# Simple modules of the $L^1$-group algebra of $SL_2(\mathbb{R})$

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Abstract

We determine the simple modules of the algebra $L^1(\mathrm{SL}_2(\mathbb{R}))$ up to equivalence
and we show that these modules are the finite rank sub-modules of the $L^p$-principal series and of the discrete series representations of $\mathrm{SL}_2(\mathbb{R})$

# Elementary definitions

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We say that $X$ is an *$A$-module* or *left* $A$-module, if there exists a non-trivial algebra homomorphism $T : A \to L(X)$.

If $X$ is equipped with a norm $\|\|_X$ then we say that the $A$-module $(T, X)$ is *bounded*, if there exists a constant $C > 0$ such that

$$\|a \cdot x\|_X \leq C\|a\|_A\|x\|_X, a \in A, x \in X.$$

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We say that the bounded $A$-module $(T, X)$ is a *Banach* module, if $X$ is a Banach space.

We say that the $A$-module $(T, X)$ is *simple*, if $X \neq \{0\}$ and if the only $A$-invariant subspaces of $X$ are the two trivial ones, namely $X$ and $\{0\}$.

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We say that the Banach $A$-module $(T, X)$ is *irreducible*, if the only $A$-invariant *closed* subspaces of $X$ are the two trivial ones, namely $X$ and $\{0\}$.
Let $(T, X)$ be an $A$-module. Let $0 \neq x \in X$. The annihilator $A_x$ of $x$ in $A$ is the left ideal

$$A_x := \{a \in A; a \cdot x = 0\}.$$

A left ideal $I$ of the algebra $A$ is called *modular*, if there exists $u \in A$, such that

$$a \cdot u - a \in I, a \in A.$$

The element $u \in A$ is called a *(right) modular unit* (for $I$) .

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## Proposition

1. *Let $(T, X)$ be an $A$-module. Let $x \neq 0$ be a cyclic vector of $X$, which means that the subspace $A \cdot x$ is equal to $X$ itself. Then the $A$ modules $(T, X)$ and $A/A_x$ are equivalent.*

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2. *Let* $(T, X)$ *be an* $A$*-module. Then* $T$ *is simple if and only if every* $x \neq 0$ *in* $X$ *is a cyclic vector.*

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2. *Let $(T, X)$ be an $A$-module. Then $T$ is simple if and only if every $x \neq 0$ in $X$ is a cyclic vector.*
3. *If $(T, X)$ is a simple $A$-module, then for every $x \neq 0$ in $X$ the annihilator $A_x$ of $x$ is a maximal modular left ideal of $A$.*

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3. *If $(T, X)$ is a simple $A$-module, then for every $x \neq 0$ in $X$ the annihilator $A_x$ of $x$ is a maximal modular left ideal of $A$.*
4. *If $I$ is a maximal modular left ideal of $A$, then the left $A$-module $A/I$ is simple.*

### Remark

Two simple $A$-modules $(T, X)$ and $(T', X')$ are equivalent, if and only if there exists $x \in X$ and $x' \in X'$ such that the annihilator ideals $A_x$ and $A_{x'}$ coincide.

# A fundamental property of simple modules

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**Theorem**

*Let $A$ be a Banach algebra and let $I$ be a proper maximal modular left ideal of $A$. Then $I$ is closed in $A$.*

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### Corollary

*Let $(T, X)$ be a simple $A$-module. Then there exists a norm $\|\|_X$ on $X$, such that $(T(X, \|\|_X)$ is Banach $A$-module.*

# Simple modules and the spectrum

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## Theorem

*Let $A$ be a Banach algebra and $a \in A$. Take any $0 \neq \lambda \in \mathbb{C}$. Then $\lambda \in \sigma(a)$ if and only there exists a simple module $(T, X)$ of $A$ and an element $x \neq 0$ in $X$ such that*

$$a \cdot x = \lambda x.$$

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## Definition

Let $A$ be an involutive Banach algebra. We say that $A$ is *symmetric* or *hermitian* if the spectrum $\sigma_A(a) \subset \mathbb{R}$ for any $a = a^*$.

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### Example

Let $G$ be a locally compact nilpotent or compactly generated group of polynomial growth. Then $L^1(G)$ is symmetric.

