The wavelet representation for shifts by wallpaper groups

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joint work with
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Thank you Professor Anthony Lau for 50 years of contributions to abstract harmonic analysis and your leadership of the Canadian team.

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Happy 75^{th} birthday.

Dedicated to the memory of **Eberhard Kaniuth**

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7. Wavelets for shifts by a crystal group Γ and a compatible expansive transformation A.
8. Simple $A\Gamma$-wavelet sets. Lots of pictures.

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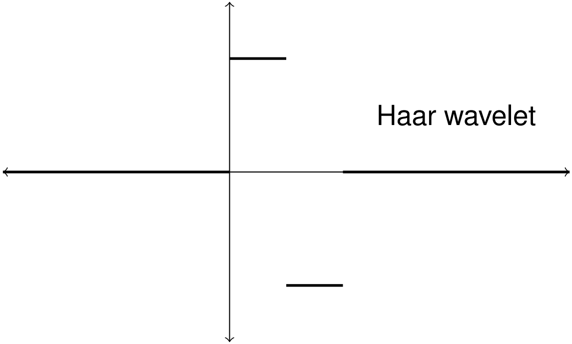

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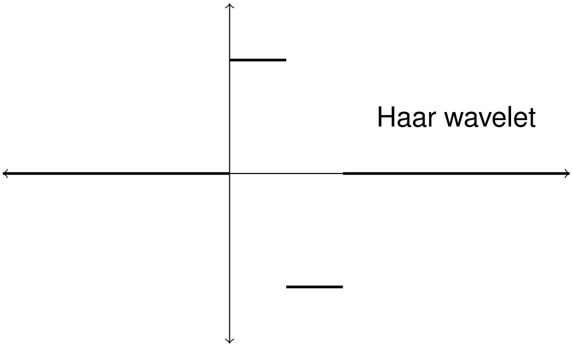

Then w is a classical wavelet called the *Haar wavelet*. It is widely used in applications even though it is not smooth.

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Thus, $\{\widehat{D_2}^j \widehat{T}_k \mathbf{1}_\Omega : j \in \mathbb{Z}, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$.

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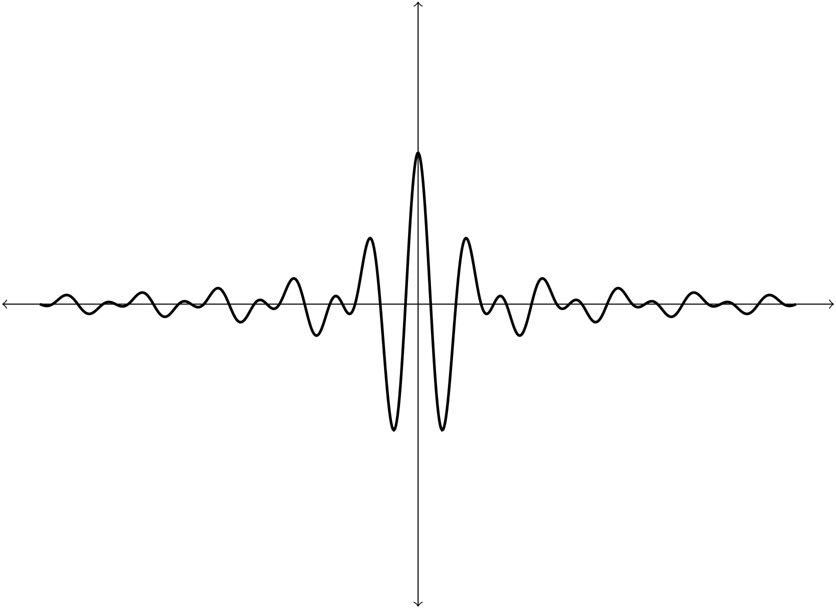
It has the disadvantage of very slow decay.

The Shannon Wavelet, graph



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See Martin and Valette: Markov Operators on the Solvable Baumslag-Solitar Groups (2000). They coined the term wavelet group.

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Definition

An *A-wavelet set* is a Borel subset Ω of $\mathbb{R}^d$ such that

$$\left\{ D_A^j T_{\underline{k}} w : \underline{k} \in \mathbb{Z}^d, j \in \mathbb{Z} \right\}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$, where $w \in L^2(\mathbb{R}^d)$ satisfies $\widehat{w} = \mathbf{1}_\Omega$.

Existence of A-wavelet sets

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2008, 2012, 2015: Kathy Merrill constructed simple A-wavelet sets for increasingly wider classes of matrices A.

