G) \subseteq \psi(H) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})} \quad \text{and} \quad \psi(H) \subseteq \phi(G) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})})$$

for some representations $(\mathfrak{H}, \varphi), (\mathfrak{H}, \psi)$ of G, H.

1. G is δ-contained in H if $\exists\ (\mathfrak{H},\phi),(\mathfrak{H},\psi)$ s.t.

$$\sup_{g\in G}\inf_{h\in H}\|\varphi(g)-\psi(h)\|\leq\delta.$$

2. G is δ-close to H if $\exists\ (\mathfrak{H},\phi),(\mathfrak{H},\psi)$ s.t.

$$d_{(\varphi,\psi)}(G,H)\leq\delta.$$

In other words, G is δ-contained in (resp. δ-close to) H if

$$\phi(G)\subseteq\psi(H)+\delta\mathbf{B}_{\mathcal{L}(\mathfrak{H})}$$

(resp.

$$\phi(G)\subseteq\psi(H)+\delta\mathbf{B}_{\mathcal{L}(\mathfrak{H})}\quad\text{and}\quad\psi(H)\subseteq\phi(G)+\delta\mathbf{B}_{\mathcal{L}(\mathfrak{H})})$$

for some representations $(\mathfrak{H},\varphi),(\mathfrak{H},\psi)$ of G,H.

G is asymptotically contained in (resp., asymptotically close to) H if it is δ-contained in (resp., δ-close to) H for each $\delta > 0$.

Thus, G is asymptotically close to $H \Leftrightarrow d(G, H) = 0$.

Question. So what?

G is asymptotically contained in (resp., asymptotically close to) H if it is δ-contained in (resp., δ-close to) H for each $\delta > 0$.

Thus, G is asymptotically close to $H \Leftrightarrow d(G, H) = 0$.

Question. So what?

G is asymptotically contained in (resp., asymptotically close to) H if it is δ-contained in (resp., δ-close to) H for each $\delta > 0$.

Thus, G is asymptotically close to $H \Leftrightarrow d(G, H) = 0$.

Question. So what?

A good example

(a) $\mathbb{Z}_2$ is not δ-contained in $\mathbb{Z}_3$ if $0 < \delta < 2 - \sqrt{3}$.

In fact, if $\mathfrak{H}$ is a Hilbert space and $u, v \in U(\mathfrak{H}) \setminus \{1\}$ with $u^2 = 1$ and $v^3 = 1$, then

$$\|1 - u\| = 2 \quad \text{and} \quad \|1 - v\| = \sqrt{3}.$$

Thus,

$$\|u - v\| \geq \|1 - u\| - \|1 - v\| \geq 2 - \sqrt{3}.$$

A good example

(a) $\mathbb{Z}_2$ is not δ-contained in $\mathbb{Z}_3$ if $0 < \delta < 2 - \sqrt{3}$.

In fact, if $\mathfrak{H}$ is a Hilbert space and $u, v \in U(\mathfrak{H}) \setminus \{1\}$ with $u^2 = 1$ and $v^3 = 1$, then

$$\|1 - u\| = 2 \quad \text{and} \quad \|1 - v\| = \sqrt{3}.$$

Thus,

$$\|u - v\| \geq \|1 - u\| - \|1 - v\| \geq 2 - \sqrt{3}.$$

A bad example

(b) Let $t \in (0, 1)$ be an irrational number.
Then

$$\theta : n \mapsto e^{2nt\pi \mathrm{i}}$$

is an injective group homomorphism

$$\text{from } \mathbb{Z} \text{ into } \mathbb{T} := U(\mathbb{C}) \; (= \{z \in \mathbb{C} : |z| = 1\})$$

with dense range.

Therefore, $\mathbb{T}$ and $\mathbb{Z}$ are asymptotically close to each other.

Note that $\mathbb{Z}$ is countable but $\mathbb{T}$ is not.

A bad example

(b) Let $t \in (0, 1)$ be an irrational number.
Then

$$\theta : n \mapsto e^{2nt\pi \mathrm{i}}$$

is an injective group homomorphism

$$\text{from } \mathbb{Z} \text{ into } \mathbb{T} := U(\mathbb{C}) \; (= \{z \in \mathbb{C} : |z| = 1\})$$

with dense range.