# Construction of simple modules

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## Definition

Let $A$ be a Banach algebra and let $(T, X)$ be a Banach module. We denote by $I_T^{fin}$ the ideal of $A$ defined by

$$I_T^{fin} := \{a \in A;\ \text{the operator } T(a) \in B(X) \text{ has finite rank}\}.$$

Let also $X^{fin}$ be the $(A)$-invariant subspace of $X$ given by

$$X^{fin} := \{\text{span}(T(a)(X)); a \in I_T^{fin}\}.$$

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*Let $A$ be a Banach algebra and let $(T, X)$ be an irreducible Banach module. Suppose that the finite rank subspace $X^{fin}$ is different from $\{0\}$. Then $X^{fin}$ is the unique simple submodule of $(T, X)$.*

# An example: The Heisenberg group

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Let $H_n = \mathbb{R}^{2n} \times \mathbb{R}$ with the multiplication

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - x' \cdot y)),$$

where $x \cdot y = x_1 y_1 + \cdots + x_n y_n$ denotes the Euclidean scalar product on $\mathbb{R}^n$.

For every $\lambda \in \mathbb{R}^*$, there exists an irreducible unitary representation $\pi_\lambda$ of $H_n$ on the Hilbert space $\mathcal{H}_\lambda = L^2(\mathbb{R}^n)$, which is given by the formula

$$\begin{aligned} \pi_\lambda(x, y, t)\xi(s) \quad &:= \quad e^{-2\pi i\lambda t - 2\pi i \frac{\lambda}{2} x\cdot y + 2\pi i \lambda s\cdot y}\xi(s - x), \\ & \qquad s \in \mathbb{R}^n, \xi \in L^2(\mathbb{R}^n), (x, y, t) \in H_n. \end{aligned}$$

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For $F \in L^1(H_n)$ the operator $\pi_\lambda(F)$ is a kernel operator with kernel function $F_\lambda$ given by

$$F_\lambda(s,t) \quad = \quad \hat{F}^{2,3}(s-t, -\frac{\lambda}{2}(s+t), \lambda), s,t\in\mathbb{R}^n.$$

Take any $\xi, \eta \in L^2(\mathbb{R})$ and let $P_{\xi,\eta}$ be the rank one operator

$$P_{\xi,\eta}(\varphi) \quad := \quad \langle \varphi, \eta \rangle \cdot \xi, \; \varphi \in L^2(H_n).$$

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If now $\pi_\lambda(F) = P_{\xi,\eta}$ for some $F \in L^1(H_n)$, then

$$\hat{F}^{2,3}(s-t, -\frac{\lambda}{2}(s+t), \lambda) \quad = \quad \xi(s)\overline{\eta(t)}, s, t \in \mathbb{R}^{2n}.$$

Hence

$$\hat{F}^3(s, u, \lambda) \quad = \quad e^{\frac{i\lambda}{4} u \cdot s} \frac{|\lambda|^n}{4} (\chi_s \cdot \hat{\xi}) * \hat{\eta}^*(\frac{\lambda}{2} u), s, u \in \mathbb{R}^n.$$

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Hence $\pi_\lambda$ admits finite rank operator and fixing a Schwartz function $\eta$ we have that

$$\mathcal{H}_\lambda^{fin} \quad = \quad \{\xi \in L^2(\mathbb{R}^n); \int_{\mathbb{R}^{2n}} |(\chi_s \cdot \hat{\xi}) * \hat{\eta}^*(u)| duds < \infty.$$

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## Proposition

*Let $A$ be a Banach algebra and take a multiplier $p$ of $A$, such that $p^2 = p$. Let $(T, X)$ be a simple $A$-module such that $X_p := T(p)X \neq \{0\}$.*

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a) *Then $(T|_{X_p}, X_p)$ is a simple $(pAp)$-module.*

b) *Take $0 \neq y \in X_p$. The ideal $A_y \subset A$ is then given by*

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c) *Let $(T, X)$ and $(T', X')$ be two simple $A$-modules. Let's assume that there is a $(pAp)$-linear isomorphism*

$$\Phi : X_p \longrightarrow X'_p,$$

*i.e. such that*

$$\Phi(T(pap)pv) = T'(pap)\Phi(pv).$$

*Then there is a unique extension of $\Phi$ to an $A$-linear isomorphism between $X$ and $X'$.*

## Proposition

a) *Assume that* $(S, Y)$ *is a simple* $(pAp)$*-module. Then* $A/A_y$ *is a simple* $A$*-module, where*

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b) *The simple* $(pAp)$*-module* $(S, Y)$ *is equivalent to* $(L|_{(p \cdot A/A_y)}, p \cdot A/A_y)$.