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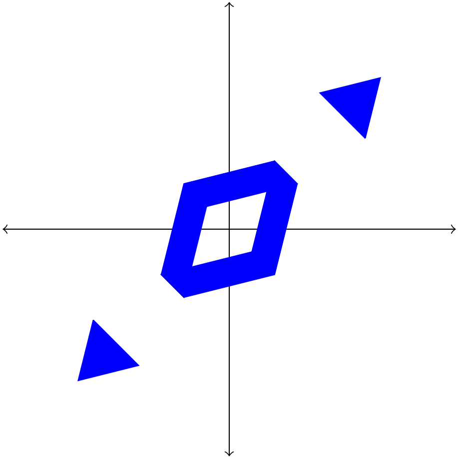

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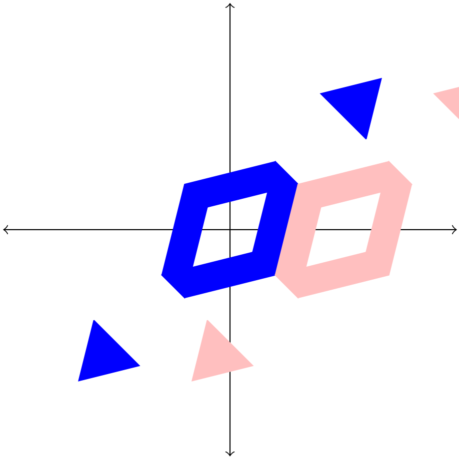

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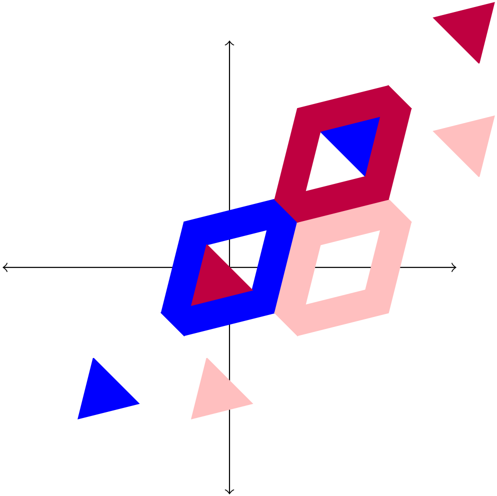

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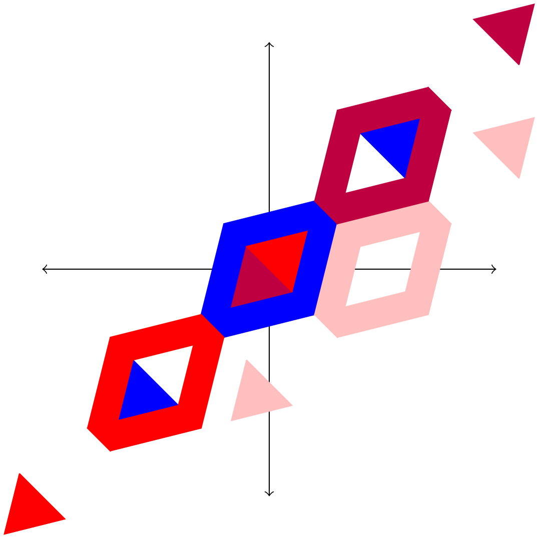

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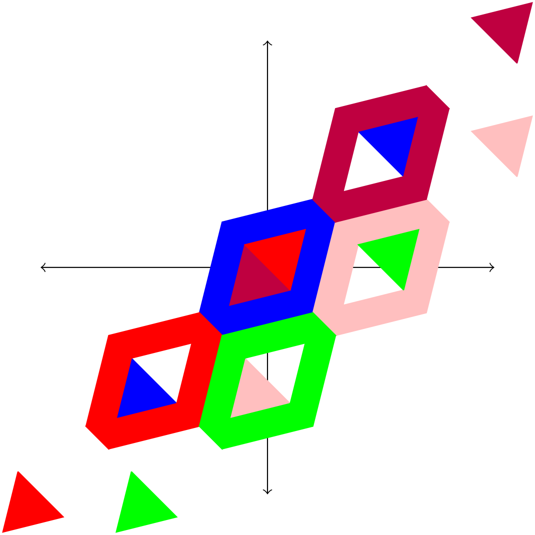

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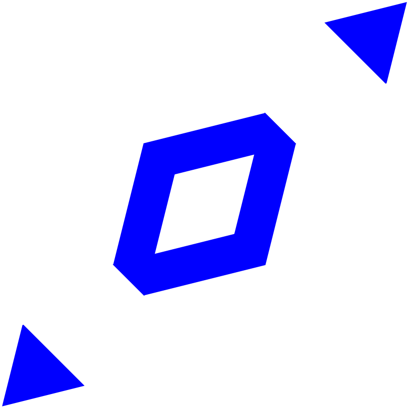

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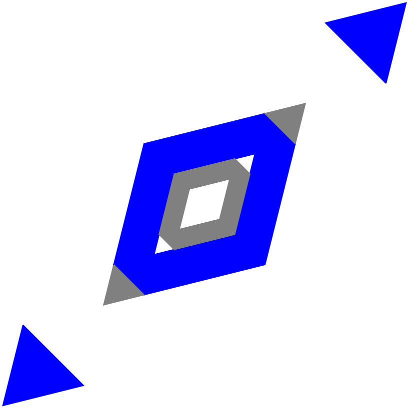

