Free Hilbert Transforms

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AHA 2018 Kaohsiung

Based mainly on a joint work with Eric Ricard

June 25, 2018

Commutative case

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Remark $\int_{\mathbb{T}} x = \hat{x}(0) = c_0$ and $\|x\|^p_{L^p(\hat{\mathbb{Z}})} = \int_{\mathbb{T}} |x|^p = \widehat{|x|^p}(0)$.

Fourier Multipliers on Free groups

$\mathbb{F}_n$: group of n free generators, $1 \leq n \leq \infty$.

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Given $x = \sum_s c_s \lambda_s$, set
$\tau x = c_e$;

$$\begin{aligned} L^p(\hat{\mathbb{F}}_n) &= \{x; \|x\|_{L^p} = (\tau|x|^p)^{\frac{1}{p}} < \infty\} \\ \|x\|_{B(\ell_2)} &= \sup_{1<p<\infty} \|x\|_{L^p} \end{aligned}$$

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- $p = 2$; $\|T_m\| = \|m\|_{\ell_\infty}$.
- $p = 1, \infty$; iff $m(s^{-1}t) = \langle Q(s), R(t)\rangle_H$ with

$$\sup_{s,t} \|Q(s)\|_H \|R(t)\|_H < C.$$

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An Example

Let $m = \chi_{\mathbb{F}_2^+}$, with $\mathbb{F}_2^+ = \{s = g_1^{k_1} g_2^{k_2} g_1^{k_3} \cdots \in \mathbb{F}_2; k_1 > 0\}$

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Ozawa (2010), asked again

Free Hilbert transforms–Main Result

Set $\mathbb{L}_0 = \{e\} \subset \mathbb{F}_\infty$;

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Theorem (M-Ricard, 17') For any $x = \sum_g c_g \lambda_g \in C_c(\mathbb{F}_\infty)$ and $1 < p < \infty$,

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Corollary

$$\|(\sum_{j=-\infty}^{\infty} |L_j x|^2)^{\frac{1}{2}}\|_{L^p(\hat{\mathbb{F}}_n)} + \|(\sum_{j=-\infty}^{\infty} |(L_j x)^*|^2)^{\frac{1}{2}}\|_{L^p(\hat{\mathbb{F}}_n)} \simeq \|x\|_{L^p(\hat{\mathbb{F}}_n)},$$

for $p \geq 2$, applying Lust-Piquard/Pisier's non commutative Khintchine's inequality.

Convergence of the Tail of Fourier Sums on $\mathbb{F}_n$.

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Yes! (M-Ricard '17) and $\lim_{|h| \mapsto \infty} \|L_h x\|_{L^p(\hat{\mathbb{F}}_n)} = 0.$

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Yes, M-Ricard '17 because
$\|[L_h, x](y)\|_{L^2(\hat{\mathbb{F}}_n)} \le \|L_h\|_{\mathcal{L}(\mathbb{F}_n) \mapsto L^4(\hat{\mathbb{F}}_n)} \|x\|_{L^4(\hat{\mathbb{F}}_n)}$ by Hölder's inequality.

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Question Is this a lucky case, or there is a general result for all Fourier multipliers associated with the first segment ?

Fourier multipliers associated with the first segment

Consider

$$T_m : \lambda_s \mapsto m(k_1(s))\lambda_s; \quad m \in \ell_\infty(\mathbb{Z}, \mathbb{C})$$

with $k_1(s)$ the first power index in the reduced word,

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Remark By taking the group homomorphism $\pi : \mathbb{F}_\infty \mapsto \mathbb{F}_2$ sending generators g_k to $a^k b a^k$, we see that $k_1(\pi(s)) = k$ iff $\pi(s) \geq g_k$. So T_m catches the information on the starting letter through π.

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Question: Characterize the c. boundedness of T_m on $L^p(\hat{\mathbb{F}}_n)$?

Remark By taking the group homomorphism $\pi : \mathbb{F}_\infty \mapsto \mathbb{F}_2$ sending generators g_k to $a^k ba^k$, we see that $k_1(\pi(s)) = k$ iff $\pi(s) \geq g_k$. So T_m catches the information on the starting letter through π.

Necessary Condition: $T_m|_{\mathbb{F}_1} : e^{ik\theta} \mapsto m(k)e^{ik\theta}$ is completely bounded on $L^p(\mathbb{T})$.