# Semi-simple Lie groups

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## Definition

Let $\chi_\rho = \chi$ be the character of $K$ corresponding to a fixed irreducible representation $\rho$ of $K$ of dimension $d_\rho$, i.e. $\chi(k) = \chi_\rho(k) = \overline{d_\rho \operatorname{tr} \rho(k)}$, $k \in K$. Normalize the Haar measure $dk$ of $K$ so that $\int_K dk = 1$. The operator $\pi(\chi)$ in $B(X)$ is a projection, since $\chi * \chi = \chi$.

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We say that the representation $(\pi, X)$ is *admissible* (for $K$) if for every character $\chi$ of $K$ the subspace $X_\chi$ is finite dimensional.

Let $\Xi(K)$ be the set of all the irreducible characters of $K$. Notice that the sum $\sum_{\chi \in \Xi} X_\chi$ is direct and that the Banach space $X$ is the closure of $\sum_\chi X_\chi$.

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## Definition

For every character $\chi$ of $K$, let $L^1_\chi = L^1(G)_\chi$ be the closed involutive subalgebra of $L^1(G)$ defined by

$$L^1(G)_\chi := \chi * L^1(G) * \chi.$$

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1. *If $\pi$ is admissible, then the dense submodule $X_{fin}$ of $\pi$ is the unique simple submodule of $\pi$.*
2. *If $\pi$ admits a simple submodule $X_0$, then $\pi$ is admissible with $dim(X_\chi) \leq d_\chi^2$ for all $\chi \in \Xi(K)$.*

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### Proposition

*Let $G$ be a connected linear semi-simple Lie group and let $(\pi, X)$ be an irreducible bounded admissible Banach representation of $G$. Then, for every element $D$ in the center of the enveloping algebra $U(\mathfrak{g})$ of $G$, the operator $d\pi(D)$ on $X^\infty$ is a multiple of the identity.*

## Remark

*Let* $(\pi, X)$ *be a simple* $L^1(G)$ *module such that* $\pi(\chi) \neq 0$ *for some character* $\chi$ *of* $K$. *Let* $x \in X_\chi$ *and choose a vector* $\xi$ *in the anti-dual space* $X'$ *of* $X$ *such that* $\langle \xi, x \rangle = 1$ *and* $\langle \xi, ker(\pi(\chi)) \rangle = \{0\}$ . *Then we obtain a coefficient*

$$c^{\pi}_{x,\xi}(g) := \langle \xi, \pi(g)x \rangle, \quad g \in G,$$

*which has the following property.*

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*Let $(\pi, X)$ be a simple $L^1(G)$ module such that $\pi(\chi) \neq 0$ for some character $\chi$ of $K$. Let $x \in X_\chi$ and choose a vector $\xi$ in the anti-dual space $X'$ of $X$ such that $\langle \xi, x \rangle = 1$ and $\langle \xi, ker(\pi(\chi)) \rangle = \{0\}$ . Then we obtain a coefficient*

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## Proposition

1. *For every central element $D \in \mathcal{U}(\mathfrak{g})$, we have that $D * c^{\pi}_{x,\xi} = \lambda c^{\pi}_{x,\xi}$ for some $\lambda \in \mathbb{C}$.*
2. *Two simple $L^1(G)$-modules $(\pi, X, )$ and $(\pi', X')$ are equivalent if and only if there exists a coefficient $c^{\pi}_{x,\xi} \neq 0$ of $\pi$ and a coefficient $c^{\pi'}_{x',\xi'}$ of $\pi$ such that $c^{\pi'}_{x',\xi'} = c^{\pi}_{x,\xi}$.*

# Induced representation

## Definition

Let $H$ be a closed subgroup of a locally compact group $G$. Let $\mathcal{E}(G/H)$ be defined by

$$\begin{aligned}\mathcal{E}(G/H) &= \{\xi : G \to \mathbb{C}; \xi(gh) = \Delta_{H,G}(h)\xi(g), \quad \forall g \in G, h \in H, \\ &\qquad \xi \text{ is continuous with compact support modulo } H\}\end{aligned}$$