Therefore, $\mathbb{T}$ and $\mathbb{Z}$ are asymptotically close to each other.

Note that $\mathbb{Z}$ is countable but $\mathbb{T}$ is not.

A bad example

(b) Let $t \in (0, 1)$ be an irrational number.
Then

$$\theta : n \mapsto e^{2nt\pi\mathrm{i}}$$

is an injective group homomorphism

$$\text{from } \mathbb{Z} \text{ into } \mathbb{T} := U(\mathbb{C}) \ (= \{z \in \mathbb{C} : |z| = 1\})$$

with dense range.

Therefore, $\mathbb{T}$ and $\mathbb{Z}$ are asymptotically close to each other.

Note that $\mathbb{Z}$ is countable but $\mathbb{T}$ is not.

An ugly example

(c) Let $n_1, n_2, \ldots$ be a sequence of relatively prime numbers. Consider the canonical injective representation from $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ into $\mathbb{T}$ defined by

$$(\bar{k}_j)_{j\in\mathbb{N}} \mapsto \prod_{j=1}^{\infty} e^{2k_j\pi \mathrm{i}/n_j}.$$

we see that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ is asymptotically contained in $\mathbb{Z}$.

In the case when $\{p_1, p_2, \ldots\}$ list all prime numbers, the image of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is dense in $\mathbb{T}$.

This means that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is asymptotically close to $\mathbb{Z}$.

Note that $\mathbb{Z}$ have a single generator but the minimal number of generators of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is infinite.

An ugly example

(c) Let $n_1, n_2, \ldots$ be a sequence of relatively prime numbers. Consider the canonical injective representation from $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ into $\mathbb{T}$ defined by

$$(\bar{k}_j)_{j\in\mathbb{N}} \mapsto \Pi_{j=1}^{\infty} e^{2k_j\pi\mathrm{i}/n_j}.$$

we see that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ is asymptotically contained in $\mathbb{Z}$.

In the case when $\{p_1, p_2, \ldots\}$ list all prime numbers, the image of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is dense in $\mathbb{T}$.

This means that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is asymptotically close to $\mathbb{Z}$.

Note that $\mathbb{Z}$ have a single generator but the minimal number of generators of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is infinite.

An ugly example

(c) Let $n_1, n_2, \ldots$ be a sequence of relatively prime numbers. Consider the canonical injective representation from $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ into $\mathbb{T}$ defined by

$$(\bar{k}_j)_{j\in\mathbb{N}} \mapsto \Pi_{j=1}^{\infty} e^{2k_j\pi \mathrm{i}/n_j}.$$

we see that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ is asymptotically contained in $\mathbb{Z}$.

In the case when $\{p_1, p_2, \ldots\}$ list all prime numbers, the image of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is dense in $\mathbb{T}$.

This means that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is asymptotically close to $\mathbb{Z}$.

Note that $\mathbb{Z}$ have a single generator but the minimal number of generators of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is infinite.

An ugly example

(c) Let $n_1, n_2, \ldots$ be a sequence of relatively prime numbers. Consider the canonical injective representation from $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ into $\mathbb{T}$ defined by

$$(\bar{k}_j)_{j\in\mathbb{N}} \mapsto \Pi_{j=1}^{\infty} e^{2k_j\pi\mathrm{i}/n_j}.$$

we see that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ is asymptotically contained in $\mathbb{Z}$.

In the case when $\{p_1, p_2, \ldots\}$ list all prime numbers, the image of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is dense in $\mathbb{T}$.

This means that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is asymptotically close to $\mathbb{Z}$.

Note that $\mathbb{Z}$ have a single generator but the minimal number of generators of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is infinite.

An ugly example

(c) Let $n_1, n_2, \ldots$ be a sequence of relatively prime numbers. Consider the canonical injective representation from $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ into $\mathbb{T}$ defined by

$$(\bar{k}_j)_{j\in\mathbb{N}} \mapsto \Pi_{j=1}^{\infty} e^{2k_j\pi\mathrm{i}/n_j}.$$

we see that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ is asymptotically contained in $\mathbb{Z}$.