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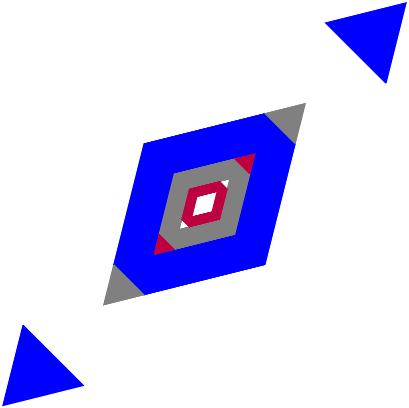

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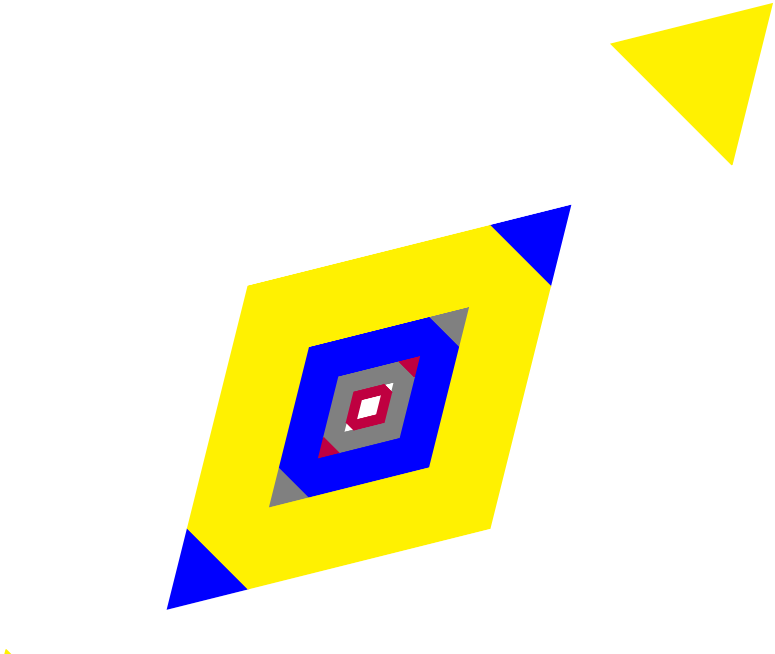

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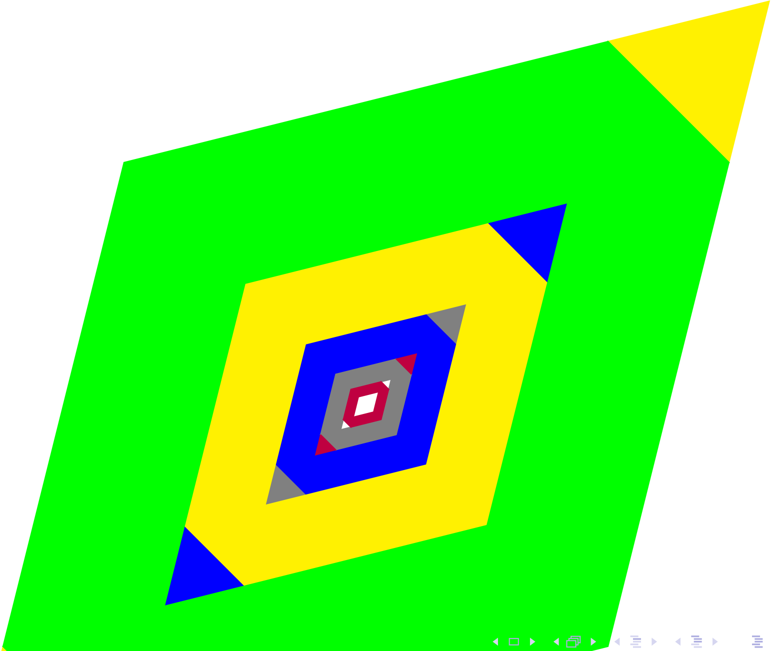

Lim, Packer and T: *A Direct Integral Decomposition of the Wavelet Representation*, PAMS **129** 3057-3067 (2001).

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In the above paper, we provided a direct integral decomposition of ρ into irreducible representations. Let me explain.

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For $\underline{\omega} \in \mathbb{R}^d$, define $\chi_{\underline{\omega}}$ in $\widehat{\mathcal{N}_A}$ by

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Thus, $\{\chi_{\underline{\omega}} : \underline{\omega} \in \mathbb{R}^d\}$ is weakly equivalent with the regular representation of $\mathcal{N}_A$.

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Saying that $\{\chi_{\underline{\omega}} : \underline{\omega} \in \mathbb{R}^d\}$ is weakly equivalent with the regular representation of $\mathcal{N}_A$ is simply saying that $\{\chi_{\underline{\omega}} : \underline{\omega} \in \mathbb{R}^d\}$ is dense in $\widehat{\mathcal{N}_A}$.

Decomposing the A-wavelet representation, II

An A-wavelet set Ω is *free* if $A^j\Omega \cap A^k\Omega = \emptyset$, for $j \neq k$. For every simple A-wavelet set, there is a free simple A-wavelet set that differs only by a null set.