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## Proposition

*Let $H$ be a closed subgroup of a locally compact group $G$. There exists a unique (up to multiplication by a positive constant) $G$-invariant positive linear functional, denoted by*

$$k \mapsto \mu_{G,H}(k) = \oint_{G/H} k(x) d\mu_{G,H}(x) = \oint_{G/H} k(x) d\dot{x},$$

*on the space $\mathcal{E}(G/H)$. We have that*

$$\int_G k(t)dt = \oint_{G/H} \left( \int_H k(th)\Delta_{G,H}(h)dh \right) d\dot{t}, \quad \forall k \in C_c(G). \tag{0.1}$$

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$$\mathcal{E}^p(G/H, T)$$

by

$$\begin{aligned} \mathcal{E}^p(G/H, T) \quad := \quad & \{\xi : G \to X;\ \xi(gh) = \Delta_{H,G}^{1/p}(h) T(h^{-1})(\xi(g)), \\ & g \in G, h \in H, \\ & \xi \ \text{ is continuous with compact support modulo } H\}. \end{aligned}$$

We remark that the space $\mathcal{E}^p(G/H, T)$ is left translation invariant and that for $\xi \in \mathcal{E}^p(G/H, T)$ the function

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is continuous with compact support modulo $H$ and satisfies the relation

$$q_\xi(xh) \quad = \quad \Delta_{H,G}(h) q_\xi(x), x \in G, h \in H,$$

and so $q_\xi \in \mathcal{E}(G/H)$. We can thus define a norm on $\mathcal{E}(G/H, \rho)$ by

$$\|\xi\|_p^p := \oint_{G/H} \|\xi(g)\|_X^p d\dot{g}.$$

## Definition

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We denote this action by $\pi_{T,p} = \mathrm{ind}_H^G(T, p)$, where

$$\pi_{T,p}(t)\xi(s) := \xi(t^{-1}s), \xi \in L^p(G/H, T), s, t \in G. \tag{0.2}$$

# The group $G = SL_2(\mathbb{R})$

In the following we consider the linear group

$$G := SL_2(\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}), ad - bc = 1 \right\}. \tag{0.3}$$

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The torus

$$K := \left\{ k_\theta = \left( \begin{array}{cc} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{array} \right), \theta \in \mathbb{R} \right\} \tag{0.4}$$

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Characters $\chi_l$, with $l \in \mathbb{Z}$, are of the form $\chi_l(k_\theta) := e^{il\theta}, \theta \in \mathbb{R}$.
Let $I$ denote the $2 \times 2$ identity matrix. We set $P := MAN \subset G$, where

$$\begin{aligned} M &:= \{\pm I\}, \\ A &:= \left\{ a_r = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}, \ r > 0 \right\}, \\ N &:= \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \ x \in \mathbb{R} \right\}. \end{aligned}$$

Modular function:

$$\Delta_{AN}\left(\begin{pmatrix} r & x \\ 0 & r^{-1} \end{pmatrix}\right) = r^{-2}, \quad r \in \mathbb{R}_+^*, \, x \in \mathbb{R}. \tag{0.5}$$

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Then

$$\begin{aligned} r &= \sqrt{a^2 + c^2} \\ \cos\theta &= \frac{a}{\sqrt{a^2 + c^2}} \\ \sin\theta &= -\frac{c}{\sqrt{a^2 + c^2}}\,. \end{aligned}$$

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$$\sigma_\pm(\varepsilon I) := \begin{cases} 1 & \text{if } \varepsilon = + \\ \varepsilon & \text{if } \varepsilon = - \end{cases}.$$

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Let $p \in [1, \infty[$. Define the space $L^p(G/P, \eta_{\tau,\pm})$ as the completion of the space

$$\begin{aligned} & C^\infty_\pm(G/P, \eta_\tau, p) \\ = \ & \{f : G \to \mathbb{C}, f \text{ smooth}, f(gma_r n) = \sigma_\pm(m) r^{-(\frac{2}{p}+i\tau)} f(g) \\ & \text{ for all } g \in G,\ m \in M,\ a_r \in A,\ n \in N\} \end{aligned}$$

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for the $L^p$-norm:

$$\|f\|_p^p = \oint_{G/P} |f(g)|^p \, d\dot{g} = \int_K |f(k)|^p \, dk, \qquad f \in C_\pm^\infty(G/P, \eta_\tau).$$

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Similarly, for $p = \infty$, we let

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The Banach space $L^\infty(G/P, \eta_{\tau,\pm})$ is by definition the closure for the infinity norm $\|f\|_\infty := \sup_{k \in K} |f(k)|$ of the space $C^{\infty}_{\tau,\pm}$.