In the case when $\{p_1, p_2, \ldots\}$ list all prime numbers, the image of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is dense in $\mathbb{T}$.

This means that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is asymptotically close to $\mathbb{Z}$.

Note that $\mathbb{Z}$ have a single generator but the minimal number of generators of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is infinite.

An ugly example

(c) Let $n_1, n_2, \dots$ be a sequence of relatively prime numbers. Consider the canonical injective representation from $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ into $\mathbb{T}$ defined by

$$(\bar{k}_j)_{j\in\mathbb{N}} \mapsto \Pi_{j=1}^{\infty} e^{2k_j\pi \mathrm{i}/n_j}.$$

we see that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{n_k}$ is asymptotically contained in $\mathbb{Z}$.

In the case when $\{p_1, p_2, \dots\}$ list all prime numbers, the image of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is dense in $\mathbb{T}$.

This means that $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is asymptotically close to $\mathbb{Z}$.

Note that $\mathbb{Z}$ have a single generator but the minimal number of generators of $\bigoplus_{k\in\mathbb{N}} \mathbb{Z}_{p_k}$ is infinite.

Another bad example

(d) Let $D := \varinjlim \mathbb{Z}_{2^k}$ and $T := \varinjlim \mathbb{Z}_{3^k}$.

Then both of them can be considered as dense subgroups of $\mathbb{T}$ and hence they are asymptotically close to each other.

Note that all elements in both D and T are of finite order, but the order of any element in D is a power of 2 while the order of any element in T is a power of 3.

Another bad example

(d) Let $D := \varinjlim \mathbb{Z}_{2^k}$ and $T := \varinjlim \mathbb{Z}_{3^k}$.

Then both of them can be considered as dense subgroups of $\mathbb{T}$ and hence they are asymptotically close to each other.

Note that all elements in both D and T are of finite order, but the order of any element in D is a power of 2 while the order of any element in T is a power of 3.

Another bad example

(d) Let $D := \varinjlim \mathbb{Z}_{2^k}$ and $T := \varinjlim \mathbb{Z}_{3^k}$.

Then both of them can be considered as dense subgroups of $\mathbb{T}$ and hence they are asymptotically close to each other.

Note that all elements in both D and T are of finite order, but the order of any element in D is a power of 2 while the order of any element in T is a power of 3.

Some positive results

Theorem 1. Any close enough finite abelian groups G, H are isomorphic.

In fact, if G, H are k-bounded abelian group, then

$$G, H \text{ are group isomorphic} \quad \Leftrightarrow \quad d(G, H) = 0.$$

Recall that a group H is called k-bounded for some integer $k \geq 2$ if the order $o(t) \leq k$, for all $t \in H$.

Some positive results

Theorem 1. Any close enough finite abelian groups G, H are isomorphic.

In fact, if G, H are k-bounded abelian group, then

$$G, H \text{ are group isomorphic} \quad \Leftrightarrow \quad d(G, H) = 0.$$

Recall that a group H is called k-bounded for some integer $k \geq 2$ if the order $o(t) \leq k$, for all $t \in H$.

Some positive results

Theorem 1. Any close enough finite abelian groups G, H are isomorphic.

In fact, if G, H are k-bounded abelian group, then

$$G, H \text{ are group isomorphic} \quad \Leftrightarrow \quad d(G, H) = 0.$$

Recall that a group H is called k-bounded for some integer $k \geq 2$ if the order $o(t) \leq k$, for all $t \in H$.

Proof.

First, we note that for any representation $(\psi, \mathfrak{H})$ of an k-bounded group H and any distinct $r, s \in H$,

$$\|\psi(r) - \psi(s)\| \geq 2 \sin \frac{\pi}{k}.$$

The assertion follows from the following inequalities.

$$\|\psi(r) - \psi(s)\| = r_\sigma(1 - \psi(r^{-1}s)) \geq \left|1 - e^{2\pi \mathrm{i}/\mathrm{o}(\mathrm{r}^{-1}\mathrm{s})}\right| \geq 2\sin(\pi/k),$$

where $r_\sigma(\cdot)$ is the spectral radius.