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Let Ω be a free A-wavelet set in $\mathbb{R}^d$. Then the A-wavelet representation ρ is unitarily equivalent to the direct integral

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and $U^{\underline{\omega}}$ is irreducible for $\underline{\omega} \in \Omega$. Moreover, $\{U^{\underline{\omega}} : \underline{\omega} \in \Omega\}$ is weakly equivalent with the left regular representation of G_A.

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We recently returned to this theme in order to explore the implications of the introduction of crystal symmetries into the theory of wavelets.

Crystal groups

For $\underline{x} \in \mathbb{R}^d$ and $B \in \mathrm{GL}_d(\mathbb{R})$, define the affine transformation $[\underline{x}, B]$ by $[\underline{x}, B]\underline{z} = B(\underline{z} + \underline{x})$, for all $\underline{z} \in \mathbb{R}^d$.

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A 2-dimensional crystal group is also called a *wallpaper group*.

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Crystal groups II

Let $\mathrm{Tran}(\mathbb{R}^d) = \{[\underline{x}, \mathrm{id}] : \underline{x} \in \mathbb{R}^d\}$, the normal subgroup of $\mathrm{Aff}(\mathbb{R}^d)$ consisting of pure translations.

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$$\{1\} \to \mathbb{Z}^d \to \Gamma \to \mathcal{D} \to \{1\}.$$

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As before $D_A f(\underline{y}) = |\det(A)|^{1/2} f(A\underline{y})$.

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Let $G_{A\Gamma} = \Gamma[A] \rtimes_{\vartheta} \mathbb{Z}$, the $A\Gamma$-*wavelet group*.

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$$G_{A\Gamma} = \Gamma[A] \rtimes_{\vartheta} \mathbb{Z} = \{([\underline{\beta}, L], j) : [\underline{\beta}, L] \in \Gamma[A], j \in \mathbb{Z}\}.$$

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The *$A\Gamma$-wavelet representation* is the map $\rho : G_{A\Gamma} \to \mathcal{U}(L^2(\mathbb{R}^d))$ given by

$$\rho([\underline{\beta}, L], j) f(\underline{x}) = T_{[\underline{\beta}, L]} D_A^j f(\underline{x}) = |\det(A)|^{j/2} f\left(A^j L^{-1} \underline{x} - A^j \underline{\beta}\right).$$

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But, let's digress again!

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Definition: MacArthur and T, 2009

Let Γ be a d-dimensional crystal group and A a compatible matrix. An $A\Gamma$-*multiwavelet* is a finite set $\{w_1, \cdots, w_\ell\} \in L^2(\mathbb{R}^d)$ such that

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Definition

A *simple* $A\Gamma$-wavelet set is an $A\Gamma$-wavelet that is a finite union of convex sets.

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I believe we have accomplished 2. for all Γ and $A = 3 \cdot \mathrm{id}$, which is compatible with all wallpaper groups, as long as we can do 1.

Fix $A = 3 \cdot \mathrm{id}$ since it is compatible with all wallpaper groups. Let Γ be a wallpaper group with point group $\mathcal{D}$ and $\underline{u}$, $\underline{v}$ as basic spanning vectors.

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Let Ω be a Borel subset of $\mathbb{R}^2$. Then Ω is an $A\Gamma$-wavelet set iff (1), (2) and (3) hold.

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(1) $\cup_{(j,k)\in\mathbb{Z}^2}(\Omega + j\underline{u} + k\underline{v})$ is co-null in $\mathbb{R}^2$ and $(j', k') \neq (k, j)$ implies $(\Omega + j\underline{u} + k\underline{v}) \cap (\Omega + j'\underline{u} + k'\underline{v})$ is a null set.

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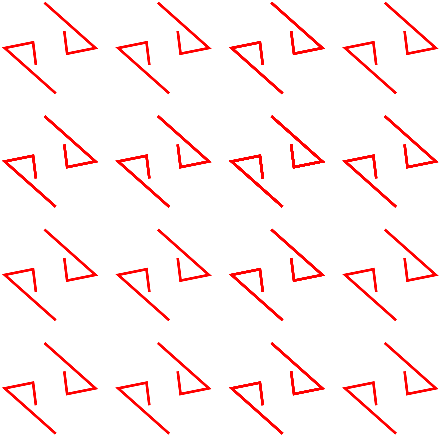
(3) $\cup_{\ell\in\mathbb{Z}} A^\ell\big(\cup_{L\in\mathcal{D}} L\Omega\big)$ is co-null in $\mathbb{R}^2$ and $\ell \neq m$ implies $A^\ell\big(\cup_{L\in\mathcal{D}} L\Omega\big) \cap A^m\big(\cup_{L\in\mathcal{D}} L\Omega\big)$ is a null set.