Fourier multipliers associated with the first segment

Theorem 1 (M-Xu, 18') Given $1 < p < \infty$, then

$$T_m : \lambda_s \mapsto m(k_1(s))\lambda_s$$

extends to a completely bounded operator on $L^p(\hat{\mathbb{F}}_\infty)$ if its restriction on $\mathbb{F}_1$

$$T_m|_{\mathbb{F}_1} : \lambda_{g_1^k} \mapsto m(k)\lambda_{g_1^k}$$

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Remark No chance for the case of $p = \infty$. Take $m(k) = \chi_{[-2,2]}(k)$, which is the symbol of a c.b multiplier on $L^\infty(\mathbb{F}_1)$. Then the multiplier T_m is the projection onto the set $\{s; |k_1(s)| \leq 2\}$ of $L^\infty(\mathbb{F}_\infty)$. To see this, let

$$x = \sum_{-N<k<N} c_k (g_1 g_2^3)^k g_1$$

and note

$$T_m(x) = \sum_{0 \leq k < N} c_k (g_1 g_2^3)^k g_1.$$

A Littlewood-Paley Theory on Free groups

Example Let $A_0 = \{0\}$

$$A_j = [2^{j-1}, 2^j), j \in \mathbb{N}; A_j = -A_{-j}, j \in -\mathbb{N}.$$

Let $\chi_j = \chi_{A_j}$ for $j \in \mathbb{Z}$. Let

$$m = \sum_{j \in \mathbb{Z}} \varepsilon_j \chi_j$$

for any $\varepsilon_j = \pm$.

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Corollary (M-Xu, 18') For $2 \leq p < \infty$,

$$\|x\|_{L^p} \simeq \|(\sum_j |T_{\chi_j}(x)|^2)^{\frac{1}{2}}\|_{L^p} + \|(\sum_j |T_{\chi_j}(x)^*|^2)^{\frac{1}{2}}\|_{L^p}.$$

The Mikhin condition

Corollary (M-Xu, 18') If m satisfies the Mikhlin condition e.g.

$$\sup_{k\in\mathbb{Z}}\{|m(k)|, k|m(k)-m(k-1)|\} < C.$$

Then

$$T_m : \lambda_s \mapsto m(k_1(s))\lambda_s$$

extends to a completely bounded linear operator on $L^p(\hat{\mathbb{F}}_\infty)$ for all $1 < p < \infty$.

Example $m(k) = k^{ti}$.

Ingredient: A Dirac operator on the free group

Let $D_\varepsilon(\lambda_e) = 0$ and

$$D_\varepsilon(\lambda_s) = \varepsilon_{i_1} k_1 \lambda_s$$

for

$$s = g_{i_1}^{k_1} g_{i_2}^{k_2} \cdots g_{i_m}^{k_m},$$

and $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n \ldots)$ a sequence of $|\varepsilon_k| = 1$.

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Theorem 2 (M-Xu, 18') For $1 < p < \infty$,

$$S_t = e^{itD_\varepsilon}, -\infty < t < \infty$$

extends to a uniformly c.b group of operators on $L^p(\hat{\mathbb{F}}_\infty)$.

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Transference $\Rightarrow$ Theorem 1 that $m(D_\varepsilon)$ is c. bounded on $L^p(\hat{\mathbb{F}}_n)$ for m being the symbol of a c.b. Fourier multiplier on $L^p(\hat{\mathbb{Z}})$, e.g.

$$sign(D) = H_\varepsilon.$$

Fourier multipliers along a geodesic path?

Proposition (Chua, 18')
Fix a geodesic path $\mathbb{P}$ on the Cayley graph of $\mathbb{F}_\infty$, then

$$T_m : \lambda_s \mapsto m(|E_{\mathbb{P}} s|)\lambda_s$$

extends to a completely bounded linear operator on $L^p(\hat{\mathbb{F}}_\infty)$ for all $1 < p < \infty$ if m has a bounded variation, i.e.

$$\sum_{k \in \mathbb{N}} |m(k-1) - m(k)| < C.$$

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Remark $|E_{\mathbb{P}} s| = |k_1(s)|$ for $s \geq g_j$ if $\mathbb{P} = \{g_j^k, k > 0\}$. The Propsition is a consequence of the boundedness of the free Hilbert transforms H_ε.

Thank you.

Happy Birthday! Tony