Proof.

First, we note that for any representation $(\psi, \mathfrak{H})$ of an k-bounded group H and any distinct $r, s \in H$,

$$\|\psi(r) - \psi(s)\| \geq 2\sin\frac{\pi}{k}.$$

The assertion follows from the following inequalities.

$$\|\psi(r) - \psi(s)\| = r_\sigma(1 - \psi(r^{-1}s)) \geq \left|1 - e^{2\pi\mathrm{i}/\mathrm{o}(\mathrm{r}^{-1}\mathrm{s})}\right| \geq 2\sin(\pi/k),$$

where $r_\sigma(\cdot)$ is the spectral radius.

Now, assume $d(G,H)=0$, and $0<\delta<\sin\frac{\pi}{k}$.

G is δ-contained in H

$\Longrightarrow$

$\exists$ representations $(\phi,\mathfrak{H}),(\psi,\mathfrak{H})$ of G,H s.t.

$$\phi(G)\subseteq\psi(H)+\delta\mathbf{B}_{\mathcal{L}(\mathfrak{H})}.$$

$\exists$ bijection $\theta:G\to H$ with

$$\|\phi(r)-\psi(\theta(r))\|<\delta,\quad\forall r\in G.$$

Furthermore, θ preserves orders of elements, i.e.,

$$o(\theta(r))=o(r),\quad\forall r\in G.$$

Now, assume $d(G, H) = 0$, and $0 < \delta < \sin \frac{\pi}{k}$.

G is δ-contained in H

$\implies$

$\exists$ representations $(\phi, \mathfrak{H}), (\psi, \mathfrak{H})$ of G, H s.t.

$$\phi(G) \subseteq \psi(H) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})}.$$

$\exists$ bijection $\theta : G \to H$ with

$$\|\phi(r) - \psi(\theta(r))\| < \delta, \quad \forall r \in G.$$

Furthermore, θ preserves orders of elements, i.e.,

$$o(\theta(r)) = o(r), \quad \forall r \in G.$$

Now, assume $d(G, H) = 0$, and $0 < \delta < \sin \frac{\pi}{k}$.

G is δ-contained in H
$\Longrightarrow$
$\exists$ representations $(\phi, \mathfrak{H}), (\psi, \mathfrak{H})$ of G, H s.t.

$$\phi(G) \subseteq \psi(H) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})}.$$

$\exists$ bijection $\theta : G \to H$ with

$$\|\phi(r) - \psi(\theta(r))\| < \delta, \quad \forall r \in G.$$

Furthermore, θ preserves orders of elements, i.e.,

$$o(\theta(r)) = o(r), \quad \forall r \in G.$$

Now suppose that the common order of G and H is

$$n = p_1^{r_1} \cdots p_k^{r_k}.$$

Then we can write G and H as direct sums of their Sylow subgroups

$$G = G(p_1) \oplus \cdots \oplus G(p_k) \quad \text{and} \quad H = H(p_1) \oplus \cdots \oplus H(p_k).$$

Since θ preserves order, θ maps bijectively from the Sylow p_i-subgroup $G(p_i)$ onto the Sylow p_i-subgroup $H(p_i)$ $(i = 1, ..., k)$. Thus, one can assume that $n = p^r$.

Now suppose that the common order of G and H is

$$n = p_1^{r_1} \cdots p_k^{r_k}.$$

Then we can write G and H as direct sums of their Sylow subgroups

$$G = G(p_1) \oplus \cdots \oplus G(p_k) \quad \text{and} \quad H = H(p_1) \oplus \cdots \oplus H(p_k).$$

Since θ preserves order, θ maps bijectively from the Sylow p_i-subgroup $G(p_i)$ onto the Sylow p_i-subgroup $H(p_i)$ $(i = 1, \ldots, k)$. Thus, one can assume that $n = p^r$.