Start with $\Gamma = p2$



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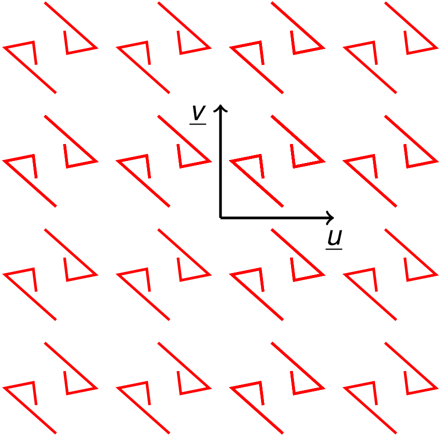

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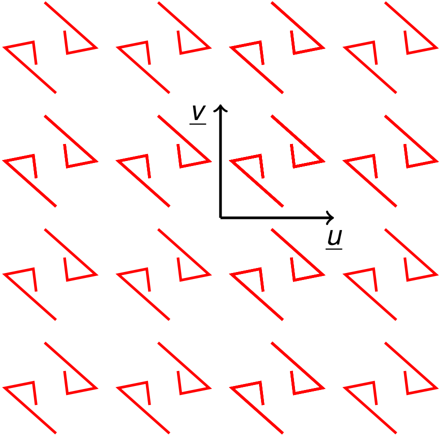

$p2 = \{[k\underline{u} + \ell\underline{v}, L] : (k,\ell) \in \mathbb{Z}^2, L \in \{\mathrm{id}, R_\pi\}\}$, where R_π is rotation through π.

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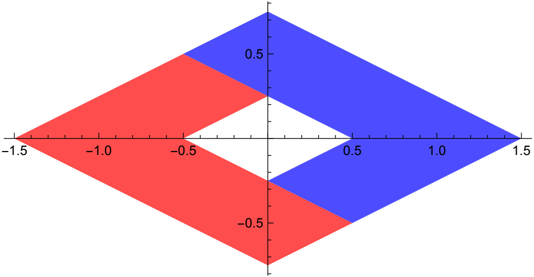

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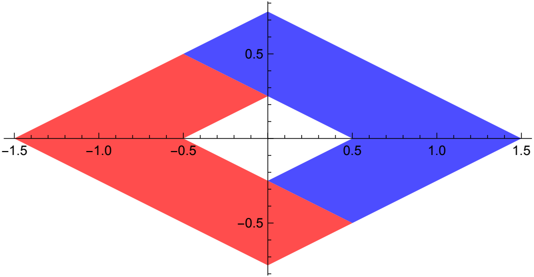

The red set is $R_\pi \Omega$.

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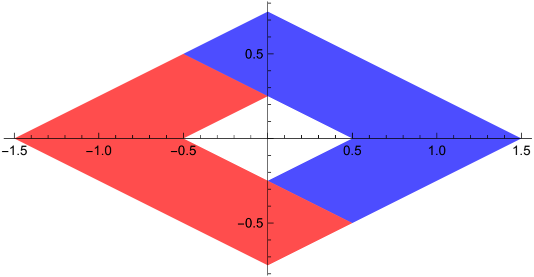

The red set is $R_\pi\Omega$. The plane is tiled by $\{3^\ell(\Omega \cup R_\pi\Omega) : \ell \in \mathbb{Z}\}$.

A simple $A\Gamma$-wavelet set for $\Gamma = p2$

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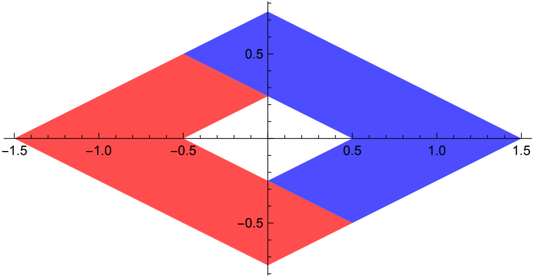

The red set is $R_\pi\Omega$. The plane is tiled by $\{3^\ell(\Omega \cup R_\pi\Omega) : \ell \in \mathbb{Z}\}$. We need to show that the blue set tiles the plane by $\mathbb{Z}^2$ translations.

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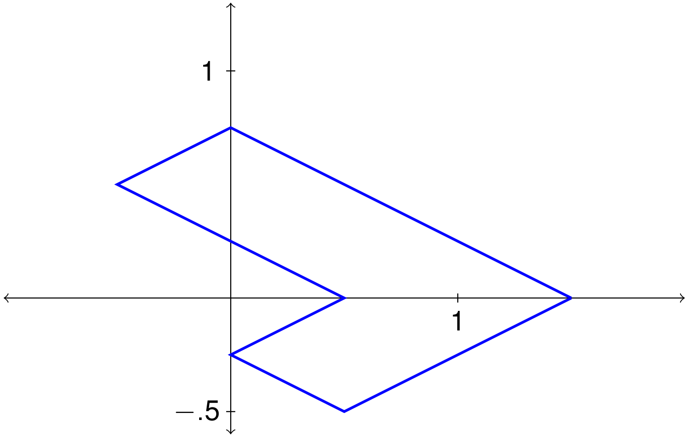

