The Jacobson Radicals of Second Dual Algebras

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- A paper: White, J. T. , The radical of the bidual of a Beurling algebra, *Quarterly Journal of Mathematics*, to appear.

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- A memoire: Dales, H. G. and Lau, A. T. –M. , The second duals of Beurling algebras, *Mem. Amer. Math. Soc.*, **177** (2005), no. 836.
- Some notation: given a Banach space E write E' for the dual of E. Hence E'' is its bidual.

Arens Products

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- Let A be a Banach algebra. Let $\Phi, \Psi \in A''$. Then there exist nets (a_α) and (b_β) in A such that, in the weak*-topology,

$$\Phi = \lim_\alpha a_\alpha, \qquad \Psi = \lim_\beta b_\beta.$$

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- Better definition:
$$\begin{aligned} \langle \lambda \cdot a, b\rangle &= \langle \lambda, ab\rangle, & \langle a \cdot \lambda, b\rangle &= \langle \lambda, ba\rangle, \\ \langle \Phi \cdot \lambda, a\rangle &= \langle \Phi, \lambda \cdot a\rangle, & \langle \lambda \cdot \Psi, a\rangle &= \langle \Psi, a \cdot \lambda\rangle, \\ \langle \Psi \Box \Phi, \lambda\rangle &= \langle \Psi, \Phi \cdot \lambda\rangle, & \langle \Psi \Diamond \Phi, \lambda\rangle &= \langle \Phi, \lambda \cdot \Psi\rangle, \end{aligned}$$
for $\Phi, \Psi \in A'', \lambda \in A', a, b \in A$.

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- $\mathrm{rad}\,(A) = \{a \in A : ba \in \mathcal{Q}(A),\; b \in A\}$ - the Jacobson radical of A.
- In particular if I is a left ideal of A such that $I^n = \{0\}$ for some $n \in \mathbb{N}$ (i.e. I is nilpotent), then $I \subset \mathrm{rad}\,(A)$.

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Question

Given a Banach algebra A, is $(A'', \Box)$ semisimple? If not, what can we say about $\mathrm{rad}\,(A'')$?

Historical Results

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Theorem (Daws, Read, 2004)

For $1 < p < \infty$ the algebra $\mathcal{B}(\ell^p)''$ is semisimple if and only if $p = 2$.

Historical Results

Theorem (Civin & Yood; Granirer; Gulick)

Let G be a locally compact group. Then $\mathrm{rad}\,(L^1(G)'')$ is non-seperable if either:

- *G is non-discrete;*
- *G is discrete, infinite, and amenable.*

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Method of Invariant Means

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Let G be a discrete group, write $A = \ell^1(G)$, and define $\varphi_0 \colon \ell^1(G) \to \mathbb{C}$ by

$$\varphi_0(f) = \sum_{s \in G} f(s) \quad (f \in \ell^1(G)).$$

Define $J = \{\Phi \in A'' : \delta_s \Box \Phi = \Phi \ (s \in G),\ \varphi_0''(\Phi) = 0\}$. Then we have

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Claim: $J^{\Box 2} = \{0\}$.
Proof
Let $\Phi \in J$. Notice that $\delta_s \Box \Phi = \varphi_0(\delta_s)\Phi$ ($s \in G$). This implies that

$$\Psi \Box \Phi = \varphi_0''(\Psi)\Phi \quad \Psi \in A''.$$

Hence, if $\Psi, \Phi \in J$ we have

$$\Psi \Box \Phi = \varphi_0''(\Psi)\Phi = 0.$$

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$J = \{\Phi \in A'' : \delta_s \Box \Phi = \Phi \ (s \in G),\ \varphi_0''(\Phi) = 0\}.$

Now suppose that G is amenable. Given two invariant means $M_1 \neq M_2$ on G, we have $M_1 - M_2 \in J$, using $\varphi_0''(M_i) = \langle M_i, 1 \rangle = 1 \ (i = 1, 2)$.
Hence $J \neq \{0\}$.

A Generalization

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Given an ideal $K \lhd \ell^1(G)''$ and an algebra homomorphism $\theta\colon \ell^1(G) \to \ell^1(G)$, define

$$J(\theta, K) = \{\Phi \in \ell^1(G)'' : \delta_s \Box \Phi = \theta(\delta_s) \Box \Phi \ (s \in G),\ \theta''(\Phi) \in K\}.$$

Then, by a similar argument to that given above, $J(\theta, K)^{\Box 2} \subset K$. Hence

$$K^{\Box n} = \{0\} \Rightarrow J(\theta, K)^{\Box (n+1)} = \{0\}.$$

Above, we had $J = J(\varphi_0, \{0\})$.

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$$\pi_i \colon (n_1, n_2, \ldots) \mapsto (n_1, \ldots, n_{i-1}, 0, n_{i+1}, \ldots).$$

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Inductively define a sequence of ideals $I_n \triangleleft \ell^1(G)''$ by

$$I_0 = \{0\}$$

and

$$I_n = J(\pi_n, I_{n-1}).$$

We see that $I_n^{\Box n} = \{0\}$ $(n \in \mathbb{N})$.

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By using invariant means coming from the factors to build new elements of the radical, we can inductively define elements $\Lambda_j \in I_j$ such that $\Lambda_j^{\Box(j-1)} \neq 0$. Hence $\mathrm{rad}\,(\ell^1(G)'')$ contains nilpotent elements of every index, and in particular itself not a nilpotent ideal.

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Question

Is there a finitely-generated example?

Branch Groups

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- They have proven to be a rich source of (counter)examples, exhibiting many interesting properties.
- Famous examples include: the Grigorchuk group, the Gupta-Sidki p-groups, the Basilica group.
- The latter examples are all finitely-generated and amenable.

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Let G be an amenable Branch group. Then $\mathrm{rad}\left(\ell^1(G)''\right)$ is not nilpotent.

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The idea: A given branch group G has a sequence of finite-index normal subgroups $(H_i)_{i\in\mathbb{N}}$, each of which admits a direct product decomposition

$$H_i = L_i^{(1)} \times \cdots \times L_i^{(k_i)},$$

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$$H_i = L_i^{(1)} \times \cdots \times L_i^{(k_i)},$$

in which the factors are isomorphic. We use invariant means coming from the subgroups $L_i^{(j)}$ to build nilpotent left ideals of arbitrarily large index in a similar fashion to what we did on $\bigoplus_{\mathbb{N}} \mathbb{Z}$.

Open Questions

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- Does there exist a locally compact group G for which $\operatorname{rad}(L^1(G)'')^{\Box 2} = 0$?
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- Is $\mathrm{rad}\,(L^1(G)'')$ nilpotent for any locally compact group G?
- Is $\mathrm{rad}\,(\ell^1(F_2)'') \neq \{0\}$?
- Does there exist a locally G, and a weight ω on G, for which $\mathrm{rad}\,(L^1(G,\omega)'', \Box) \neq \mathrm{rad}\,(L^1(G,\omega)'', \Diamond)$?

Thank you!