Now suppose that the common order of G and H is

$$n = p_1^{r_1} \cdots p_k^{r_k}.$$

Then we can write G and H as direct sums of their Sylow subgroups

$$G = G(p_1) \oplus \cdots \oplus G(p_k) \quad \text{and} \quad H = H(p_1) \oplus \cdots \oplus H(p_k).$$

Since θ preserves order, θ maps bijectively from the Sylow p_i-subgroup $G(p_i)$ onto the Sylow p_i-subgroup $H(p_i)$ $(i = 1, ..., k)$. Thus, one can assume that $n = p^r$.

In this case, $G = \bigoplus_{k=1}^{i} \mathbb{Z}_{p^{m_k}}$ and $H = \bigoplus_{l=1}^{j} \mathbb{Z}_{p^{n_l}}$, where

$$m_1 \leq \ldots \leq m_i \text{ and } n_1 \leq \ldots \leq n_j.$$

It is clear that $m_i = n_j$.

By using some counting arguments, we will see $m_{i-1} = n_{j-1}$.

Inductively, one has $i = j$ and $m_k = n_k$ $(k = 1, \ldots, i)$.

In this case, $G = \bigoplus_{k=1}^{i} \mathbb{Z}_{p^{m_k}}$ and $H = \bigoplus_{l=1}^{j} \mathbb{Z}_{p^{n_l}}$, where

$$m_1 \leq ... \leq m_i \text{ and } n_1 \leq ... \leq n_j.$$

It is clear that $m_i = n_j$.

By using some counting arguments, we will see $m_{i-1} = n_{j-1}$.

Inductively, one has $i = j$ and $m_k = n_k$ $(k = 1, ..., i)$.

In this case, $G = \bigoplus_{k=1}^{i} \mathbb{Z}_{p^{m_k}}$ and $H = \bigoplus_{l=1}^{j} \mathbb{Z}_{p^{n_l}}$, where

$$m_1 \leq ... \leq m_i \text{ and } n_1 \leq ... \leq n_j.$$

It is clear that $m_i = n_j$.

By using some counting arguments, we will see $m_{i-1} = n_{j-1}$.

Inductively, one has $i = j$ and $m_k = n_k$ $(k = 1, ..., i)$.

Theorem 2.

Suppose that $k \in \{2, 3, 4, ...\}$, and G and H are two groups such that H is k-bounded. Take any $\delta \in (0, \frac{1}{2} \sin \frac{\pi}{k})$.

1. If G is δ-contained in H, then G is isomorphic to a subgroup of H.
2. If G is δ-close to H, then they are isomorphic.

Theorem 2.

Suppose that $k \in \{2, 3, 4, ...\}$, and G and H are two groups such that H is k-bounded. Take any $\delta \in (0, \frac{1}{2} \sin \frac{\pi}{k})$.

1. If G is δ-contained in H, then G is isomorphic to a subgroup of H.
2. If G is δ-close to H, then they are isomorphic.

Theorem 2.
Suppose that $k \in \{2, 3, 4, ...\}$, and G and H are two groups such that H is k-bounded. Take any $\delta \in (0, \frac{1}{2} \sin \frac{\pi}{k})$.

1. If G is δ-contained in H, then G is isomorphic to a subgroup of H.
2. If G is δ-close to H, then they are isomorphic.

(a) A subset $E \subseteq G$ is asymptotically included in another $F \subseteq G$ if $\forall \delta > 0$, $\exists(\mathfrak{H}, \phi)$ with

$$\phi(E) \subseteq \phi(F) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})}.$$

(b) G is asymptotically abelian if $C_G := \{s^{-1}r^{-1}sr : r, s \in G\}$ is asymptotically included in $\{e\}$.

(c) G is pairwise asymptotically abelian if $\{s^{-1}r^{-1}sr\}$ asymptotically included in $\{e\}$, $\forall r, s \in G$.

(a) A subset $E \subseteq G$ is asymptotically included in another $F \subseteq G$ if $\forall \delta > 0$, $\exists (\mathfrak{H}, \phi)$ with

$$\phi(E) \subseteq \phi(F) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})}.$$

(b) G is asymptotically abelian if $C_G := \{s^{-1}r^{-1}sr : r, s \in G\}$ is asymptotically included in $\{e\}$.