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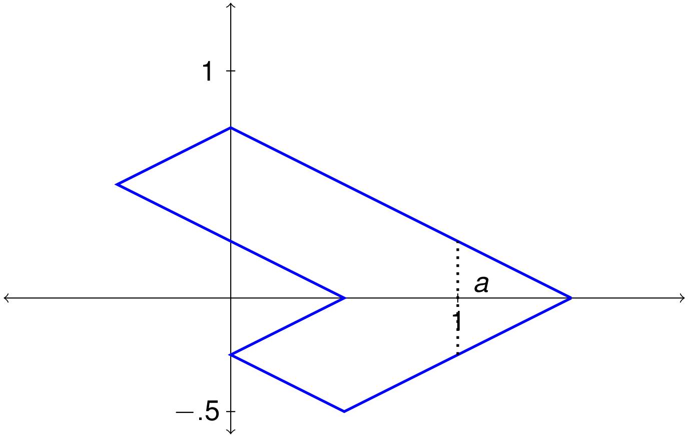

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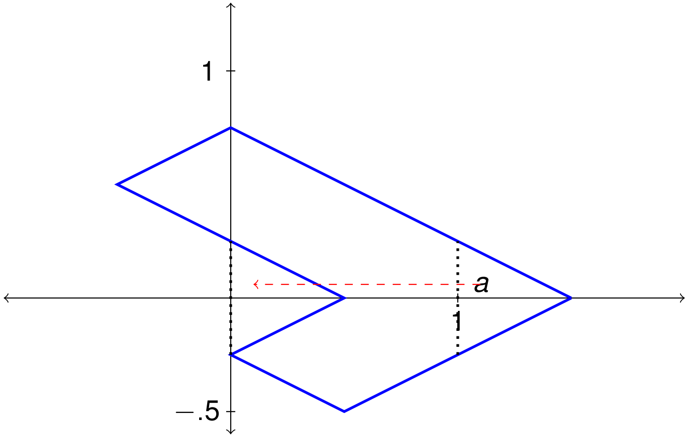

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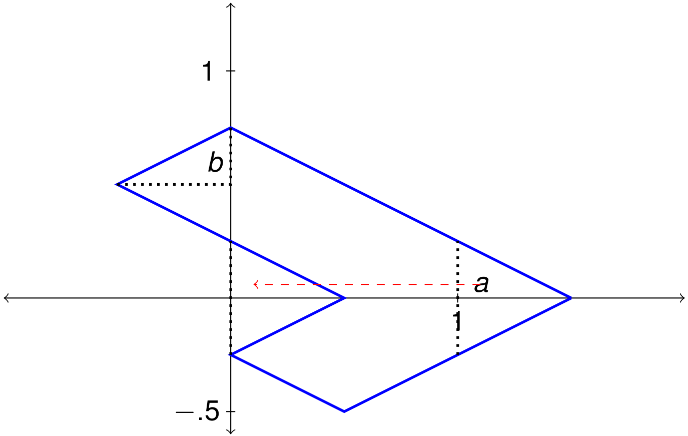

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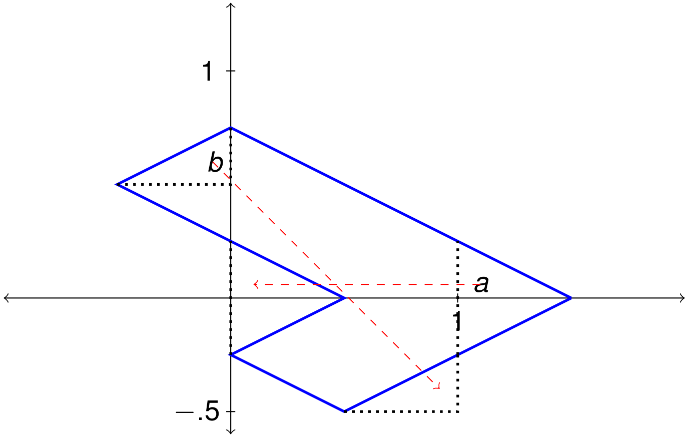

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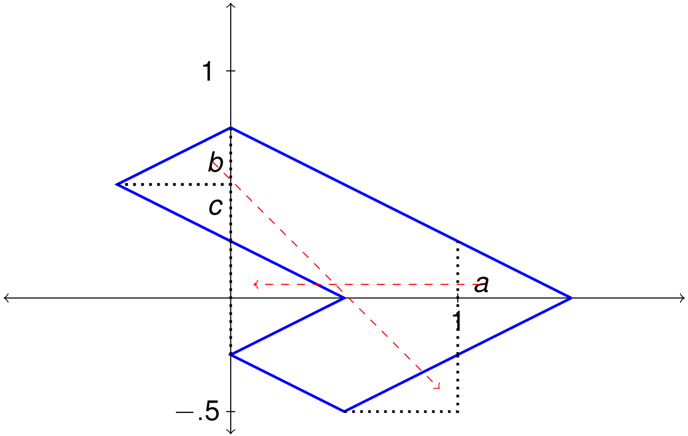