## Definition

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$$\pi^{p}_{\tau,\pm} = \pi_{s,\pm} := \mathrm{ind}_P^G(\eta_{\tau,\pm}, p)$$

be the induced representation for $P = MAN$ and the character $\eta_{\tau,\pm}$, which acts by left translation on the space $L^p(G/P, \eta_{\tau,\pm})$.

For the composition series of $\pi^p_{\tau,\pm}$, consider for $l \in \mathbb{Z}$ the function $\chi^p_{\tau,l}$ defined by

$$\chi^p_{\tau,l}(kan) := \chi_{-l}(k)\eta_\tau(a)\Delta^{1/p}_{AN}(an), \qquad k \in K,\ a \in A,\ n \in N$$

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Then the functions $\chi^p_{\tau,l}$, $l \in 2\mathbb{Z}$, form a total subset of $L^p_{\tau,+}$ and the functions $\chi^p_{\tau,l}$, $l \in 2\mathbb{Z}+1$, form a total subset of $L^p_{\tau,-}$.

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$$E^- := \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad E^+ := \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad W := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

form a basis for the complexification $\mathfrak{g}_\mathbb{C} := \mathfrak{sl}_2(\mathbb{C})$ of the Lie algebra $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{R})$ of $G$.

We know from [La85], VI.5, that we have

$$(0.7)\qquad \begin{aligned} d\pi^p_{\tau,\pm}(W)\chi^p_{\tau,l} &= il\,\chi^p_{\tau,l}, \\ d\pi^p_{\tau,\pm}(E^-)\chi^p_{\tau,l} &= \Big(\frac{2}{p}+i\tau-l\Big)\chi^p_{\tau,l-2}, \\ d\pi^p_{\tau,\pm}(E^+)\chi^p_{\tau,l} &= \Big(\frac{2}{p}+i\tau+l\Big)\chi^p_{\tau,l+2}. \end{aligned}$$

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These relations hold true for the case $p=\infty$ by setting $2/p := 0$.
The formulas (0.7) show that the representations $\pi^p_{\tau,\pm}$ are irreducible if $\frac{2}{p}+i\tau \notin \mathbb{Z}$. For $\frac{2}{p}+i\tau \in \mathbb{Z}$ we have special cases.

# The discrete series.

For $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, let

$$(0.8) \qquad \alpha(x) := \frac{1}{2}(a + d - ic + ib)\,, \qquad \beta(x) := \frac{1}{2}(c + b - ia + id)\,.$$

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We denote them by $\pi_s$.
The $K$-eigenvalues of the spanning functions $\overline{\xi_{s,r}}$ are again the characters $\xi_{-s-2r}$, $r \in \mathbb{N}$, $s \leq -2$.

# A list of simple module

Proposition

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*1) If* $s := \frac{2}{p} + i\tau - 1 \notin \mathbb{Z}$, $p \in [1, \infty]$, *then to* $(p, \tau)$ *correspond two simple* $L^1(G)$*-modules:* $((\pi^p_{\tau,+})^{fin}, (L^p_{\tau,+})^{fin})$ *and* $((\pi^p_{\tau,-})^{fin}, (L^p_{\tau,-})^{fin})$.

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## Proposition

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*2) To $(p, \tau) = (\infty, 0)$, there correspond 4 simple $L^1(G)$-modules:*

- $((\pi^\infty_{0,-})^{fin}, (L^\infty_{0,-})^{fin})$,
- *the trivial one dimensional module* $f \in L^1(G) \to \int_G f(g)\, dg$,
- *the module* $((\pi^\infty_{0,+,+})^{fin}, (L^\infty_{0,+,+})^{fin})$, *where*

$$L^\infty_{0,+,+} = span\{\chi^\infty_{0,l}; l \in 2\mathbb{N}\} \ mod\ \mathbb{C}\chi^\infty_{0,0}$$

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5) For every $s \in \mathbb{N}$, $s \geq 2$ or $s \in -\mathbb{N}, s \leq -2$, we have the simple $L^1(G)$-modules $(\pi_s^{fin}, (L^2_s(G))^{fin})$ inside $L^2(G)$.