(c) G is pairwise asymptotically abelian if $\{s^{-1}r^{-1}sr\}$ asymptotically included in $\{e\}$, $\forall r, s \in G$.

(a) A subset $E \subseteq G$ is asymptotically included in another $F \subseteq G$ if $\forall \delta > 0$, $\exists (\mathfrak{H}, \phi)$ with

$$\phi(E) \subseteq \phi(F) + \delta \mathbf{B}_{\mathcal{L}(\mathfrak{H})}.$$

(b) G is asymptotically abelian if $C_G := \{s^{-1}r^{-1}sr : r, s \in G\}$ is asymptotically included in $\{e\}$.

(c) G is pairwise asymptotically abelian if $\{s^{-1}r^{-1}sr\}$ asymptotically included in $\{e\}$, $\forall r, s \in G$.

Theorem 3.

1. If $r, s \in G$ have finite orders and $\{r\}$ is asymptotically included in $\{s\}$, then $o(r) = o(s)$.
2. If every element in G has a finite order and G is pairwise asymptotically abelian, then G is abelian.
3. If G is asymptotically contained in an abelian group, then G is asymptotically abelian.
4. If G is asymptotically contained in an abelian group, then G does not contain any finite non-abelian subgroup.

Theorem 3.

1. If $r, s \in G$ have finite orders and $\{r\}$ is asymptotically included in $\{s\}$, then $o(r) = o(s)$.
2. If every element in G has a finite order and G is pairwise asymptotically abelian, then G is abelian.
3. If G is asymptotically contained in an abelian group, then G is asymptotically abelian.
4. If G is asymptotically contained in an abelian group, then G does not contain any finite non-abelian subgroup.

Theorem 3.

1. If $r, s \in G$ have finite orders and $\{r\}$ is asymptotically included in $\{s\}$, then $o(r) = o(s)$.
2. If every element in G has a finite order and G is pairwise asymptotically abelian, then G is abelian.
3. If G is asymptotically contained in an abelian group, then G is asymptotically abelian.
4. If G is asymptotically contained in an abelian group, then G does not contain any finite non-abelian subgroup.

Theorem 3.

1. If $r, s \in G$ have finite orders and $\{r\}$ is asymptotically included in $\{s\}$, then $o(r) = o(s)$.
2. If every element in G has a finite order and G is pairwise asymptotically abelian, then G is abelian.
3. If G is asymptotically contained in an abelian group, then G is asymptotically abelian.
4. If G is asymptotically contained in an abelian group, then G does not contain any finite non-abelian subgroup.

Theorem 3.

1. If $r, s \in G$ have finite orders and $\{r\}$ is asymptotically included in $\{s\}$, then $o(r) = o(s)$.
2. If every element in G has a finite order and G is pairwise asymptotically abelian, then G is abelian.
3. If G is asymptotically contained in an abelian group, then G is asymptotically abelian.
4. If G is asymptotically contained in an abelian group, then G does not contain any finite non-abelian subgroup.

If $G_1, ..., G_n$ are groups in a class $\mathcal{G}$ such that G_i is asymptotically close to G_{i+1} for every $i = 1, .., n-1$, then we say that G_1 and G_n are asymptotically equivalent inside $\mathcal{G}$.

When $\mathcal{G}$ is the class of all groups, we simple say that G_1 and G_n are *asymptotically equivalent*.

If $G_1, ..., G_n$ are groups in a class $\mathcal{G}$ such that G_i is asymptotically close to G_{i+1} for every $i = 1, .., n-1$, then we say that G_1 and G_n are asymptotically equivalent inside $\mathcal{G}$.

When $\mathcal{G}$ is the class of all groups, we simple say that G_1 and G_n are *asymptotically equivalent*.

Theorem 4.