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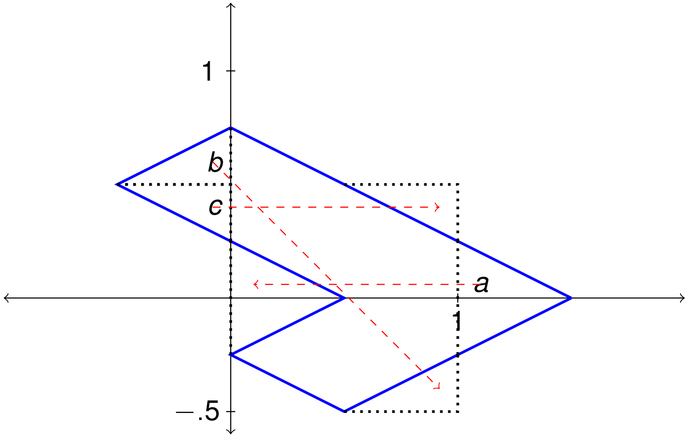

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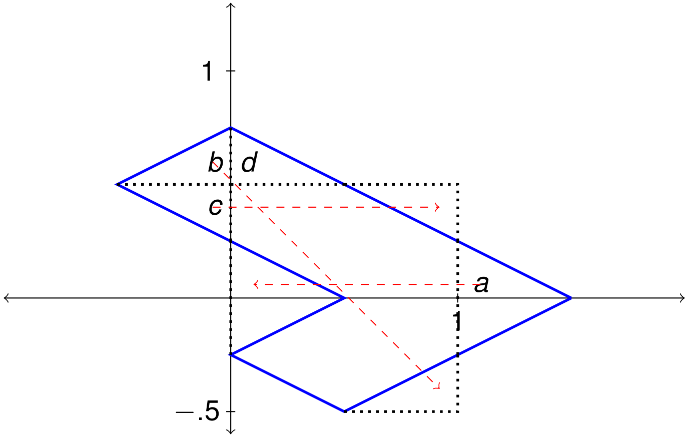

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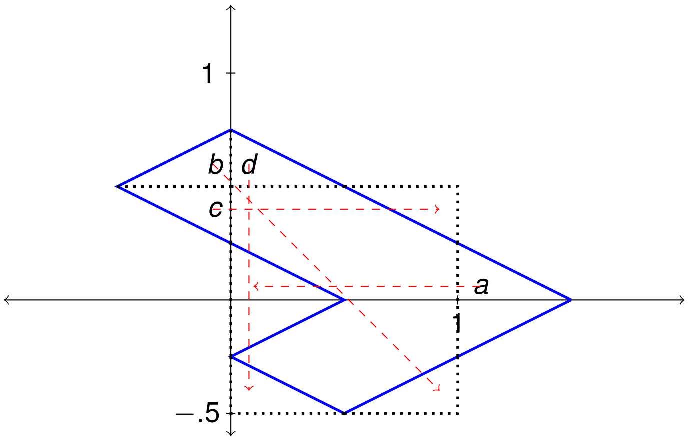

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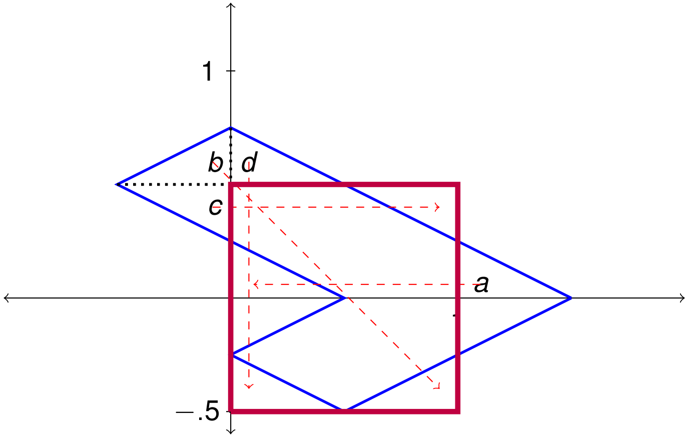

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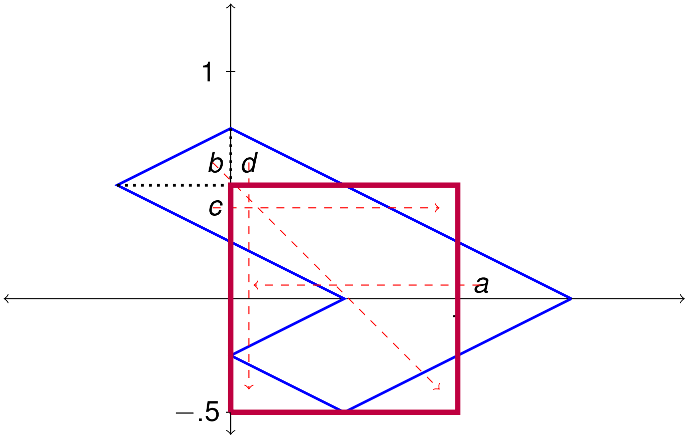
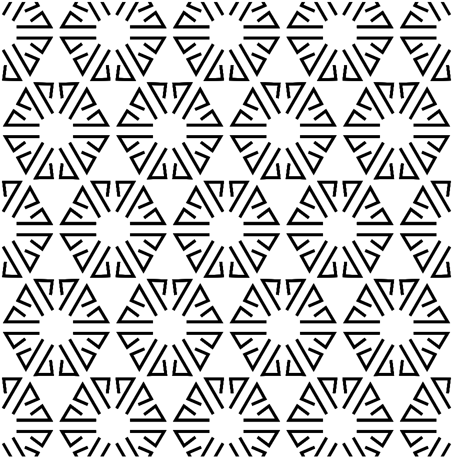


Thus, the set interior to the blue line tiles the plane by $\mathbb{Z}^2$ translations.