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Let $s + 1 := \frac{2}{p} + i\tau \in [-1, 1] + i\mathbb{R}$ and let $l \in \mathbb{Z}$. We obtain the coefficient $c_{s,l}$ of the representation $\pi^p_{\tau,\pm}$ by

# Questions:

# When is $\pi^{fin}_{s,\varepsilon}$ equivalent to $\pi^{fin}_{s',\varepsilon'}$?

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$$c_{s,l}(g) := \langle \pi^p_\tau(g)\chi^p_{\tau,l}, \chi^q_{\tau,l} \rangle, \; g \in G. \tag{0.9}$$

For $r \in \mathbb{R}_+^*$ we then have that

$$\begin{aligned} c_{s,l}(a_r) &= \int_K \chi_{\tau,l}^p(a_r^{-1}k)\overline{\chi_{\tau,l}^q(k)}\, dk \\ &= 2\int_{-\pi/2}^{\pi/2}\Big(\frac{1}{\sqrt{r^2\sin^2\psi + \frac{\cos^2\psi}{r^2}}}\Big)^{2/p+i\tau}(\cos\psi + i\sin\psi)^l \\ &\qquad \times\Big(\frac{\cos\psi}{r\sqrt{r^2\sin^2\psi+\frac{\cos^2\psi}{r^2}}} + i\frac{r\sin\psi}{\sqrt{r^2\sin^2\psi+\frac{\cos^2\psi}{r^2}}}\Big)^{-l}\ \frac{d\psi}{2\pi} \end{aligned}$$

Furthermore, we have

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Hence $c_{s,l}$ is $K$-invariant.

# The characters of the algebra $L^1(G)_l$

Let $l \in \mathbb{Z}$. To simplify the notation, we write $L^1(G)_l$ for the subalgebra $L^1(G)_{\chi_l} = \overline{\chi}_l * L^1(G) * \overline{\chi}_l = \chi_{-l} * L^1(G) * \chi_{-l}$ of $L^1(G)$.

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Hence the simple $L^1(G)$-modules are now determined by the characters of the abelian algebras $L^1(G)_l$, $l \in \mathbb{Z}$.

### Definition

Let $\phi : G \to \mathbb{C}$ a nonzero $C^\infty$-function on $G$. We say that $\phi$ is an *l-spherical function* provided it satisfies

$$\int_K \phi(gkg')\chi_l(k^{-1})dk = \phi(g)\phi(g') \tag{0.14}$$

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## Lemma

*Let $\phi$ be an I-spherical function. Then*

$$\phi(kgk') = \chi_I(kk')\phi(g)$$

*for all $g \in G$ and $k, k' \in K$. Consequently,*

$$\begin{aligned} \phi(k) &= \chi_I(k) \quad \text{for all } k \in K\,, \\ \phi(I) &= 1\,. \end{aligned}$$

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## Proposition

*Integration against the I-spherical functions gives the characters of the commutative convolution algebra*

$$\begin{aligned} C_{c,I}(G) &:= \overline{\chi}_I * C_c(G) * \overline{\chi}_I \\ &= \{f \in C_c(G), f(kgk') = \overline{\chi_I(kk')}f(g) \text{ for all } g \in G \text{ and } k, k' \in K\}\,. \end{aligned}$$

The following proposition is standard knowledge.

## Proposition

1. *The subspace*

$$L^\infty(G)_l := \{\phi \in L^\infty(G), \phi(kgk') = \overline{\chi_l(kk')}\phi(g),\ \ k, k' \in K, g \in G\}$$

   *of $L^\infty(G)$ represents the algebraic dual space of the Banach space $L^1(G)_l$.*

2. *The characters of the commutative Banach algebra $L^1(G)_l$ are given by the bounded l-spherical functions.*

# A family of characters

We now define for $s \in \mathbb{C}$ and $l \in \mathbb{Z}$ the functions

$$(0.15) \qquad \rho_{s,l}(ka_r n) := r^{-(s+1)}\overline{\chi_l(k)}, \qquad k \in K,\ r \in \mathbb{R}_+^*,\ n \in N.$$

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Notice that for all $g \in G$, $r \in \mathbb{R}_+^*$ and $k \in K$, we have

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By Lemma 31 and Proposition 29, each function $\phi_{s,l}$ determines by integration on $G$ a character of the commutative algebra $C_{c,l}(G)$.