1. If G and H are asymptotically equivalent inside the class of groups whose elements are all of finite orders, then G is abelian if and only if H is abelian.
   Note that G, H need not be isomorphic (see Example (d)).
2. If G is k-bounded and is asymptotically equivalent to H, then G is isomorphic to H.
3. If two groups are asymptotically equivalent, then either they are both finite and isomorphic or they are both infinite.
4. Let $\mathcal{H}$ be the class of all finitely generated infinite abelian groups. Any two elements in $\mathcal{H}$ are asymptotically equivalent inside $\mathcal{H}$.

Theorem 4.

1. If G and H are asymptotically equivalent inside the class of groups whose elements are all of finite orders, then G is abelian if and only if H is abelian.
   Note that G, H need not be isomorphic (see Example (d)).
2. If G is k-bounded and is asymptotically equivalent to H, then G is isomorphic to H.
3. If two groups are asymptotically equivalent, then either they are both finite and isomorphic or they are both infinite.
4. Let $\mathcal{H}$ be the class of all finitely generated infinite abelian groups. Any two elements in $\mathcal{H}$ are asymptotically equivalent inside $\mathcal{H}$.

Theorem 4.

1. If G and H are asymptotically equivalent inside the class of groups whose elements are all of finite orders, then G is abelian if and only if H is abelian.
   Note that G, H need not be isomorphic (see Example (d)).
2. If G is k-bounded and is asymptotically equivalent to H, then G is isomorphic to H.
3. If two groups are asymptotically equivalent, then either they are both finite and isomorphic or they are both infinite.
4. Let $\mathcal{H}$ be the class of all finitely generated infinite abelian groups. Any two elements in $\mathcal{H}$ are asymptotically equivalent inside $\mathcal{H}$.

Theorem 4.

1. If G and H are asymptotically equivalent inside the class of groups whose elements are all of finite orders, then G is abelian if and only if H is abelian.
   Note that G, H need not be isomorphic (see Example (d)).
2. If G is k-bounded and is asymptotically equivalent to H, then G is isomorphic to H.
3. If two groups are asymptotically equivalent, then either they are both finite and isomorphic or they are both infinite.
4. Let $\mathcal{H}$ be the class of all finitely generated infinite abelian groups. Any two elements in $\mathcal{H}$ are asymptotically equivalent inside $\mathcal{H}$.

Theorem 4.

1. If G and H are asymptotically equivalent inside the class of groups whose elements are all of finite orders, then G is abelian if and only if H is abelian.
   Note that G, H need not be isomorphic (see Example (d)).
2. If G is k-bounded and is asymptotically equivalent to H, then G is isomorphic to H.
3. If two groups are asymptotically equivalent, then either they are both finite and isomorphic or they are both infinite.
4. Let $\mathcal{H}$ be the class of all finitely generated infinite abelian groups. Any two elements in $\mathcal{H}$ are asymptotically equivalent inside $\mathcal{H}$.

Theorem 4.

1. If G and H are asymptotically equivalent inside the class of groups whose elements are all of finite orders, then G is abelian if and only if H is abelian.
   Note that G, H need not be isomorphic (see Example (d)).
2. If G is k-bounded and is asymptotically equivalent to H, then G is isomorphic to H.
3. If two groups are asymptotically equivalent, then either they are both finite and isomorphic or they are both infinite.
4. Let $\mathcal{H}$ be the class of all finitely generated infinite abelian groups. Any two elements in $\mathcal{H}$ are asymptotically equivalent inside $\mathcal{H}$.

Results presented here can be found from

> *Chi-Wai Leung, Chi-Keung Ng and Ngai-Ching Wong, On a notion of closeness of groups, Ann. Funct. Anal.* **7***(1) (2016), 24–32.*

Please visit

`http:\\www.math.nsysu.edu.tw\ ~wong`

to explore more of Wong.

Thank you!

Results presented here can be found from

> *Chi-Wai Leung, Chi-Keung Ng and Ngai-Ching Wong, On a notion of closeness of groups, Ann. Funct. Anal.* **7***(1) (2016), 24–32.*

Please visit

`http:\\www.math.nsysu.edu.tw\ ~wong`

to explore more of Wong.

Thank you!