Hexagonal symmetry



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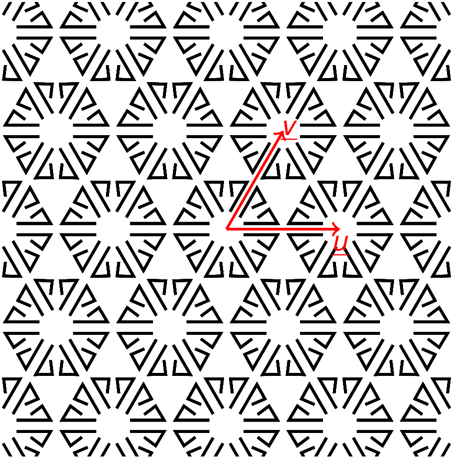

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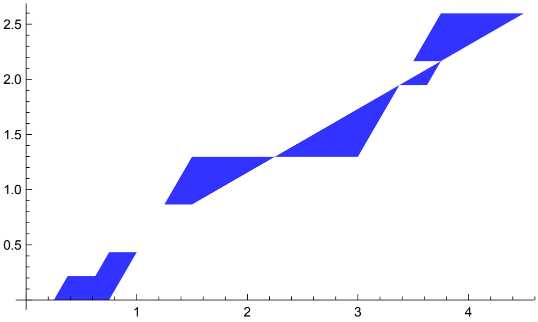
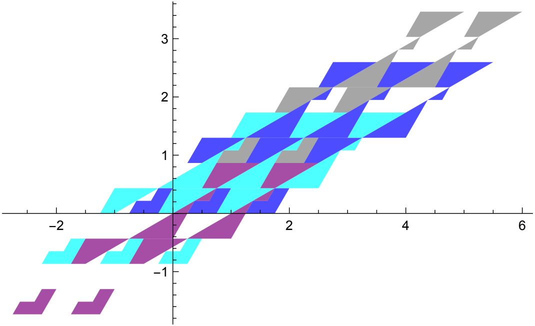
Here is an example.

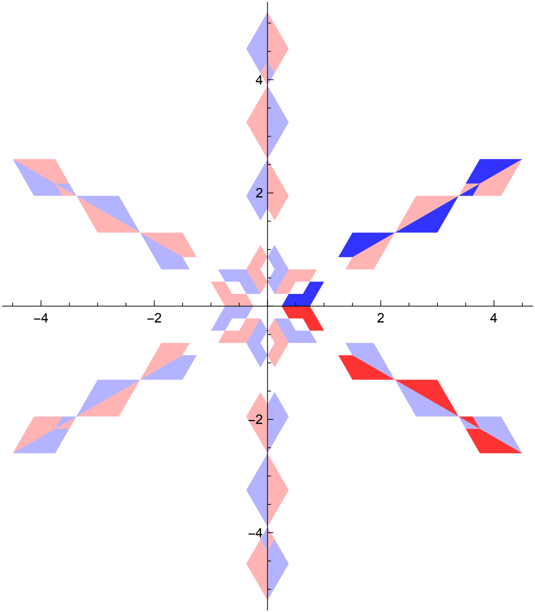
An $A\Gamma$-wavelet set, $\Gamma = p6m$



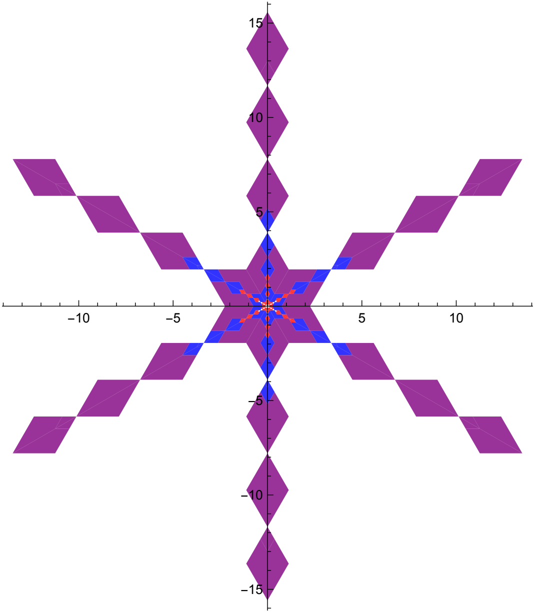
Tiling by integer translations



Union under action of point group



Tiling the plane with dilations by powers of 3



Concluding remarks

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The End

Thank you!