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For their explicit expression, observe that because of Lemma 27 and the decomposition $G = KAK$ of $G$, the functions $\phi_{s,l}$ are uniquely determined by their restriction to $A$. Remarking that the function $\phi \mapsto \chi_l(k_\psi)\rho_{s,l}(a_r^{-1}k_\psi)$ is $\pi$-periodic, we get:

$$\begin{aligned}
\phi_{s,l}(a_r) &= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\psi + i\sin\psi)^l \rho_{s,l}(a_r^{-1}k_\psi)\,\frac{d\psi}{2\pi} \\
&= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\psi + i\sin\psi)^l \frac{1}{\left(\sqrt{r^2\sin^2\psi + \frac{\cos^2\psi}{r^2}}\right)^{s+1}} \\
&\qquad \left(\frac{\cos\psi}{r\sqrt{r^2\sin^2\psi + \frac{\cos^2\psi}{r^2}}} + i\frac{r\sin\psi}{\sqrt{r^2\sin^2\psi + \frac{\cos^2\psi}{r^2}}}\right)^{-l}\frac{d\psi}{2\pi}\,. \qquad (0.18)
\end{aligned}$$

## Proposition

*Let $s \in \mathbb{C}$ and $l \in \mathbb{Z}$ .*

1. *For any $s \in \mathbb{C}$ and $l \in \mathbb{Z}$ we have that*

$$\phi_{s,l} = \phi_{-s,l}.$$

2.

$$c_{s,l} \quad = \quad \phi_{s,l}, s + 1 \in [-1, 1] + i\mathbb{R}.$$

# Koornwinder's list

### Proposition

*Let $l \in \mathbb{Z}$. Every bounded $l$-spherical function is of the form $\phi_{s,l}$ for some $s \in \mathbb{C}$.*

# Behaviour at infinity

We must study $\lim_{r\to\infty} \phi_{s,l}(r)$. Starting from the expression (0.18) of $\phi_{s,l}$, we get for $a_r \in A$

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Hence, if Re $s > 1$, we see that

$$\lim_{r\to\infty}\frac{\phi_{s,l}(a_r)}{r^{s-1}} = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-il\arctan(v)}\left(\frac{1}{v^2+1}\right)^{\frac{s+1}{2}} dv.$$

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and if Re $s<-1$ then

$$\lim_{r\to 0}\frac{\phi_{s,l}(a_r)}{r^{s+1}} = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{il\arctan(v)}\left(\frac{1}{v^2+1}\right)^{\frac{-s+1}{2}} dv.$$

Therefore, if $\mathrm{Re}(s) > 1$, a necessary condition for $\phi_{s,l}$ to be bounded is that the number

$$I_{s,l} := \int_{-\infty}^{\infty} e^{-il\arctan(v)} \left(\frac{1}{v^2+1}\right)^{\frac{s+1}{2}} dv \tag{0.19}$$

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similarly for Re $s < -1$ the number

$$I_{s,l} := \int_{-\infty}^{\infty} e^{il\arctan(v)} \left(\frac{1}{v^2+1}\right)^{\frac{-s+1}{2}} dv \tag{0.20}$$

must be 0.

## Proposition

*For every* $s \in \mathbb{C} \setminus \mathbb{Z}$ *with* $\mathrm{Re}(s) > 1$ *or* $\mathrm{Re}(s) < -1$*, the integral* $I_{s,l}$ *is nonzero. In particular the functions* $\phi_{s,l}$ *are not bounded if* $\mathrm{Re}(s) > 1$ *or* $\mathrm{Re}(s) < -1$ *and* $s \notin \mathbb{Z}$*.*

We can now formulate the main theorem.

### Theorem

*Every simple module of the Banach algebra* $L^1(\mathrm{SL}_2(\mathbb{R}))$ *is equivalent to one of the simple modules listed in Proposition 24. Two simple modules with the parameters* $(s,\varepsilon)$ *resp.* $(s',\varepsilon')$ *are equivalent if and only if* $\varepsilon=\varepsilon'$ *and* $s'=s$ *or* $s'=-s$.

# Mautner's group, a question

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Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let

$$\begin{aligned} M = M_\theta &= \mathbb{R} \ltimes \mathbb{C}^2 \\ (t,u,v)\cdot(t',u',v') &= (t+t'), e^{-it'}u+u', e^{-i\theta t'}v+v') \end{aligned}$$

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# Question: What is $Simple(M)$?

Thank you

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